MAT 127 Lec 9 M Feb 19

This week: • No quiz
• Short WebAssign (due next Mon)
• Wed: Review for midterm
• Next Mon: Practice midterm + Q/A.

Recall: We discussed how to represent some fcns as power series

$$\boxed{\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n \quad |x|<1}$$

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n \quad |x|<1$$

$$\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n} \quad |x|<1$$

$$\log(1+x)=\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} \quad |x|<1$$

We will now do this in general.

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for $|x-a| < R$.

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

If $x = a$ then $\boxed{f(a) = c_0}$

We can differentiate to obtain

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

If $x=a$ then $\boxed{f'(a) = c_1}$

Continue to differentiate

$$f''(x) = 2c_2 + 2 \cdot 3 \cdot c_3(x-a) + \cdots$$

If $x=a$ then $f''(a) = 2c_2$

$$\Leftrightarrow \boxed{\frac{f''(a)}{2} = c_2}$$

Again:

$$f'''(x) = 2\cdot 3\cdot c_3 + 2\cdot 3\cdot 4\cdot c_4 (x-a) + \cdots$$

If $x=a$ then $f'''(a) = 2\cdot 3\cdot c_3$

$$\Leftrightarrow \boxed{\frac{f'''(a)}{3!} = c_3}$$

In general: ← n-th derivative

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

Thm: If f has a power series representation at $x=a$, that is,

if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, $|x-a| < R$

then the coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

TAYLOR SERIES $\quad |x-a| < R$

In the special case $a=0$ we get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

MACLAURIN SERIES $\quad |x| < R$

<u>Ex</u>: We have already seen the Maclaurin Series for $\log(1+x)$, $\frac{1}{1-x}$, $\frac{1}{1+x}$, $\frac{1}{1+x^2}$, ... !

Let's find a new one.

Ex: $f(x) = e^x$. Find its Maclaurin series.

$f'(x) = e^x$, $f''(x) = e^x$ ..

$f^{(n)}(x) = e^x$ for all n.

$f(0) = e^0 = 1$, so $f^{(n)}(0) = 1$ for all n.

Therefore $c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$

for all n.

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x}$$

$x=1$ gives the interesting sum

$$\boxed{e = \sum_{n=0}^{\infty} \frac{1}{n!}}$$

When is a function equal to its Taylor series?

As with any conv. series we need to look at the partial sums.

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N.$$

This is called the N-th degree Taylor polynomial

The function $f(x)$ is equal to its Taylor series if

$$f(x) = \lim_{N\to\infty} T_N(x) \quad |x-a| < R$$

Can consider the difference

$f(x) - T_N(x)$ for any N.

This is of course not 0 in general but is some remainder term

$$R_N(x) = f(x) - T_N(x).$$

$$\Leftrightarrow \quad f(x) = T_N(x) + R_N(x).$$

If we can show

$\lim_{N\to\infty} R_N(x) = 0$ somehow, we would be able to conclude that

$$\lim_{N\to\infty} T_N(x) = \lim_{N\to\infty} \left(f(x) - R_N(x)\right)$$

$$= f(x) - \lim_{N\to\infty} R_N(x) = f(x).$$

This results in :

Thm: If $f(x) = T_N(x) + R_N(x)$

deg N Taylor poly

and $\lim_{N\to\infty} R_N(x) = 0$ for $|x-a|<R$

then f is equal to its Taylor series for $|x-a|<R$

Thm (Taylor's inequality)

If $|f^{(N+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder of the Taylor series satisfies the inequality

$$|R_N(x)| \le \frac{M}{(N+1)!}|x-a|^{N+1} \text{ for } |x-a| \le d.$$

A consequence of this inequality is (roughly) that if all derivatives of f exists, then the inequality will hold, and therefore ← exponential

$$|R_N(x)| \le M \frac{|x-a|^{N+1}}{(N+1)!}$$ ← factorial

$$\lim_{N\to\infty} M \cdot \frac{|x-a|^{N+1}}{(N+1)!} = 0$$ for all $|x-a| \le d$ so

$$\lim_{N\to\infty} R_N(x) = 0$$ & so

f will be equal to its Taylor series.

<u>Ex</u>: Let's consider $f(x) = \sin(x)$. We will find the Maclaurin series.

<u>Derivatives of f at $x=0$.</u>

$f(x) = \sin x, \quad f(0) = 0$

$f'(x) = \cos x, \quad f'(0) = 1$

$f''(x) = -\sin x, \quad f''(0) = 0$

$f'''(x) = -\cos x, \quad f'''(0) = -1$

$f^{(4)}(x) = \sin x, \quad f^{(4)}(0) = 0$

Pattern: 0, 1, 0, −1, 0, 1, 0, −1, ...

n = 0 1 2 3 4 5 6 7

$$\sin x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x.$$