MAT 127 MIDTERM II

PRACTICE PROBLEMS

Reference page.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \quad |x| < 1$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad x \in \mathbb{R}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad x \in \mathbb{R}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad |x| < 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = \binom{k}{0} + \binom{k}{1} x + \binom{k}{2} x^2 \binom{k}{3} x^3 + \cdots, \quad |x| < 1$$

$$\binom{k}{0} = 1, \quad \binom{k}{n} = \frac{k(k-1)\cdot \cdots \cdot (k-n+1)}{n!}$$

The **Taylor series of** $f(x)$ **centered at** $x = a$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

1. (a) (10 pts) Calculate the degree 4 Taylor polynomial $T_4(x)$ of $f(x) = \cos(2x)$ centered around $x = \pi$.

Solution. By the definition we have

$$T_4(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \frac{f'''(\pi)}{3!}(x-\pi)^3 + \frac{f^{(4)}(\pi)}{4!}(x-\pi)^4.$$

We the compute the derivatives at 0:

$$\begin{aligned} f(x) &= \cos(2x), \quad f(\pi) = 1 \\ f'(x) &= -2\sin(2x), \quad f'(\pi) = 0 \\ f''(x) &= -4\cos(2x), \quad f''(\pi) = -4 \\ f'''(x) &= 8\sin(2x), \quad f'''(\pi) = 0 \\ f^{(4)}(x) &= 16\cos(2x), \quad f^{(4)}(\pi) = 16, \end{aligned}$$

which gives $T_4(x) = 1 - \frac{4}{2!}(x-\pi)^2 + \frac{16}{4!}(x-\pi)^4 = 1 - 2(x-\pi)^2 + \frac{2}{3}(x-\pi)^4$. □

(b) (10 pts) Find a power series representation of the integral

$$\int e^{-x^2}\, dx.$$

Solution. Using the Maclaurin series for e^x we get

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

We now compute the integral:

$$\begin{aligned} \int e^{-x^2}\, dx &= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}\, dx \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}. \end{aligned}$$

□

2. (a) (10 pts) Verify that $y = Ce^{-x} + x^2 - 2x$ for any value of the constant C is a solution to the second order ODE $y'' + y' = 2x$.

Solution. We differentiate and find

$$y' = -Ce^{-x} + 2x - 2, \quad y'' = Ce^{-x} + 2.$$

Then $y'' + y' = (Ce^{-x} + 2) + (-Ce^{-x} + 2x - 2) = 2x$. □

(b) (10 pts) Find a solution to the initial-value problem

$$y'' + y' = 2x, \quad y(0) = 1.$$

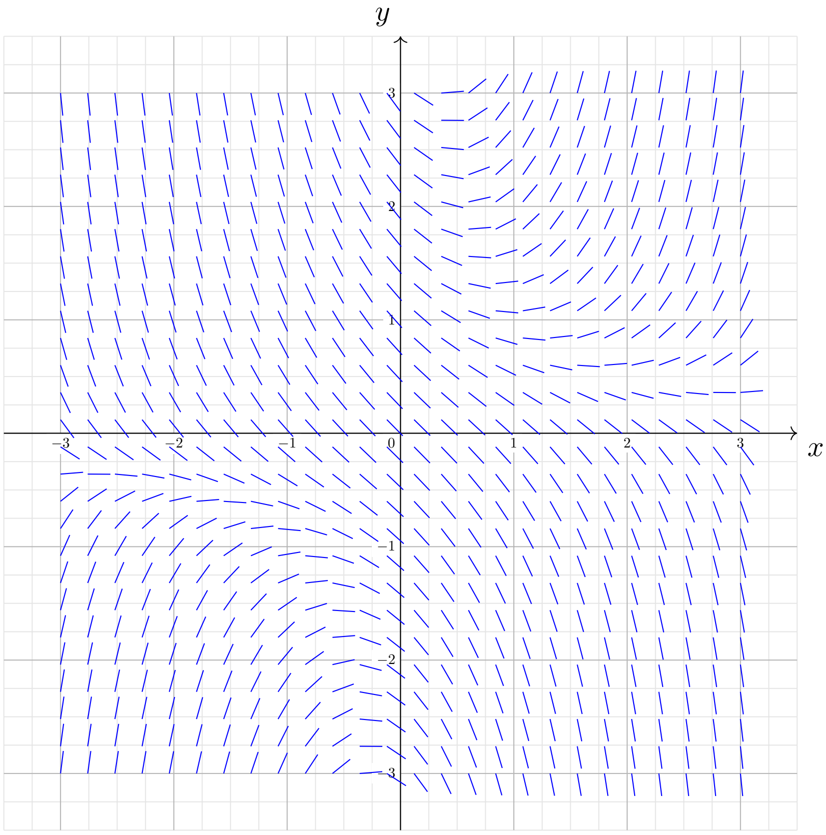
(You may use the result from part (a), even if you did not solve part (a).)

Solution. From part (a) we know that $y = Ce^{-x} + x^2 - 2x$ is a solution to the ODE $y'' + y' = 2x$. The specific solution that is sought satisfies $y(0) = 1$, which means

$$y(0) = Ce^0 + 0 - 0 = C = 1.$$

so a specific solution to the initial-value problem is $y = e^{-x} + x^2 - 2x$. □

3. (a) (10 pts) Sketch the slope field for the first order ODE $y' = xy - 1$.



(b) (10 pts) Use Euler's method with step size 1 to estimate $y(3)$ where $y(x)$ is the solution to the initial-value problem

$$y' = x + y + 1, \quad y(0) = 0.$$

Solution. We have $y(0) = 0$, so $(x_0, y_0) = (0, 0)$. The function for the slope is given by $F(x, y) = x + y + 1$. The first step is $x_1 = 1$, and

$$y_1 = y_0 + hF(x_0, y_0) = 0 + 1 \cdot (0 + 0 + 1) = 1.$$

Then we have $x_2 = 2$ and

$$y_2 = y_1 + hF(x_1, y_1) = 1 + 1 \cdot (1 + 1 + 1) = 4.$$

For the last step we have $x_3 = 3$ and

$$y_3 = y_2 + hF(x_2, y_2) = 4 + 1 \cdot (2 + 4 + 1) = 11,$$

so $y(3) \approx 11$. □

4. Find the general solution to the following separable first order ODEs

(a) (10 pts) $3y^2y' = \frac{1}{x}$

Solution. Write $y' = \frac{dy}{dx}$. Then we get

$$3y^2\frac{dy}{dx} = \frac{1}{x} \Leftrightarrow 3y^2\,dy = \frac{1}{x}\,dx \Leftrightarrow \int \frac{3}{y}^2\,dy = \int \frac{1}{x}\,dx \Leftrightarrow y^3 = \log|x| + C$$
$$\Leftrightarrow y = \sqrt[3]{\log|x| + C}.$$

□

(b) (10 pts) $\frac{dy}{dx} = -xe^{-y}$

Solution. We separate the variables and integrate.

$$\frac{dy}{dx} = -xe^{-y} \Leftrightarrow e^y\,dy = -x\,dx \Leftrightarrow \int e^y\,dy = \int -x\,dx \Leftrightarrow e^y = -\frac{x^2}{2} + C$$
$$\Leftrightarrow y = \log\left|-\frac{x^2}{2} + C\right|.$$

□

5. After drinking a cup of coffee containing 100 mg of caffeine, the amount of caffeine in a person's body t hours after drinking the cup is described by the differential equation

$$\frac{dC}{dt} = -\frac{7}{50}C(t).$$

(a) (10 pts) Solve the initial-value problem

$$\frac{dC}{dt} = -\frac{7}{50}C(t), \quad C(0) = 100.$$

Solution. The ODE is separable, so we first find a general solution of it.

$$\frac{1}{C}\,dC = -\frac{7}{50}\,dt \Leftrightarrow \log|C| = -\frac{7t}{50} + D \Leftrightarrow C(t) = e^{-\frac{7t}{50}+D} = Ee^{-\frac{7t}{50}}.$$

Since we are given $C(0) = 100$ we can find the constant E. Namely $C(0) = Ee^0 = E = 100$. Therefore our solution is $C(t) = 100e^{-\frac{7t}{50}}$. □

(b) (10 pts) Solve the equation $C(t) = 50$. (Leave the answer in its exact form.) This is approximately the half-life of caffeine.

Solution. The equation is equivalent to $100e^{-\frac{7t}{50}} = 50 \Leftrightarrow e^{-\frac{7t}{50}} = \frac{1}{2}$. Then we take logarithms on both sides:

$$-\frac{7t}{50} = \log\left(\frac{1}{2}\right) = -\log 2 \Leftrightarrow t = \frac{50\log 2}{7} \approx 4.95 \text{ hours}.$$

□