MAT 127 Lec 16 W Mar 20

Recall:

General ODE:

Equation involving a function $y = y(x)$, one or more of its derivatives, and x.

The order of an ODE is the highest order of a derivative that occurs.

Ex: $y' = 2xy$ 1st order ODE.

Let's verify that $y = Ce^{x^2}$ is a solution for any $C > 0$.

$$y' = Ce^{x^2} \cdot 2x = 2x\underbrace{(Ce^{x^2})}_{=y} = 2xy \checkmark$$

When solving ODE's we are

Usually interested in finding a specific solution, satisfying some extra requirement. In many physical problems we need to find the specific solution satisfying a condition of the form

$y(t_0) = y_0$. "initial condition"

Initial-value problem (IVP)
= the problem of finding a sol of an ODE satisfying an initial condition.

<u>Ex</u>: Solve the IVP

$y' = 2xy$, $y(0) = 3$.

We already know that $y = Ce^{x^2}$ is a general sol of this ODE.
The initial condition $y(0) = 3$

determines the constant:

$y(0) = Ce^{0^2} = \boxed{C = 3}$ & so the specific solution to the IVP is $\boxed{y = 3e^{x^2}}$.

Slope fields & Euler's method

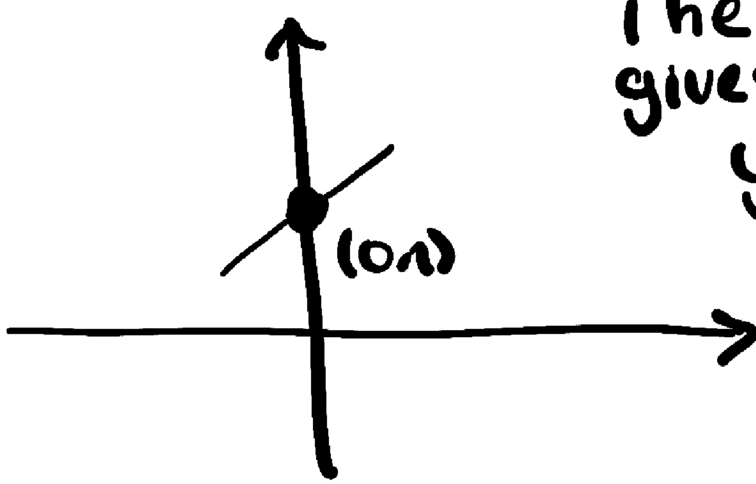
(§7.2 Stewart)

In general, can't solve ODE's unless they are of specific forms. Can still learn a lot about them using a graphical approach (slope field) or numerical approach (Euler's method).

Ex: Suppose we need to sketch the graph of the solution to the IVP $y' = x + y$, $y(0) = 1$.

The equation tells us what the <u>slope</u> is of y at the point $(x, y(x))$.

The initial cond. tells us that the solution passes through $(0,1)$ $(y(0)=1)$.



The ODE then gives

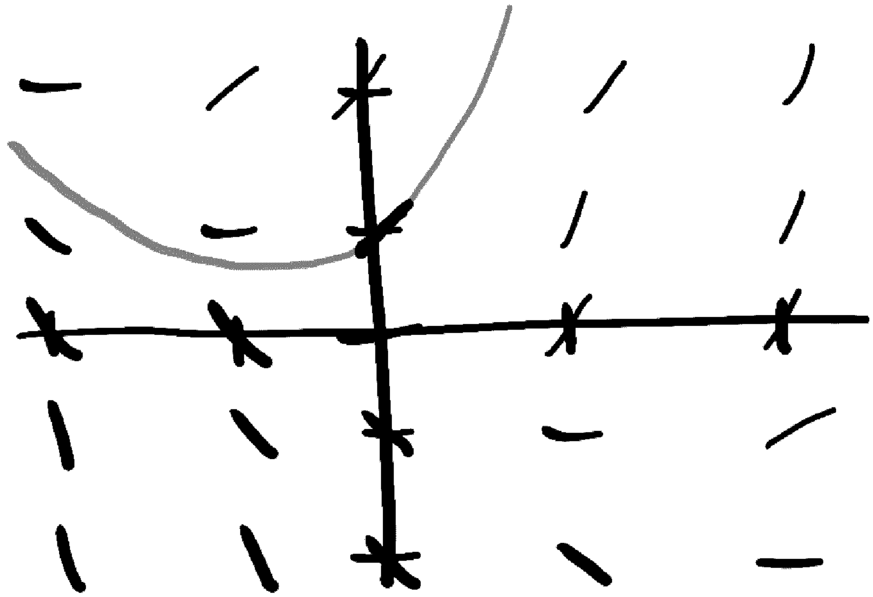
$$y' = x + y = 0 + 1 = 1$$

at this pt.

So the solution, whatever it is, will have slope 1 at $(0,1)$.

Sketch of more slopes:

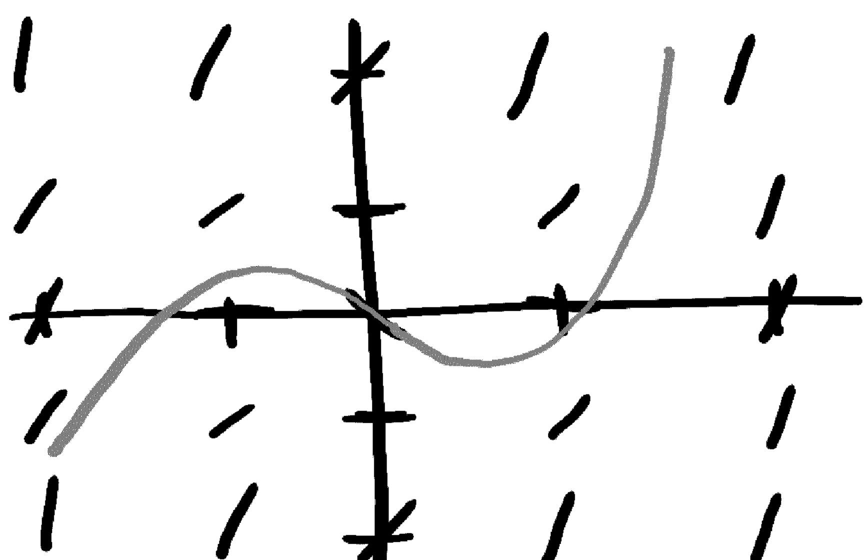
$y \backslash x$	-2	-1	0	1	2
-2	-4	-3	-2	-1	0
-1	-3	-2	-1	0	1
0	-2	-1	0	1	2
1	-1	0	1	2	3
2	0	1	2	3	4



Ex: Sketch slope field of

$y' = x^2 + y^2 - 1$ & sketch the graph of the solution with $y(0) = 0$

y \ x	−2	−1	0	1	2
−2	7	4	3	4	7
−1	4	1	0	1	4
0	3	0	−1	0	3
1	4	1	0	1	4
2	7	4	3	4	7

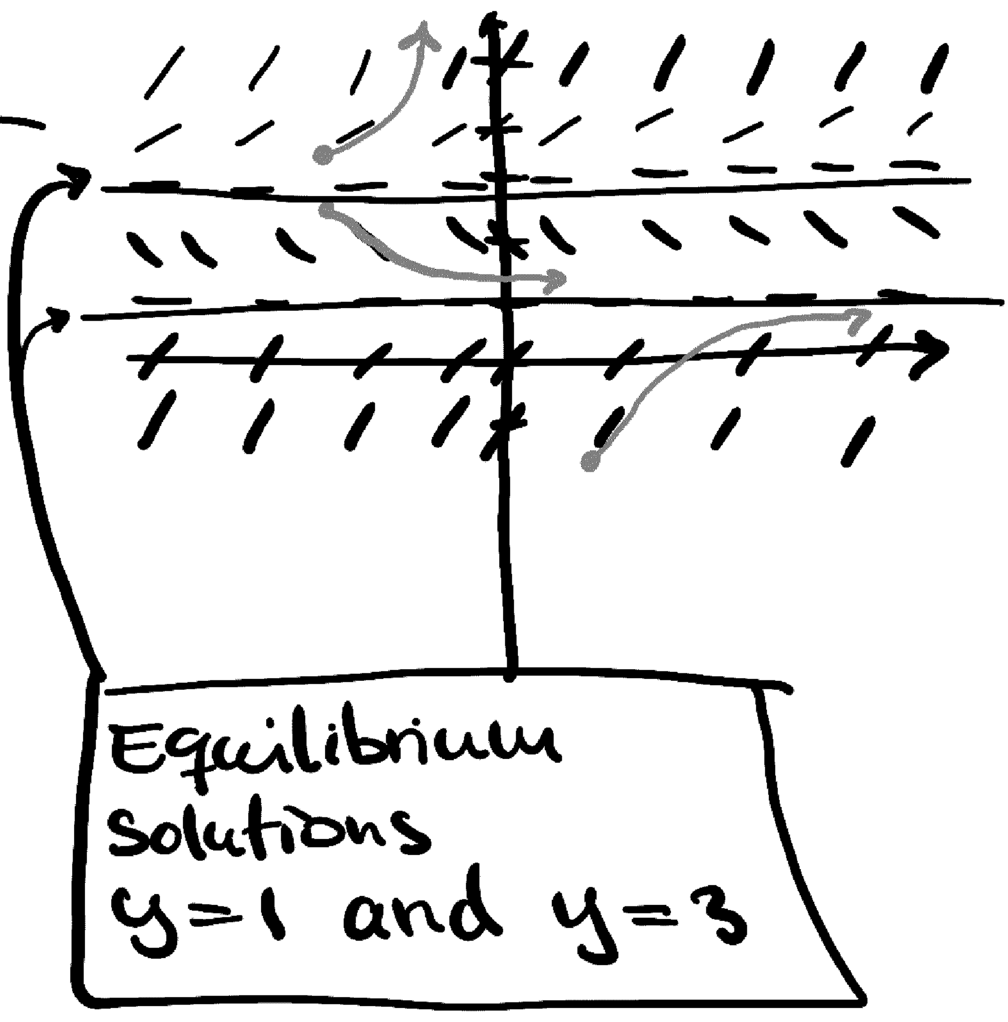


<u>Ex</u>: $y' = (y-1)(y-3)$.

Let's sketch slope field.

Note that the RHS does <u>not</u> depend on x, so slope only dep on y.

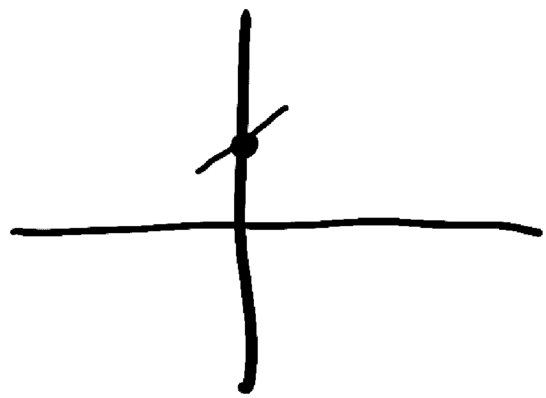
y	y'
-1	8
0	3
1	0
2	-1
3	0
4	3
5	8



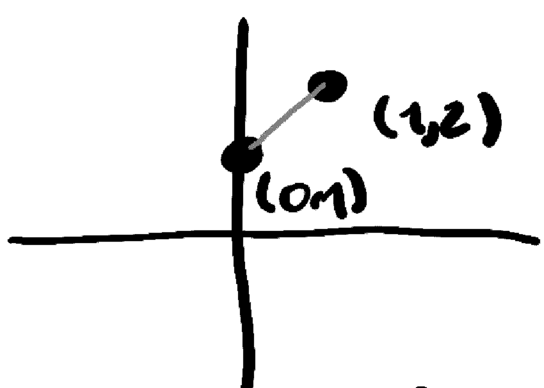
<u>Euler's method</u>

Consider $y' = x+y$, $y(0) = 1$ again.

We already know that $y'(0) = 1$ by plugging in $(x,y) = (0,1)$ into the ODE.



Now let's approximate the solution by taking a small step in the direction of this slope.

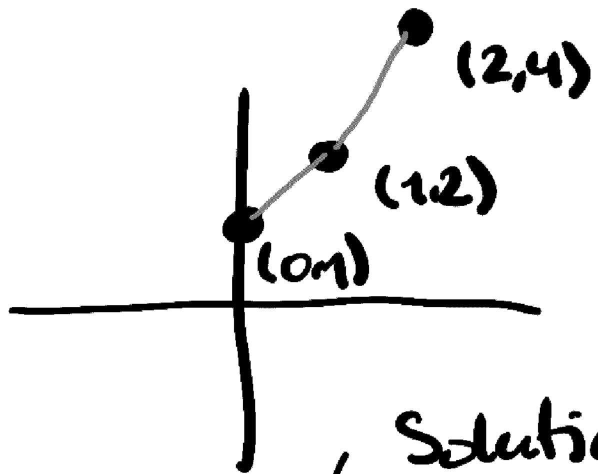


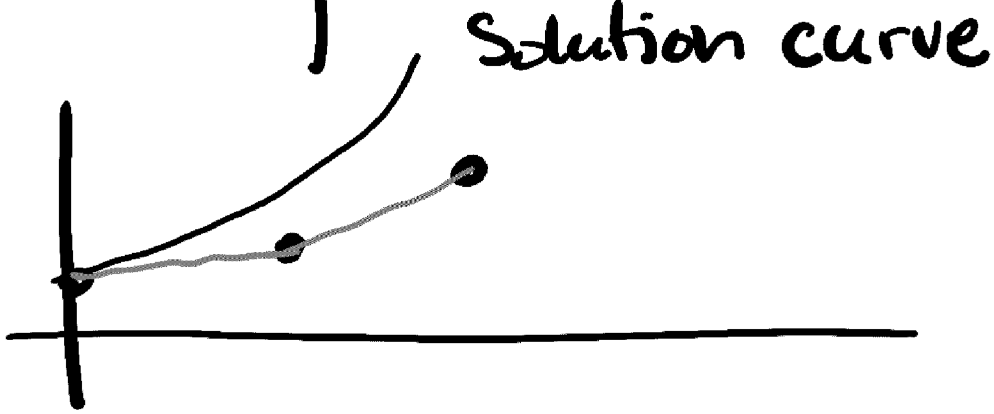
Took a step of size 1 (from $x=0$ to $x=1$)

Now adjust slope according to

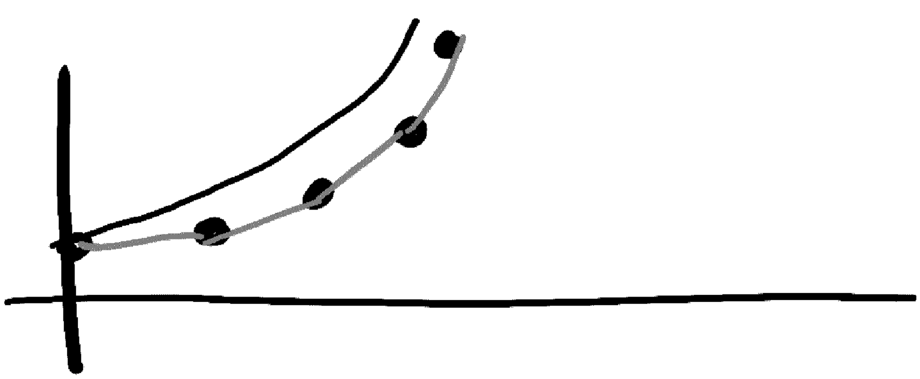
the ODE $y' = x + y$.

New slope $= y'(1) = 1+2 = 3$





Can use smaller step sizes to get better approximation



Suppose we have a general ODE now:

$$\boxed{y' = F(x,y)}$$

Let (x_0, y_0) be our start point.
Let h be our <u>step size</u>.

$$\begin{cases} x_1 = x_0 + h, \text{ then} \\ y_1 = y_0 + h \cdot F(x_0, y_0) \end{cases}$$

$$\begin{cases} x_2 = x_1 + h \\ y_2 = y_1 + h \cdot F(x_1, y_1) \end{cases}$$

$\vdots$ Euler's method

$$\boxed{\begin{aligned} &x_n = x_{n-1} + h \\ &y_n = y_{n-1} + h \cdot F(x_{n-1}, y_{n-1}) \\ &n = 1, 2, 3, \ldots \end{aligned}}$$

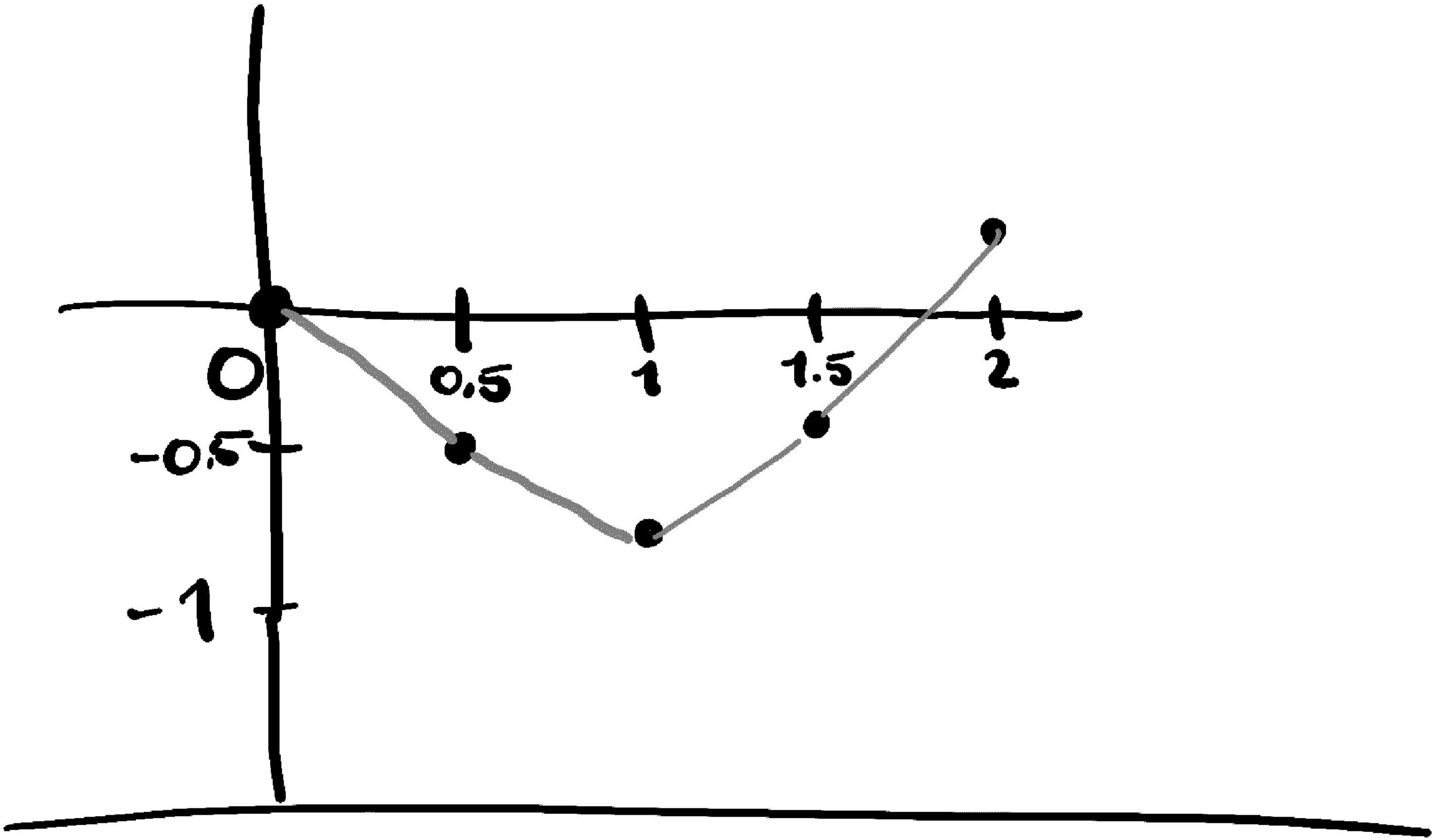
<u>Ex</u>: Consider $y' = x^2 + y^2 - 1$, $y(0) = 0$

Let's create a table of numerical

approximations at the sol from $x=0$ to $x=2$ w/ step size $h=0.5$

$(x_0, y_0) = (0,0)$

x	y
$x_0 = 0$	0
$x_1 = 0.5$	$y_1 = y_0 + h \cdot F(x_0, y_0)$ $= 0 + 0.5 \cdot (0^2 + 0^2 - 1)$ $= -0.5$
$x_2 = 1$	$y_2 = y_1 + h \cdot F(x_1, y_1)$ $= -0.5 + 0.5\left(0.5^2 + (-0.5)^2 - 1\right)$ $= -0.75$
$x_3 = 1.5$	$y_3 = y_2 + h \cdot F(x_2, y_2)$ $= -0.75 + 0.5\left(1^2 + (-0.75)^2 - 1\right)$ $= -0.46875$
$x_4 = 2$	$y_4 = y_3 + h \cdot F(x_3, y_3)$ $= -0.46875$ $+ 0.5\left(1.5^2 + (-0.46875)^2 - 1\right)$ ≈ 0.266


0
0.5
1
1.5
2
-0.5
-1