MAT 127 Lec 25 M Apr 22

<u>Recall</u>:

We are interested in solving 2^{nd} order equations of the form

$$\boxed{\begin{array}{c} ay''+by'+cy=0 \\ a,b,c \text{ constants} \end{array}}$$

Have seen that Solutions are often of the form

$$y=C_1e^{r_1x}+C_2e^{r_2x}$$

where r_1, r_2 are two constants determined by the equation.

<u>Ex</u> $y''-4y=0$. Let's find solutions of the form e^{rx}.

$$y=e^{rx}, \quad y'=re^{rx}, \quad y''=r^2e^{rx}$$

$y''-4y = r^2e^{rx}-4e^{rx} = e^{rx}(r^2-4)$

$= 0$ for all x

means $r^2-4=0$

$\Rightarrow r = \pm 2$, so both e^{2x} and e^{-2x} are solutions to $y''-4y=0$. More generally

$\boxed{y = c_1e^{2x}+c_2e^{-2x}}$ is the general solution

In fact we can solve all equations in a similar way.

$$ay''+by'+cy=0 \quad , a \neq 0.$$

then if we look for solutions

of the form $y=e^{rx}$ we get

$y'=re^{rx}$; $y''=r^2e^{rx}$

$$ay''+by'+cy=ar^2e^{rx}+bre^{rx}+ce^{rx}$$

$$=e^{rx}(ar^2+br+c)=0$$

$$\Leftrightarrow \boxed{ar^2+br+c=0}$$

So if r is a root of the polynomial ar^2+br+c then $y=e^{rx}$ is a solution to $ay''+by'+cy=0$.

The equation

$$ar^2+br+c=0 \text{ is}$$

called the "characteristic equation".

Ex: $2y''-3y'+y=0$.

Characteristic eqn:

$$2r^2-3r+1=0$$

$$r^2-\frac{3}{2}r+\frac{1}{2}=0$$

$$r=\frac{3}{4}\pm\sqrt{\left(\frac{3}{4}\right)^2-\frac{1}{2}}$$

$$=\frac{3}{4}\pm\sqrt{\frac{9}{16}-\frac{8}{16}}=\frac{3}{4}\pm\frac{1}{4}$$

$r_1=1$, $r_2=\frac{1}{2}$.

e^x and $e^{\frac{x}{2}}$ are solutions to $2y''-3y'+y=0$ &

$\boxed{y=c_1e^x+c_2e^{\frac{1}{2}x}}$ is the general solution

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Ex $y''+y=0$. We have already

solved this equation before using power series. The general solution is

$$y = c_1 \cos x + c_2 \sin x. \quad (*)$$

Now suppose we consider the characteristic equation:

$r^2 + 1 = 0$. It's solutions are $r = \pm i$.

Then e^{ix} and e^{-ix} are solutions! So the general solution is

$$y = c_1 e^{ix} + c_2 e^{-ix}$$

But how is this consistent w/ $(*)$?

Remember Euler's formula

$$e^{ix} = \cos x + i \sin x.$$

$$\rightarrow e^{-ix} = e^{i(-x)} = \cos(-x) + i\sin(-x)$$

$$= \cos x - i \sin x$$

$$\boxed{\begin{gathered}\cos(-x) = \cos x \\ \sin(-x) = -\sin x\end{gathered}}$$

So

$$y = C_1 e^{ix} + C_2 e^{-ix}$$

$$= C_1(\cos x + i \sin x)$$

$$+ C_2(\cos x - i \sin x)$$

$$= (C_1 + C_2)\cos x + i(C_1 - C_2)\sin x$$

$$= D_1 \cos x + D_2 \sin x$$

CONSTANTS

More generally:

If r_1, r_2 are two complex roots of ar^2+br+c of the form

$$r_1 = \mu + \lambda i, \quad r_2 = \mu - \lambda i$$

then the general solution is

$$y = C_1 e^{(\mu+\lambda i)x} + C_2 e^{(\mu-\lambda i)x}$$

We will not present this function like this, but we will use Euler's formula like before:

$$C_1 e^{(\mu+\lambda i)x} + C_2 e^{(\mu-\lambda i)x}$$

$$= C_1 e^{\mu x} \cdot e^{i(\lambda x)} + C_2 e^{\mu x} e^{i(-\lambda x)}$$

$$= e^{\mu x}\Big(C_1(\cos(\lambda x) + i\sin(\lambda x))$$
$$+ C_2(\cos(\lambda x) - i\sin(\lambda x))\Big)$$

$$= e^{\mu x}\left((C_1+C_2)\cos(\lambda x) + i(C_1-C_2)\sin(\lambda x)\right)$$

$$= e^{\mu x}\left(D_1\cos(\lambda x)+D_2\sin(\lambda x)\right)$$

constants

<u>Ex</u>: $y''-6y'+10y=0$.

Characteristic eqn

$$r^2-6r+10=0$$

$$r=3\pm\sqrt{3^2-10}=3\pm i$$

$$\left|\begin{array}{l}\mu=\mathrm{Re}(r)=3\\ \lambda=\mathrm{Im}(r)=1\end{array}\right.$$

$\Rightarrow$ General solution is

$$y=e^{3x}(C_1\cos x+C_2\sin x).$$

To summarize:

$$ay''+by'+cy=0$$

if characteristic eqn

$$ar^2+br+c=0$$

has two real solutions r_1, r_2

general sol is

$$y=c_1e^{r_1x}+c_2e^{r_2x}$$

if it has two non-real solutions $r=\mu\pm\lambda i$ then general sol is

$$y=e^{\mu x}(c_1\cos(\lambda x)+c_2\sin(\lambda x))$$

Remaining case of a single solution (double root) is discussed next time!