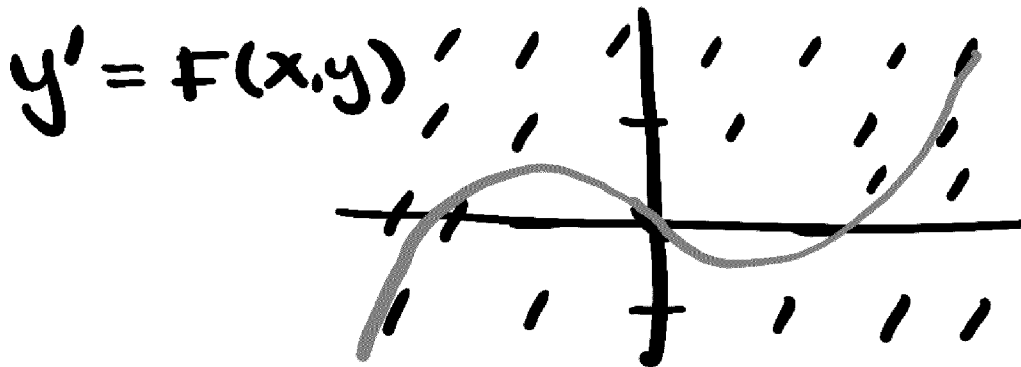


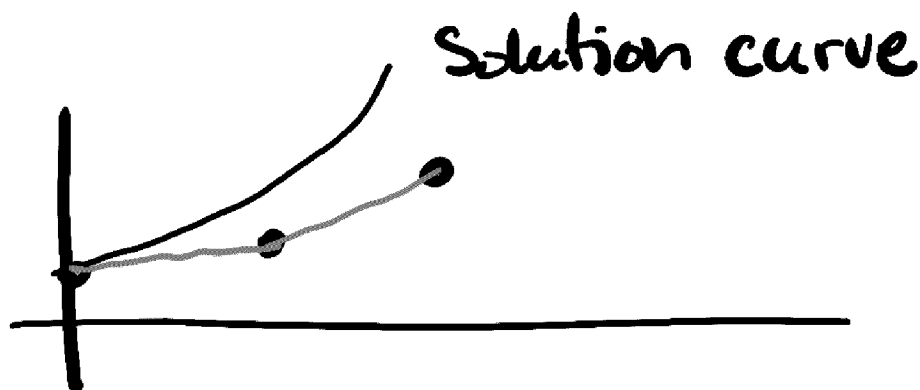
Recall • Slope fields:



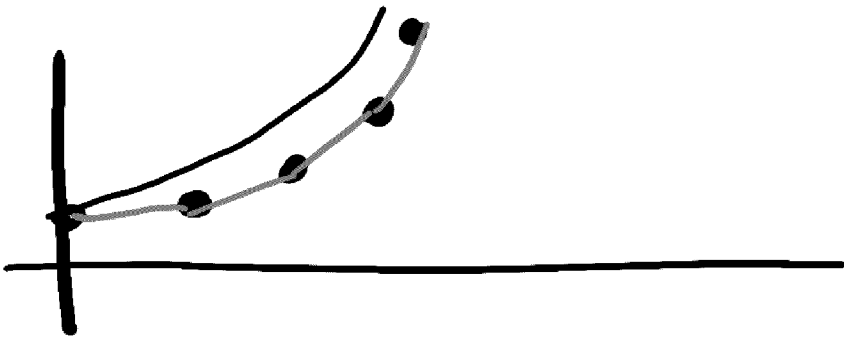
Used to graphically see general features of general solutions

• Euler's method:

Numerically approximate a sol by small straight line steps.



Can use smaller step sizes to get better approximation



Suppose we have a general ODE now:

$$y' = F(x, y)$$

Let (x_0, y_0) be our start point.
Let h be our step size.

$$\begin{cases} x_1 = x_0 + h, & \text{then} \\ y_1 = y_0 + h \cdot F(x_0, y_0) \end{cases}$$

$$\begin{cases} x_2 = x_1 + h \\ y_2 = y_1 + h \cdot F(x_1, y_1) \end{cases}$$

Euler's method

$$x_n = x_{n-1} + h$$

$$y_n = y_{n-1} + h \cdot F(x_{n-1}, y_{n-1})$$

$$n = 1, 2, 3, \dots$$

Ex: Consider $y' = x^2 + y^2 - 1$, $y(0) = 0$

Let's create a table of numerical approximations at the ~~set~~ from $x=0$ to $x=2$ w/ step size

$$h = 0.5$$

$$(x_0, y_0) = (0, 0)$$

| x | y |
|-------------|--|
| $x_0 = 0$ | 0 |
| $x_1 = 0.5$ | $y_1 = y_0 + h \cdot F(x_0, y_0)$ $= 0 + 0.5 \cdot (0^2 + 0^2 - 1)$ $= -0.5$ |

$$x_2 = 1$$

$$y_2 = y_1 + h \cdot F(x_1, y_1)$$

$$= -0.5 + 0.5(0.5^2 + (-0.5)^2 - 1)$$

$$= -0.75$$

$$x_3 = 1.5$$

$$y_3 = y_2 + h \cdot F(x_2, y_2)$$

$$= -0.75 + 0.5(1^2 + (-0.75)^2 - 1)$$

$$= -0.46875$$

$$x_4 = 2$$

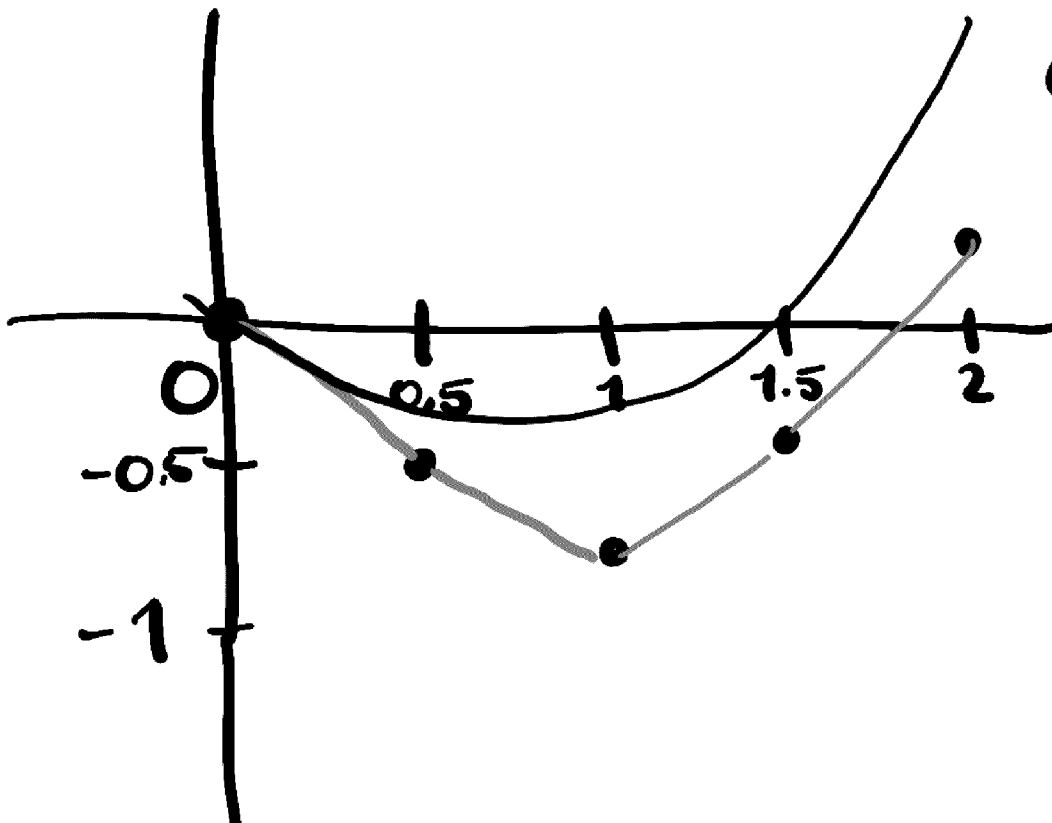
$$y_4 = y_3 + h \cdot F(x_3, y_3)$$

$$= -0.46875$$

$$+ 0.5(1.5^2 + (-0.46875)^2 - 1)$$

$$\approx 0.266$$

Solution
Curve



Separable ODEs

We will now learn how to solve some ODEs.

Def: A separable ODE is a first order ODE that can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

Meaning the RHS can be "separated" into a function that only depends on x and a function that only depends on y .

Ex: • $y' = x^2 y$ is separable

• $x^2 y' = y$ is also separable!

Because its equivalent to

$$y' = \frac{y}{x^2} = \frac{1}{x^2} \cdot y$$

- $y' = x^2 + y^2 - 1$ is not separable, because the RHS is not a product of the form $g(x)f(y)$.
-

To solve a general separable ODE we use the following trick:

$y' = g(x)f(y)$. If $f(y) \neq 0$ write

$$h(y) = \frac{1}{f(y)} \text{ so } \boxed{\frac{dy}{dx} = \frac{g(x)}{h(y)}}$$

Rewrite it in the differential form

$$\underbrace{h(y) dy}_{\text{only } y} = \underbrace{g(x) dx}_{\text{only } x}$$

Then integrate:

$$\int h(y) dy = \int g(x) dx$$

$$\underline{\text{Ex:}} \quad x^2 y' = y \Leftrightarrow \frac{dy}{dx} = \frac{y}{x^2}$$

$$\Leftrightarrow \frac{1}{y} dy = \frac{1}{x^2} dx \Leftrightarrow \int \frac{1}{y} dy = \int \frac{1}{x^2} dx$$

$$\Leftrightarrow \ln|y| = -\frac{1}{x} + C$$

$$\Leftrightarrow y(x) = e^{-\frac{1}{x} + C} = e^C e^{-\frac{1}{x}} = D e^{-\frac{1}{x}}$$

Ex: Solve the IVP

$$y' y^2 = x^2, \quad y(0) = 2.$$

The equation is separable, so we first solve it.

$$\frac{dy}{dx} y^2 = x^2 \Leftrightarrow y^2 dy = x^2 dx$$

$$\Leftrightarrow \int y^2 dy = \int x^2 dx$$

$$\Leftrightarrow \frac{y^3}{3} = \frac{x^3}{3} + C$$

$$\Leftrightarrow y^3 = x^3 + D$$

$$\Leftrightarrow y = \sqrt[3]{x^3 + D}$$

GENERAL
SOLUTION

To find specific solution we look at the initial condition.

$$y(0) = 2 \Leftrightarrow \sqrt[3]{0 + D} = 2$$

$$\Leftrightarrow D = 2^3 = 8$$

Solution to the IVP is

therefore

$$y(x) = \sqrt[3]{x^3 + 8}$$

Ex: Find a general solution to

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$$

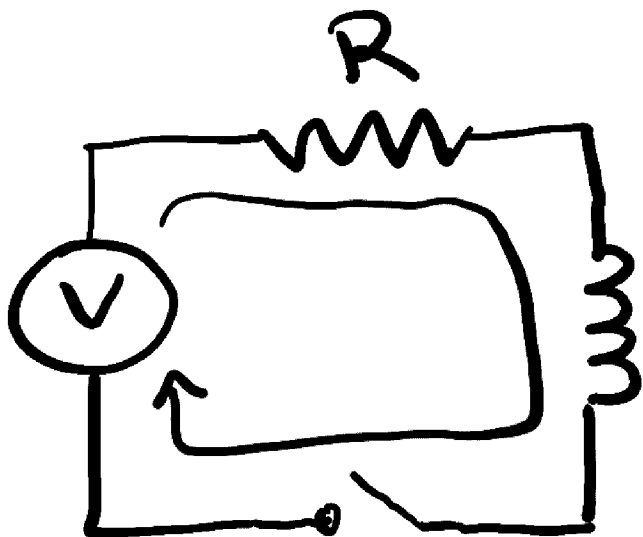
$$2y + \cos y \, dy = 6x^2 \, dx$$

$$\int 2y + \cos y \, dy = \int 6x^2 \, dx$$

$$\Leftrightarrow y^2 + \sin y = 2x^3 + C$$

Can not solve for y , so this equation defines y implicitly as a function of x .

Ex: Simple electric circuit:



- Resistor w/ resistance R

- Inductor w/ inductance L

Want to model the current I through the circuit as a function of time when switch is closed.

The voltage "drops" as it passes through the two components

The voltage drop over the resistor is given by Ohm's law

$$V = R \cdot I$$

The voltage drop over the inductor is $V(t) = L \frac{dI}{dt}$

Kirchhoff's laws says that the total voltage drop is equal to the total supplied voltage

$$\leadsto V = L \frac{dI}{dt} + RI(t)$$

Now let

$$R = 12 \Omega$$

$$L = 4 \text{ H}$$

$$V = 60 \text{ V}$$

$$60 = 4 \frac{dI}{dt} + 12 I(t)$$

$$\Leftrightarrow 15 = \frac{dI}{dt} + 3I(t)$$

$$\Leftrightarrow \frac{dI}{dt} = 15 - 3I$$

$$\Leftrightarrow \frac{1}{15-3I} dI = dt$$

$$\Leftrightarrow \int \frac{1}{15-3I} dI = \int dt$$

$$\Leftrightarrow -\frac{1}{3} \ln|15-3I| = t + C$$

$$\Leftrightarrow \ln|15-3I| = -3t + D$$

$$\Leftrightarrow 15-3I = e^{-3t+D} = E e^{-3t}$$

$$\Leftrightarrow 3I = 15 - E e^{-3t}$$

$$\boxed{I = 5 - F \cdot e^{-3t}}$$

The specific solution with $I(0) = 0$ is given by:

$$I(0) = 5 - E \cdot e^0 = 5 - F = 0$$

$\Leftrightarrow F = 5$ meaning

$$I(t) = 5 - 5e^{-3t}.$$
