MAT 127 Lec 14 W Mar 6

Absolute and conditional convergence (§8.4 in Stewart)

<u>Recall</u>: $\sum_{n=1}^{\infty} a_n$ converges if

$$\lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} a_n}_{\text{partial sum}} = \lim_{N\to\infty} S_N \text{ exists.}$$

<u>Def</u>: $\sum_{n=1}^{\infty} a_n$ is called <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

<u>Note</u>: If the terms a_n are positive then $|a_n| = a_n$ so convergence and absolute convergence is the same for those series

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, then $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$

p-series w/ $p = 2 > 1$ so it's conv.

Therefore $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by the alternating series test

$\left(\sum_{n=1}^{\infty} (-1)^n a_n, \quad a_n = \frac{1}{n} \text{ and } a_n \text{ is decreasing } \& \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0 \right)$

However $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is div

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent, but not absolutely convergent.

Def: $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it's conv but not absolutely convergent

Previously we have used the alt. series test to deal with alternating series, but sometimes we can use absolute convergence.

Theorem. If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, it is convergent.

Note: The reverse implication convergence $\Rightarrow$ absolute convergence is false as we saw above with the alternating harmonic series.

Ex: Determine whether $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges or not.

Note $-1 \leq \cos n \leq 1$ & series "looks like" $\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series w/ $p = 2 > 1$ which converges.

So we <u>would</u> like to use the comparison test, but we can't since $\cos n$ is sometimes negative.

Instead $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$

$|\cos n| \leq 1$ and now it's a positive series. Comparison test:

$$\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \leq \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\underline{\text{CONV}}}$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ convergent

means $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ — " — .

Recall ratio test: $\sum_{n=1}^{\infty} a_n$

$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

- $\rho > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ div

Now if $\rho < 1$, the ratio test actually tells us that $\sum a_n$ is absolutely convergent.

($\rho = 1$ is still inconclusive).

Ex: Is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ absolutely conv? conv?

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ p-series w/ $p = \frac{1}{2} < 1$

So this series diverges

$\rightsquigarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is not abs. conv.

Is it convergent? Yes by the alt. series test.

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n, \quad \boxed{a_n = \frac{1}{\sqrt{n}}}$$

decreasing: $\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$ ✓

$\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ ✓

$\Rightarrow$ Convergent by alternating series test.

<u>Ex</u>: Determine if $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$ is absolutely convergent.

$$\sum_{n=1}^{\infty}\left|\frac{(-2)^n}{n^2}\right| = \sum_{n=1}^{\infty}\left|\frac{(-1)^n 2^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{2^n}{n^2}$$

<u>div test</u>: $\lim_{n\to\infty}\frac{2^n}{n^2} = \infty$

$\Rightarrow \sum_{n=1}^{\infty}\frac{2^n}{n^2}$ is divergent, so $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ is not absolutely conv.

Is it conv? No!

Div test is already conclusive

$\lim_{n\to\infty} \frac{(-1)^n 2^n}{n^2}$ does not exist, so

$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ is divergent.
