MAT 127 Lec 13 M Mar 4

Recall:

- Maclaurin series of a function $f(x)$

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

- Binomial series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad |x|<1$$

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$$ Binomial coefficient.

$(k \geq n)$, $\binom{k}{n} = 0$ for $n > k$

and $\binom{k}{0} = 1$

Other Maclaurin series we have seen:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \quad |x|<1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{all } x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{all } x$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad |x|<1$$

$$\log(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad |x|<1$$

will not have to memorize these for exams, but they should feel familiar.

<u>Ex</u>: $f(x) = \sqrt{1+x} = (1+x)^{1/2}$

$\boxed{k = \frac{1}{2}}$ $\binom{1/2}{0} = 1$

$$\binom{1/2}{1} = \frac{1}{2}, \quad \binom{1/2}{2} = \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)}{2} = -\frac{1}{8}$$

$$\binom{1/2}{3} = \frac{\frac{1}{2}\cdot\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} = \frac{\left(-\frac{1}{4}\right)\left(-\frac{3}{2}\right)}{3!} = \frac{1}{16}$$

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$$

<u>Ex</u> $g(x) = \frac{1}{\sqrt{4-x}} = (4-x)^{-\frac{1}{2}}$

$$= \left(4\left(1-\frac{x}{4}\right)\right)^{-1/2} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1+\left(-\frac{x}{4}\right)\right)^{-\frac{1}{2}}, \quad \boxed{k = -\frac{1}{2}}$$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1+\left(-\frac{x}{4}\right)\right)^{-\frac{1}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n$$

$\binom{-1/2}{0} = 1$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{(-1)^n x^n}{4^n}$$

$$\binom{-1/2}{1} = -\frac{1}{2}, \quad \binom{-1/2}{2} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2} = \frac{3}{8}$$

$$\binom{-1/2}{3} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!} = \frac{\frac{3}{4}\left(-\frac{5}{2}\right)}{6}$$

$$= -\frac{15}{48} \qquad \frac{-\frac{1}{2}\cdot\frac{-1-2}{2}}{2} =$$

$$\frac{1}{\sqrt{4-x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{15x^3}{48} + \cdots$$

Converges for $\left|-\frac{x}{4}\right| < 1 \iff |x| < 4$.

Applications:

Ex: Compute $\lim\limits_{x \to 0} \dfrac{2\cos x - 2 + x^2}{x^4}$.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

gives

$$\lim_{x \to 0} \frac{2\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - 2 + x^2}{x^4}$$

$$= \lim_{x \to 0} \frac{\left(2 - x^2 + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots\right) - 2 + x^2}{x^4}$$

$$= \lim_{x \to 0} \frac{\frac{2x^4}{4!} - \frac{2x^6}{6!} + \cdots}{x^4}$$

$$=\lim_{x\to 0} \frac{2}{4!}-\frac{2x^2}{6!}+\frac{2x^4}{8!}-\cdots = \frac{2}{4!}=\frac{1}{12}$$

Ex: $\displaystyle\lim_{x\to 0} \frac{\sin(x^2)-x^2}{x^6}$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\Rightarrow \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

$$\lim_{x\to 0} \frac{\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\right) - x^2}{x^6}$$

$$= \lim_{x\to 0} \frac{-\frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots}{x^6}$$

$$= \lim_{x\to 0} -\frac{1}{3!} + \frac{x^4}{5!} - \cdots = -\frac{1}{3!} = -\frac{1}{6}.$$

Ex: $\displaystyle\lim_{x\to 0} \frac{\arctan(x)-x}{x^3}$.

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

$$\lim_{x \to 0} \frac{\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right) - x}{x^3}$$

$$= \lim_{x \to 0} \frac{-\frac{x^3}{3} + \frac{x^5}{5} - \cdots}{x^3} = \lim_{x \to 0} -\frac{1}{3} + \frac{x^2}{5} - \cdots$$

$$\boxed{= -\frac{1}{3}}$$

<u>Ex</u>: Find $\int x^2 \sin(x^2)\,dx$ as an infinite series.

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

$$\int x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+1)!}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int x^{2n+3}\,dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \frac{x^{2n+4}}{2n+4}$$

<u>Ex</u> Approximate $\int_0^{1/2} xe^{-x^2}\,dx$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ so } e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

$$\int_0^{1/2} xe^{-x^2}\,dx = \int_0^{1/2} x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx$$

$$= \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{1/2} x^{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{x^{2n+2}}{2n+2}\right]_0^{1/2}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2n+2} \cdot \frac{1}{2^{2n+2}} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+2)4^n}$$

$$= \frac{1}{4}\left(\frac{1}{1\cdot 2\cdot 1} + \frac{-1}{1\cdot 4\cdot 4} + \frac{1}{2\cdot 6\cdot 16} - \cdots\right)$$

$$= \frac{1}{4}\left(\frac{1}{2} - \frac{1}{16} + \frac{1}{192} - \cdots\right)$$

$$\approx \frac{1}{8} - \frac{1}{48} + \frac{1}{768}$$

$$= \frac{27}{256} \approx 0.1055$$

<u>Bonus exercise</u>: Can compute this integral exactly via u-substitution & compare to our approximated result.

<u>Ex</u> Approximate $\int_0^1 \sqrt{1+x^4}\, dx$

Recall from earlier

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n, \text{ so}$$

$$\sqrt{1+x^4} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^{4n}.$$

$$\int_0^1 \sqrt{1+x^4}\, dx = \int_0^1 \sum_{n=0}^{\infty} \binom{1/2}{n} x^{4n}\, dx$$

$$= \sum_{n=0}^{\infty} \binom{1/2}{n} \int_0^1 x^{4n}\, dx = \sum_{n=0}^{\infty} \binom{1/2}{n} \left[\frac{x^{4n+1}}{4n+1}\right]_0^1 .$$

$$= \sum_{n=0}^{\infty} \binom{1/2}{n} \frac{1}{4n+1} = \underbrace{\binom{1/2}{0}}_{=1} \cdot \frac{1}{1} + \underbrace{\binom{1/2}{1}}_{=1/2} \cdot \frac{1}{5}$$

$$+ \underbrace{\binom{1/2}{2}}_{=-1/8} \cdot \frac{1}{9} + \underbrace{\binom{1/2}{3}}_{=1/16} \cdot \frac{1}{13} + \cdots$$

$$= 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} + \cdots$$

$$\approx 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} = \frac{10211}{9360} \approx 1.0909$$
