

Recall:

- Geometric Series

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ if } |r| < 1}$$

- Divergence test:

- If $\lim_{n \rightarrow \infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\lim_{n \rightarrow \infty} a_n = 0$ we cannot draw any conclusion.

Ratio test [§8.4 Stewart]

A geometric series $\sum_{n=1}^{\infty} r^n$ conv
if and only if $|r| < 1$.

In such a series $a_n = ar^n$
 $a_{n+1} = ar^{n+1}$

$$\leadsto \left| \frac{a_{n+1}}{a_n} \right| = |r|$$

Geometric series converges if this quotient is < 1 .

What if we have $\sum_{n=0}^{\infty} b_n$. Does

$\left| \frac{b_{n+1}}{b_n} \right|$ tell us anything useful about convergence? Yes!

Thm (The ratio test)

For the series $\sum_{n=1}^{\infty} b_n$, let

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right|.$$

- If $\rho < 1$ then $\sum_{n=1}^{\infty} b_n$ converges
- If $\rho > 1$ then $\sum_{n=1}^{\infty} b_n$ diverges
- If $\rho = 1$ we do not know.

Ex Does $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ Converge?

First check divergence test:

$\lim_{n \rightarrow \infty} \frac{n^3}{3^n} = 0$ (exponentials grow faster than polynomials)
 $\rightarrow$ Can not draw any conclusion.

Try the ratio test. $a_n = \frac{n^3}{3^n}$

$$\underline{\underline{S}} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3 / 3^{n+1}}{n^3 / 3^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3 \cdot 3^n}{n^3 \cdot 3^{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \left(\frac{n+1}{n} \right)^3 \cdot \frac{1}{3} \right|$$

$$= \frac{1}{3} \cdot \left(\lim_{n \rightarrow \infty} \frac{n+1}{n} \right)^3 = \frac{1}{3} \cdot 1^3 = \frac{1}{3} < \underline{\underline{1}}$$

so ratio tells us that the series converges.

Ex: Does $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ Converge?

① Div test: $\lim_{n \rightarrow \infty} \frac{2^n}{n!} = 0$. No conclusion can be drawn.

② Ratio test: $a_n = \frac{2^n}{n!}$

$$\underline{L} = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}/(n+1)!}{2^n/n!} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1} \cdot n!}{2^n \cdot (n+1)!} \right|$$

$$= 2 \cdot \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = 2 \cdot \lim_{n \rightarrow \infty} \frac{\overbrace{n \cdot (n-1) \cdots 2 \cdot 1}^{\text{CANCEL}}}{(n+1) \cdot \underbrace{n \cdot (n-1) \cdots 2 \cdot 1}_{\text{CANCEL}}}$$

$$= 2 \cdot \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < \underline{1}$$

Ratio test gives that $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Converges

Ex Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converge?

① Div test: $\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$. No conclusion can be drawn.

② Ratio test: $a_n = \frac{1}{n^2}$.

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^2 = 1$$

→ Can not draw any conclusion.

(In fact, by other means $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$)

Ex Does $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$ converge?

Ratio test:

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} ((n+1)!)^2 / (2(n+1))!}{(-1)^n (n!)^2 / (2n)!} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)! \cdot (n+1)! \cdot (2n)!}{n! \cdot n! \cdot (2n+2)!}$$

$$= \lim_{n \rightarrow \infty} \frac{[n+1]n! [n+1]n! \cdot (2n)!}{n! \cdot n! \cdot [(2n+2) \cdot (2n+1) \cdot (2n)!]}$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n \rightarrow \infty} \frac{n^2+2n+1}{4n^2+6n+2} \\
 &= \lim_{n \rightarrow \infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{4 + \frac{6}{n} + \frac{2}{n^2}} = \underline{\underline{\frac{1}{4} < 1}}
 \end{aligned}$$

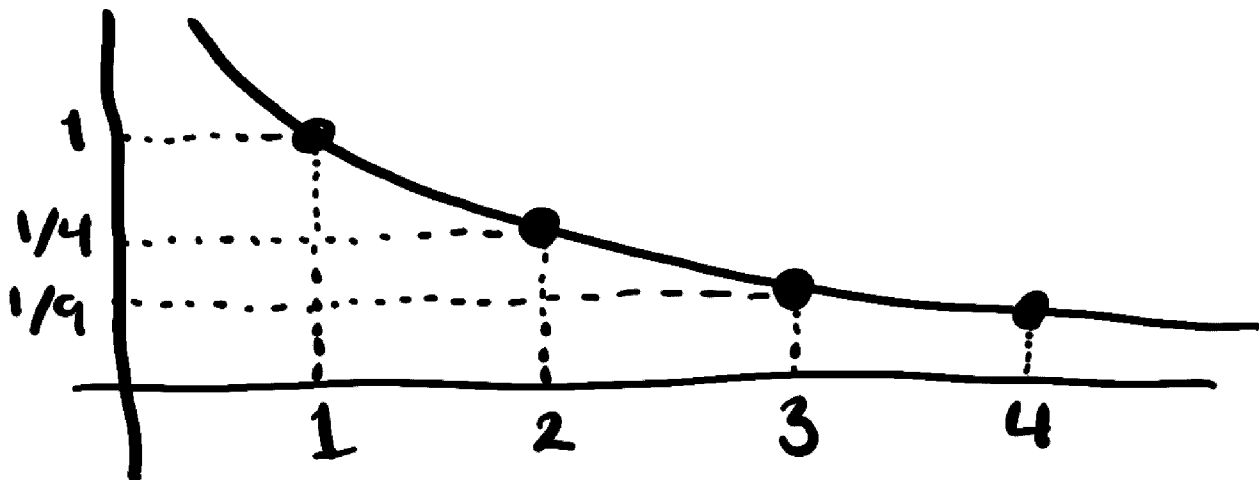
→ Series converges.

Integral test

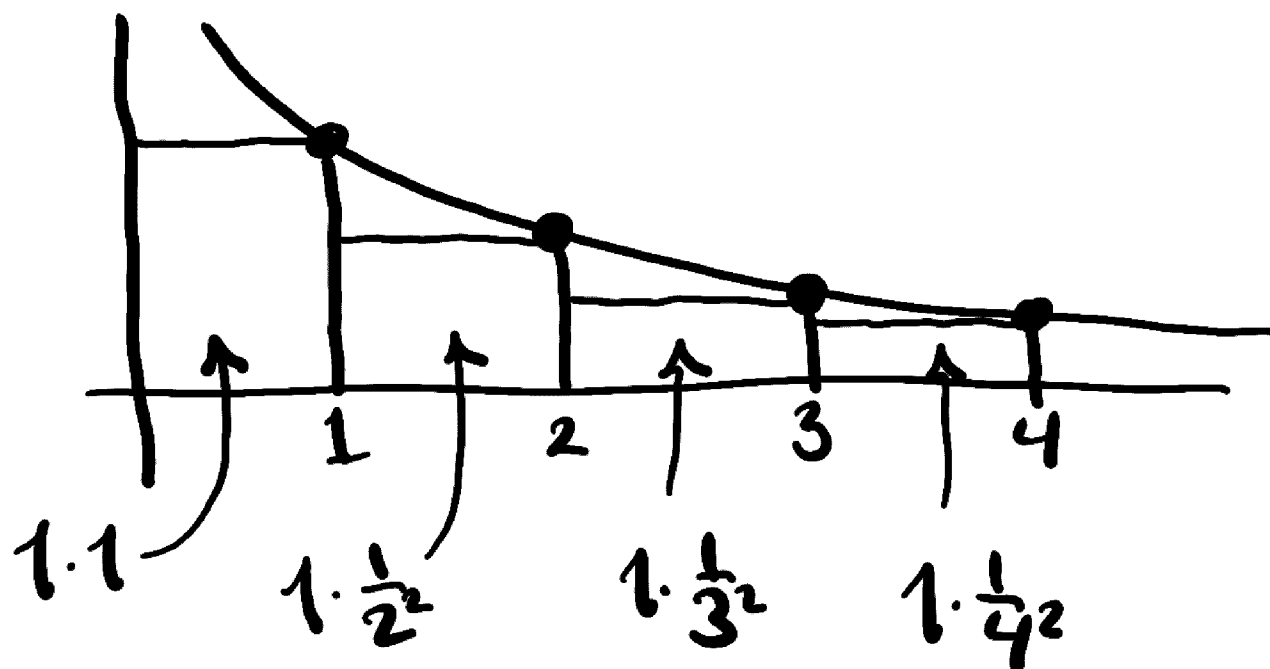
§8.3 Stewart

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Will try to understand it geometrically.

Look at the graph $y = \frac{1}{x^2}$



Drawing rectangles under the graph:

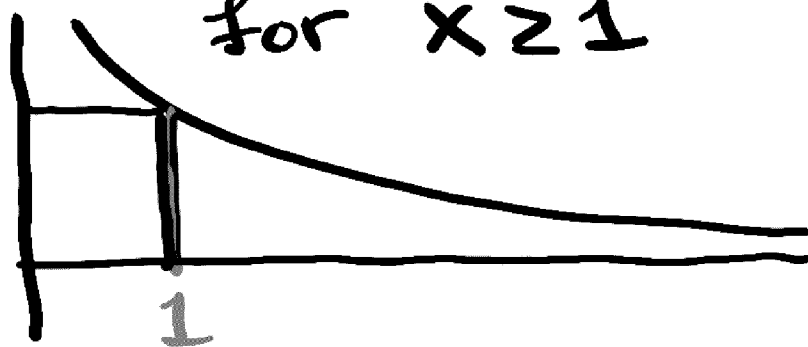


Sum of areas of all such rectangles

$$= \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Remember that

$$\int_1^{\infty} \frac{1}{x^2} dx = \text{area under graph for } x \geq 1$$



Comparing the figures we arrive at

$$0 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq \underline{\underline{1 + \int_1^{\infty} \frac{1}{x^2} dx}}$$

Recall:

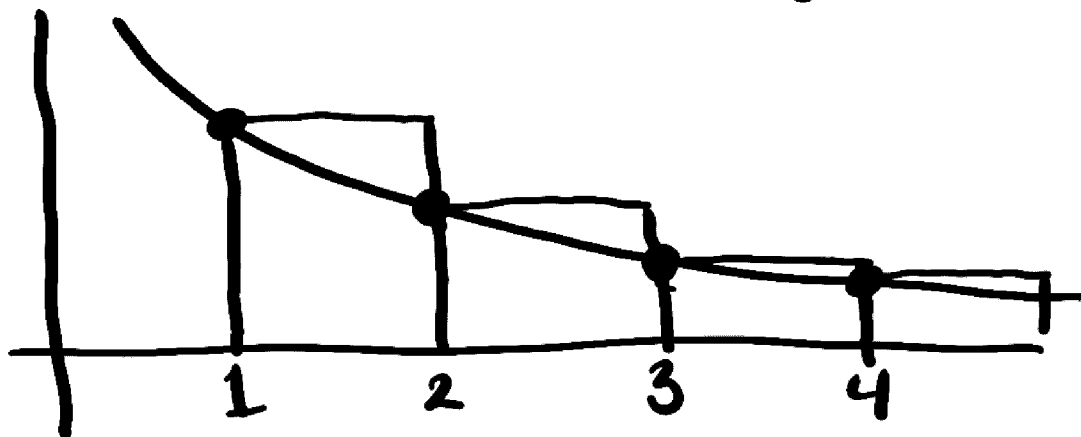
$$\int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^{\infty} = -\lim_{x \rightarrow \infty} \frac{1}{x} + 1$$

$$= 1$$

$$\text{So } 0 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2 \text{ and}$$

this series must converge!

Similar way of summing rectangles:



$$\sum_{n=1}^{\infty} \frac{1}{n^2} \geq \int_1^{\infty} \frac{1}{x^2} dx = 1.$$

If this integral would diverge, then the series would too.

Thm (The integral test)

If f is continuous, positive, and decreasing on $[1, \infty)$.

(a) If $\int_1^{\infty} f(x) dx$ is convergent, then $\sum_{n=1}^{\infty} f(n)$ is convergent.

(b) If $\int_1^{\infty} f(x) dx$ is divergent, then $\sum_{n=1}^{\infty} f(n)$ is divergent.

Consider $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for different values of p .

Easy values:

If $p < 0$ then $\lim_{n \rightarrow \infty} \frac{1}{n^p} = \infty$

So div. test $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

If $p = 0$ then $\frac{1}{n^0} = 1$

and $\sum_{n=1}^{\infty} \frac{1}{n^0} = \sum_{n=1}^{\infty} 1$ diverges.

If $p > 0$ we use the integral test:

$$\int_1^{\infty} \frac{1}{x^p} dx$$

If $p = 1$ $\int_1^{\infty} \frac{1}{x} dx = [\ln|x|]_1^{\infty}$

$$= (\lim_{x \rightarrow \infty} \ln|x|) - 1 = \infty$$

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (of course, we already know this!)

If $p \neq 1$,

$$\int_1^{\infty} \frac{1}{x^p} = \int_1^{\infty} x^{-p} dx = \left[\frac{x^{-p+1}}{-p+1} \right]_1^{\infty}$$

$$= \left(\lim_{x \rightarrow \infty} \frac{1}{1-p} \cdot \frac{1}{x^{p-1}} \right) - \frac{1}{1-p}$$

$$= \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & 0 < p \leq 1 \end{cases}$$

so the integral test gives:

→ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent for $p > 1$.
divergent for $0 < p < 1$