MAT 127 Lec 22 W Apr 10

## Power series solutions to ODEs

It's not always possible to find a solution to an ODE in terms of elementary functions.

For example

$$\boxed{y'' - 2xy' + y = 0}$$

But it's important to be able to solve them.

Can obtain a power series representation of the solution $y$.

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Ex. Let's consider

$y'' + y = 0$ and let's try to use power series to solve it.

Assume there's a solution near $x=0$ and that

$$y(x)=\sum_{n=0}^{\infty} b_n x^n.$$

Thus

$$\begin{cases} y'(x)=\sum_{n=1}^{\infty} n\, b_n x^{n-1} \\ y''(x)=\sum_{n=2}^{\infty} n(n-1) b_n x^{n-2}. \end{cases}$$

To compare the series for $y$ and $y''$ we shift the index:

$$y''(x)=\sum_{n=2}^{\infty} n(n-1) b_n x^{n-2}$$

$$=\sum_{n=0}^{\infty} (n+2)(n+1) b_{n+2} x^n.$$

Therefore

$$y''+y=\sum_{n=0}^{\infty} (n+2)(n+1) b_{n+2} x^n$$

$$+\sum_{n=0}^{\infty} b_n x^n$$

$$=\sum_{n=0}^{\infty}\left[(n+2)(n+1)\,b_{n+2}+b_n\right]x^n=0$$

This power series being <u>zero</u> must mean we need to find the coefficients $b_0, b_1, b_2, \ldots$ so that

$$\boxed{(n+2)(n+1)\,b_{n+2}+b_n=0} \quad \text{for all } n$$

<u>recursion relation</u>

$$\boxed{b_{n+2}=-\frac{b_n}{(n+2)(n+1)}}$$

<u>$n=0$</u> $\quad b_2=-\dfrac{b_0}{2\cdot 1}$

<u>$n=1$</u> $\quad b_3=-\dfrac{b_1}{3\cdot 2}$

<u>$n=2$</u> $\quad b_4=-\dfrac{b_2}{4\cdot 3}=-\dfrac{1}{4\cdot 3}\cdot\left(-\dfrac{b_0}{2\cdot 1}\right)$

$$= \frac{b_0}{4\cdot 3\cdot 2\cdot 1} = \frac{b_0}{4!}$$

$\underline{n=3}$ $$b_5 = -\frac{b_3}{5\cdot 4} = \left(-\frac{1}{5\cdot 4}\right)\left(-\frac{b_1}{3\cdot 2}\right)$$

$$= \frac{b_1}{5!}$$

$\underline{n=4}$ $$b_6 = -\frac{b_4}{6\cdot 5} = -\frac{b_0}{6\cdot 5\cdot 4!} = -\frac{b_0}{6!}$$

<u>Pattern</u>:

$$\begin{cases} b_{2n} = (-1)^n \dfrac{b_0}{(2n)!} \\ b_{2n+1} = (-1)^n \dfrac{b_1}{(2n+1)!} \end{cases} \qquad (*)$$

Let's check:

$$b_2 = (-1)^1 \frac{b_0}{2!} = -\frac{b_0}{2!}$$

$$b_4 = (-1)^2 \frac{b_0}{4!} = \frac{b_0}{4!} \qquad \text{Ok}$$

$$b_6 = (-1)^3 \frac{b_0}{6!} = \frac{b_0}{6!}$$

$$b_3 = (-1)^1 \frac{b_1}{3!} = -\frac{b_1}{3!}$$

ok.

$$b_5 = (-1)^2 \frac{b_1}{5!} = \frac{b_1}{5!}$$

Now let's plug in (*) into

$$y(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$$

$$= b_0 + b_1 x - \frac{b_0}{2!} x^2 - \frac{b_1}{3!} x^3 + \frac{b_0}{4!} x^4 + \frac{b_1}{5!} x^5 + \cdots$$

$$= \left( b_0 - \frac{b_0}{2!} x^2 + \frac{b_0}{4!} x^4 - \frac{b_0}{6!} x^6 + \cdots \right) + \left( b_1 x - \frac{b_1}{3!} x^3 + \frac{b_1}{5!} x^5 - \cdots \right)$$

$$= b_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)$$

$$+ b_1\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)$$

$$= b_0\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}+b_1\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= b_0\cos x + b_1 \sin x.$$

Therefore $y = b_0\cos x + b_1 \sin x$ for any constants $b_0, b_1$ is the general solution to

$y'' + y = 0$

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<u>Note</u>: Usually we can not solve the recurrence relation involving the coefficients in the power series.

**Ex**: $y'' - 2xy' + y = 0$

Let's use power series to find a power series representation of the general solution.

shift $n$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$y'' = \sum_{n=1}^{\infty} (n+1) n c_{n+1} x^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

shift $n$

multiply

$$2xy' = 2x \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$= 2 \sum_{n=0}^{\infty} (n+1) c_{n+1} x^{n+1}$$

$$= 2\sum_{n=1}^{\infty} n c_n x^n$$

shift n

$$= 2\sum_{n=0}^{\infty} n c_n x^n$$

add back (n=0)-term (which is =0)

Therefore

$$y'' - 2xy' + y = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - 2\sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=0}^{\infty} \left[ (n+2)(n+1) c_{n+2} - 2n c_n + c_n \right] x^n$$

$$= \sum_{n=0}^{\infty} \left[ (n+2)(n+1) c_{n+2} - (2n-1) c_n \right] x^n$$

$$= 0$$

we get the recurrence

$$(n+2)(n+1)C_{n+2} - (2n-1)C_n = 0$$

$$\boxed{C_{n+2} = \frac{(2n-1)C_n}{(n+2)(n+1)}}$$

Let's find a pattern.

$\underline{n=0}$ $\quad C_2 = \frac{-C_0}{2 \cdot 1} = -\frac{C_0}{2!}$

$\underline{n=1}$ $\quad C_3 = \frac{1 \cdot C_1}{3 \cdot 2} = \frac{C_1}{3!}$

$\underline{n=2}$ $\quad C_4 = \frac{3 \cdot C_2}{4 \cdot 3} = -\frac{3 \cdot C_0}{4 \cdot 3 \cdot 2!} = -\frac{3C_0}{4!}$

$\underline{n=3}$ $\quad C_5 = \frac{5C_3}{5 \cdot 4} = \frac{5 \cdot C_1}{5 \cdot 4 \cdot 3!} = \frac{5 \cdot C_1}{5!}$

$\underline{n=4}$ $\quad C_6 = \frac{7 \cdot C_4}{6 \cdot 5} = -\frac{7 \cdot 3 \cdot C_0}{6 \cdot 5 \cdot 4!} = -\frac{7 \cdot 3 \cdot C_0}{6!}$

$\underline{n=5}$ $\quad C_7 = \frac{9 \cdot C_5}{7 \cdot 6} = \frac{9 \cdot 5 \cdot C_1}{7 \cdot 6 \cdot 5!} = \frac{9 \cdot 5 \cdot C_1}{7!}$

The patterns are

$$\begin{cases} c_{2n} = \dfrac{-3\cdot 7\cdot 11\cdot \cdots \cdot (4n-5)c_0}{(2n)!} \\ c_{2n+1} = \dfrac{5\cdot 9\cdot 13\cdot \cdots (4n-3)\, c_1}{(2n+1)!} \end{cases}$$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$= c_0 + c_1 x - \frac{c_0}{2!}x^2 + \frac{c_1}{3!}x^3$$

$$- \frac{3c_0}{4!}x^4 + \frac{5\cdot c_1}{5!}x^5 - \frac{3\cdot 7 c_0}{6!}x^6 + \cdots$$

$$= c_0\left(1 - \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{3\cdot 7x^6}{6!} - \cdots\right)$$

$$+ c_1\left(x + \frac{x^3}{3!} + \frac{5x^5}{5!} + \frac{5\cdot 9x^7}{7!} + \cdots\right)$$

This is the answer & these power series do not have a description in terms of "elementary functions"

(like $\sin x$, $\cos x$, etc)

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<u>Ex</u>: $y' = x^2 y$. It's separable, so we can solve it that way, but let's use power series.

$$y = \sum_{n=0}^{\infty} c_n x^n$$

shift $n$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

multiply

$$x^2 y = x^2 \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+2}$$

$$= \sum_{n=2}^{\infty} c_{n-2} x^n$$

$$y' - x^2 y = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=2}^{\infty} c_{n-2} x^n \quad (*)$$

To gather the terms we need

the series to start at the same index. To "fix" it, we write out the first two terms in the former series.

$$(*) = C_1 X + 2C_2 X^2 + \sum_{n=2}^{\infty} (n+1) C_{n+1} X^n - \sum_{n=2}^{\infty} C_{n-2} X^n$$

$$= \underbrace{C_1}_{=0} + \underbrace{2C_2}_{=0} X + \sum_{n=2}^{\infty} \left[ (n+1) C_{n+1} - C_{n-2} \right] X^n$$

$$= 0$$

Immediately get $\boxed{C_1 = 0 \,,\ C_2 = 0.}$

Then $\boxed{(n+1) C_{n+1} - C_{n-2} = 0 \quad n \geq 2}$

$$C_{n+1} = \frac{C_{n-2}}{n+1} \quad n \geq 2$$

<u>$n=2$</u> $\quad C_3 = \dfrac{C_0}{3}$

$\underline{n=3}$ $c_4 = \frac{c_1}{4} = 0$

$\underline{n=4}$ $c_5 = \frac{c_2}{5} = 0$

$\underline{n=6}$ $c_6 = \frac{c_3}{6} = \frac{c_0}{6 \cdot 3} = \frac{c_0}{3^2 \cdot 2 \cdot 1} = \frac{c_0}{3^2 \cdot 2!}$

$\underline{n=7}$ $c_7 = \frac{c_4}{7} = 0$

$\underline{n=8}$ $c_8 = \frac{c_5}{8} = 0$

$\underline{n=9}$ $c_9 = \frac{c_6}{9} = \frac{c_0}{9 \cdot 3^2 \cdot 2!} = \frac{c_0}{3^3 \cdot 3!}$

Pattern seems to be

$c_{3n} = \frac{c_0}{3^n \cdot n!}$ and $c_n = 0$ for $n$ not a multiple of 3.

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \cdots$$

$$= c_0 + c_3 x^3 + c_6 x^6 + c_9 x^9 + \cdots$$

$$= c_0 \left( 1 + \frac{x^3}{3 \cdot 1!} + \frac{x^6}{3^2 \cdot 2!} + \frac{x^9}{3^3 \cdot 3!} + \cdots \right)$$

$$= c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 \sum_{n=0}^{\infty} \frac{\left(\frac{x^3}{3}\right)^n}{n!}$$

$$= c_0 e^{\frac{x^3}{3}}$$

This is also the solution obtained via separation of variables!

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