MAT 127 Lec 26 W Apr 24

Recall:

Want to solve

$$ay''+by'+cy=0.$$

First find solutions to the characteristic equation

$$\boxed{ar^2+br+c=0}$$

- If r_1, r_2 are two real solutions, the general solution is

  $$y(x)=C_1e^{r_1x}+C_2e^{r_2x}$$

  C_1, C_2 are constants

- If $r=\mu\pm\lambda i$ are non-real solutions, the general solution is

$$\boxed{y(x) = e^{\mu x}\left(C_1 \cos(\lambda x) + C_2 \sin(\lambda x)\right)}.$$

<u>Ex</u>: $y'' + 2y' + 5y = 0$

<u>Char</u> <u>eqn</u>: $r^2 + 2r + 5 = 0$.

<u>solutions</u>:

$$r = -1 \pm \sqrt{1-5} = -1 \pm \sqrt{-4}$$
$$= -1 \pm 2i$$

$\mu = -1, \ \lambda = 2$

General solution is:

$$\boxed{y(x) = e^{-x}\left(C_1 \cos(2x) + C_2 \sin(2x)\right)}$$

There is one remaining case: The characteristic eqn
$ar^2 + br + C = 0$ can have a double/repeated solution

<u>Ex</u> $y'' + 2y' + y = 0$

<u>Char equ</u>:

$r^2 + 2r + 1 = (r+1)^2 = 0$

has the only solution

$r = -1$.

$y_1(x) = e^{-x}$ is a solution

to $y'' + 2y' + y = 0$. Can

verify this by differentiating:

$$\begin{cases} y_1'(x) = -e^{-x} \\ y_1''(x) = e^{-x} \end{cases}$$

$\rightsquigarrow$ $e^{-x} + 2(-e^{-x}) + e^{-x} = 0$ ✓

Can we guess another solution?

It's tricky!

In fact another solution is $y_2(x) = xe^{-x}$.

$$y_2'(x) = e^{-x} - xe^{-x}$$

$$y_2''(x) = -e^{-x} - (e^{-x} - xe^{-x})$$

$$= -2e^{-x} + xe^{-x}$$

$$y_2''(x) + 2y_2'(x) + y_2(x)$$

$$= (\cancel{-2e^{-x}} + xe^{-x}) + \cancel{2(e^{-x}} - xe^{-x})$$

$$+ (xe^{-x})$$

$$= xe^{-x} - 2xe^{-x} + xe^{-x} = 0 \checkmark$$

General solution is therefore

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

$$= (c_1 + c_2 x) e^{-x}.$$

This happens in general!

<u>Ex</u>: $y'' - 4y' + 4y = 0$

characteristic equation is

$$r^2 - 4r + 4 = 0$$
$$\quad\quad = (r-2)^2$$

$r = 2$ repeated solution

General solution is therefore

$$\boxed{y(x) = (c_1 + c_2 x)\, e^{2x}.}$$

<u>Let's summarize!</u>

To solve the ODE

$ay'' + by' + cy = 0$ for

a, b, c constants

we first solve the characteristic equation

$$\boxed{ar^2+br+c=0}$$

Three cases:

① If r_1, r_2 are two different real roots, the general sol is

$$\boxed{y(x)=c_1e^{r_1x}+c_2e^{r_2x}}$$

② If $r=\mu\pm\lambda x$ are two non-real roots, the general sol is

$$\boxed{y(x)=e^{\mu x}(c_1\cos(\lambda x)+c_2\sin(\lambda x))}$$

③ If r is a repeated root, the general sol is

$$\boxed{y(x)=(c_1+c_2x)e^{rx}}$$

<u>Ex</u>: Solve the initial-value problem

$$\boxed{2y''+y'-y=0,\ y(0)=0,\ y'(0)=1}$$

Char eqn

$$2r^2+r-1=0 \Leftrightarrow r^2+\frac{1}{2}r-\frac{1}{2}=0$$

$$r=-\frac{1}{4}\pm\sqrt{\frac{1}{16}+\frac{1}{2}}=-\frac{1}{4}\pm\sqrt{\frac{1+8}{16}}$$

$$=-\frac{1}{4}\pm\frac{3}{4}=\frac{-1\pm3}{4}$$

$$r_1=-1,\ r_2=\frac{1}{2}$$

General solution is

$$y(x)=c_1e^{-x}+c_2e^{\frac{x}{2}}.$$

Initial conditions:

$$y(0)=c_1e^0+c_2e^0=c_1+c_2=0$$

$$y'(x)=-c_1e^{-x}+\frac{c_2}{2}e^{\frac{x}{2}}$$

$$y'(0) = -C_1 e^0 + \frac{C_2}{2} e^0 = -C_1 + \frac{C_2}{2}$$
$$= 1$$

Two equations:

$$\begin{cases} C_1 + C_2 = 0 \\ -C_1 + \frac{C_2}{2} = 1 \end{cases}$$

First eq $\rightsquigarrow$ $C_1 = -C_2$

Insert in second eqn:

$$-(-C_2) + \frac{C_2}{2} = 1$$

$$\Leftrightarrow \frac{3C_2}{2} = 1 \qquad \boxed{C_2 = \frac{3}{2}}$$

$$\Rightarrow \boxed{C_1 = -\frac{3}{2}}$$

The specific solution is

$$y(x) = -\frac{3}{2} e^{-x} + \frac{3}{2} e^{\frac{x}{2}}.$$

Ex: Solve the IVP

$$\boxed{\begin{aligned} y''-4y'+4y=0, \quad y(0)&=1 \\ y'(0)&=0 \end{aligned}}$$

Characteristic eq:

$$r^2-4r+4=0$$

$$\Leftrightarrow (r-2)^2=0$$

repeated solution $r=2$.

General solution:

$$y(x)=(C_1+C_2x)e^{2x}.$$

Initial conditions:

$$y(0)=(C_1+C_2\cdot 0)e^0=\boxed{C_1=1}$$

$$y'(x)=C_2e^{2x}+2(C_1+C_2x)e^{2x}$$

$$y'(0) = c_2 e^0 + 2(c_1 + c_2 \cdot 0)e^0$$

$$= c_2 + 2c_1 = c_2 + 2 = 0$$

$c_1 = 1$

$\Rightarrow \boxed{c_2 = -2}$

Specific solution is

$$\boxed{y(x) = (1-2x)e^{2x}}$$

<u>Ex</u> Solve the IVP

$$\boxed{y'' - 6y' + 10y = 0, \ y(0) = 1, \ y'(0) = 0}$$

Characteristic eqn:

$$r^2 - 6r + 10 = 0$$

$$r = 3 \pm \sqrt{9-10} = 3 \pm \sqrt{-1} = 3 \pm i$$

General solution is

$$y(x) = e^{3x}(c_1\cos(x) + c_2\sin(x))$$

Initial conditions:

$$y(0) = e^0(c_1\cos(0) + c_2\sin(0))$$

$$= \boxed{c_1 = 1}$$

$$y'(x) = 3e^{3x}(c_1\cos x + c_2\sin x)$$

$$+ e^{3x}(-c_1\sin x + c_2\cos x)$$

$$y'(0) = 3e^0(c_1\cos 0 + c_2\sin 0)$$

$$+ e^0(-c_1\sin 0 + c_2\cos 0)$$

$$= 3c_1 + c_2 \underset{c_1=1}{=} 3 + c_2 = 0$$

$$\Rightarrow \boxed{c_2 = -3}$$

Specific solution is

$$\boxed{y(x) = e^{3x}(\cos x - 3\sin x)}$$