# MAT 127 MIDTERM I

TUESDAY FEBRUARY 27, 2024
8:30–9:50PM

Name: ______________________________ ID: ______________

Lecture (circle yours):

| L01 | L02 | L03 |
|---|---|---|
| MW 4:00–5:20pm | TuTh 10:00–11:20am | TuTh 5:30–6:50pm |

**Instructions.**

(1) Fill in your name and Stony Brook ID number and circle your lecture number at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no electronic devices.

(3) You have 80 minutes to complete this exam.

(4) Leave all answers in exact form (that is, *do not* approximate $\pi$, square roots, etc.)

(5) You must justify all your answers and show all your work. Even a correct answer without any justification will result in no credit.

**1.** Determine whether the following sequences converge or diverge.

(a) (10 pts)

$$a_n = \frac{8n^5 - 5n^2 + 2}{4n^5 + n^4 - n^2 + 9}$$

*Solution.* Since $a_n$ is described by the function $f(n) = \frac{8n^5-5n^2+2}{4n^5+n^4-n^2+9}$, we compute the limit by dividing the numerator and denominator by $n^5$ (the dominant factor)

$$\lim_{n\to\infty} \frac{8n^5 - 5n^2 + 2}{4n^5 + n^4 - n^2 + 9} = \lim_{n\to\infty} \frac{8 - \frac{5}{n^3} + \frac{2}{n^5}}{4 + \frac{1}{n} - \frac{1}{n^3} + \frac{9}{n^5}} = \frac{8 - 0 + 0}{4 + 0 - 0 + 0} = 2,$$

and therefore the sequence **converges to** 2. □

(b) (10 pts)

$$a_n = \frac{2^n + \pi^n}{3^n + e^n}$$

*Solution.* Since $a_n$ is described by the function $f(n) = \frac{2^n+\pi^n}{3^n+e^n}$, we compute the limit by dividing the numerator and denominator by $\pi^n$ (the dominant factor)

$$\lim_{n\to\infty} \frac{2^n + \pi^n}{3^n + e^n} = \lim_{n\to\infty} \frac{\left(\frac{2}{\pi}\right)^n + 1}{\left(\frac{3}{\pi}\right)^n + \left(\frac{e}{\pi}\right)^n} = \infty.$$

Since $2 < e < 3 < \pi$, we have $\left(\frac{2}{\pi}\right)^n + 1 \to 1$ as $n \to \infty$ in the numerator and $\left(\frac{3}{\pi}\right)^n + \left(\frac{e}{\pi}\right)^n \to 0$ as $n \to \infty$ in the denominator, and therefore the quotient tends to $\infty$ as as $n \to \infty$. The sequence $\{a_n\}_{n=1}^{\infty}$ is **divergent**. □

**2.** Determine whether the following series converge or diverge.

(a) (10 pts)

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{5^n}{6^{n+1}}\right)$$

*Solution.* We split the sum up into two separate terms

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{5^n}{6^{n+1}}\right)=\sum_{n=1}^{\infty}\frac{1}{n}+\sum_{n=1}^{\infty}\frac{5^n}{6^{n+1}}.$$

The first term is the harmonic series (it is a $p$-series with $p=1$), so it is divergent. The second term is a geometric series with $a=\frac{1}{6}$ and $r=\frac{5}{6}$:

$$\sum_{n=1}^{\infty}\frac{5^n}{6^{n+1}}=\sum_{n=1}^{\infty}\frac{5^n}{6\cdot 6^n}=\frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^n=\frac{1}{6}\cdot\frac{1}{1-\frac{5}{6}}=\frac{1}{6}\cdot 6=1$$

The second term is therefore convergent. The sum of a divergent and a convergent series is divergent, and therefore $\sum_{n=1}^{\infty}\left(\frac{1}{n}+\frac{5^n}{6^{n+1}}\right)$ is **divergent**. □

(b) (10 pts)

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

*Solution.* The series almost looks like the series $\sum_{n=1}^{\infty}\frac{1}{4n^2}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$, which we recognize as a $p$-series with $p=2$, so we will guess that it converges. We use the comparison test. Note that $4n^2-1\geq 4n^2-3n^2=n^2$, because $1\leq 3n^2$ for all $n\geq 1$. Therefore

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\leq\sum_{n=1}^{\infty}\frac{1}{4n^2-3n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$ is a positive series, and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, we conclude by the direct comparison test that $\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$ **converges**.

An alternative solution would be to note that $4n^2-1=(2n+1)(2n-1)$ and use partial fraction decomposition to obtain $\frac{1}{4n^2-1}=\frac{1}{2}\cdot\frac{1}{2n-1}-\frac{1}{2}\cdot\frac{1}{2n+1}$. Looking at the partial sums we note that this series is telescoping:

$$\begin{aligned}S_N&=\frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\\&=\frac{1}{2}\left(\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{2N-1}-\frac{1}{2N+1}\right)\right)=\frac{1}{2}\left(1-\frac{1}{2N+1}\right)\end{aligned}$$

which gives $\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\lim_{N\to\infty}S_N=\frac{1}{2}$, so it is **convergent**. □

**3.** (20 pts) Is the series $\displaystyle\sum_{n=9}^{\infty} \frac{(\log n)^2}{n}$ convergent or divergent? [*Hint: Use the integral test and note the starting point of the series.*]

*Solution.* As per the suggestion we use the integral test. We first have that the function $f(x) = \frac{(\log x)^2}{x}$ is positive for all $x \geq 9$. We next need to show that $f$ is decreasing. Namely, we compute the derivative

$$f'(x) = \frac{x \cdot 2(\log x)\frac{1}{x} - (\log x)^2}{x^2} = \frac{2\log x - (\log x)^2}{x^2} = \frac{\log x(2 - \log x)}{x^2}.$$

The denominator is always positive for $x \geq 9$. In the numerator we always have $\log x > 0$ for $x \geq 9$. Therefore the quotient is negative when $2 - \log x < 0 \Leftrightarrow \log x > 2 \Leftrightarrow x > e^2$, in particular for $x \geq 9 > e^2$. Therefore $f'(x) < 0$ for all $x \geq 9$, and the integral test is applicable. We compute the indefinite integral

$$\int_9^{\infty} \frac{(\log x)^2}{x} dx = \begin{bmatrix} u = \log x \\ du = \frac{1}{x} dx \\ x = 9 \Rightarrow u = \log 9 \\ x \to \infty \Rightarrow u \to \infty \end{bmatrix} = \int_{u=\log 9}^{\infty} u^2 du = \left[\frac{u^3}{3}\right]_{u=\log 9}^{\infty}$$

$$= \left(\lim_{u\to\infty} \frac{u^3}{3}\right) - \frac{(\log 9)^3}{3} = \infty.$$

Since this indefinite integral diverges, we conclude that $\sum_{n=9}^{\infty} \frac{(\log n)^2}{n}$ **diverges** by the integral test. □

**4.** Find the radius and interval of convergence of the following power series.

(a) (10 pts)

$$\sum_{n=1}^{\infty} \frac{(3x)^n}{6^n}$$

*Solution.* We can first simplify: $\frac{(3x)^n}{6^n} = \frac{3^n x^n}{2^n \cdot 3^n} = \frac{x^n}{2^n}$. Next, we use the ratio test, with the terms $a_n = \frac{x^n}{2^n}$.

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{2^{n+1}}}{\frac{x^n}{2^n}} \right| = \lim_{n\to\infty} \frac{1}{2}|x| = \frac{|x|}{2}.$$

By the ratio test we conclude that the series **converges** when $\frac{|x|}{2} < 1 \Leftrightarrow |x| < 2$, and **diverges** when $\frac{|x|}{2} > 1 \Leftrightarrow |x| > 2$. We check the two boundary cases $x = \pm 2$ separately.

$x = 2$**:** $\sum_{n=1}^{\infty} \frac{6^n}{6^n} = \sum_{n=1}^{\infty} 1 = 1 + 1 + \cdots$. This **diverges** by the divergence test because $\lim_{n\to\infty} 1 = 1 \neq 0$.

$x = -2$**:** $\sum_{n=1}^{\infty} \frac{(-6)^n}{6^n} = \sum_{n=1}^{\infty} (-1)^n = -1 + 1 - \cdots$. This **diverges** by the divergence test because $\lim_{n\to\infty} (-1)^n$ does not exist.

Therefore the interval of convergence is $(-2, 2)$, and the radius of convergence is $R = 2$. □

(b) (10 pts)

$$\sum_{n=1}^{\infty} \frac{(-1)^n n!(x-1)^n}{n}$$

*Solution.* We can first simplify: $\frac{(-1)^n n!(x-1)^n}{n} = \frac{(-1)^n n \cdot (n-1)!(x-1)^n}{n} = (-1)^n (n-1)!(x-1)^n$. Next, we use the ratio test, with the terms $a_n = (-1)^n (n-1)!(x-1)^n$.

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} n!(x-1)^{n+1}}{(-1)^n (n-1)!(x-1)^n} \right| = \lim_{n\to\infty} n \cdot |x-1| = \begin{cases} \infty, & |x-1| \neq 0 \\ 0, & |x-1| = 0 \end{cases}.$$

Therefore by the ratio test we conclude that the series **converges** when $|x-1| = 0 \Leftrightarrow x = 1$, and **diverges** otherwise. The interval of convergence is therefore the set $\{1\}$ and the radius of convergence is $R = 0$. □

**5.** Find a power series representation of the following functions. Also indicate the interval of convergence.

(a) (10 pts) $f(x) = \log(1+x)$ [*Hint: What is the derivative of $f(x)$?*]

*Solution.* We know that the function $\frac{1}{1-x}$ has the power series representation $\sum_{n=0}^{\infty} x^n$ for $|x| < 1$ (this is the geometric series). Next we have that

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n,$$

for $|x| < 1$. We integrate both sides and we obtain

$$f(x) + C = \int \sum_{n=0}^{\infty}(-1)^n x^n dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.$$

We determine the value of the constant $C$ by plugging in the value $x = 0$. We know that $f(0) = \log(1+0) = \log 1 = 0$. Setting $x = 0$ in the right hand side gives $\sum_{n=0}^{\infty}(-1)^n \frac{0^{n+1}}{n+1} = 0$, so we conclude that $C = 0$. Therefore

$$f(x) = \log(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.$$

The interval of convergence is the same as that of $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$, namely $(-1, 1)$. □

(b) (10 pts) $g(x) = (1+x)\log(1+x) - x$ [*Hint: What is the derivative of $g(x)$?*]

*Solution.* By the hint, we note that $g'(x) = \log(1+x)$. We know from part (a) that

$$g'(x) = \log(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.$$

We integrate both sides to obtain

$$g(x) + C = \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+2}}{(n+1)(n+2)}.$$

We again determine the value of $C$ by plugging in $x = 0$ in this equation, noting that $g(0) = 0$:

$$C = \sum_{n=0}^{\infty}(-1)\frac{0^{n+2}}{(n+1)(n+2)} = 0.$$

Therefore

$$(1+x)\log(1+x) - x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+2}}{(n+1)(n+2)} = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \cdots.$$

The interval of convergence is the same as that of $\log(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$, namely $(-1, 1)$. □