MAT 127 Lec10 W Feb 21

Review for the midterm:

Sequences: All of §8.1

$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots, a_n, \ldots$

n-th term

- Describe n-th by identifying patterns:

$1, \frac{1}{2}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \ldots \quad a_n = \frac{(-1)^{n+1}}{n^2}$.

- Convergence of limits:

a_n described by function f, then

$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$.

Remember: (type of function, ordered by "speed" at which it increases)

logarithm $\ll$ polynomial $\ll$ exponential $\ll$ factorial

eg $\lim_{n\to\infty} \frac{(\log n)^{10000}}{n} = 0$

$$\lim_{n\to\infty} \frac{n!}{e^n} = \infty, \quad \lim_{n\to\infty} \frac{n^{7000}}{2^n} = 0$$

etc. Common technique: divide numerator and denominator w/ dominant term.

$$\lim_{n\to\infty} \frac{e^n + n^3}{n! + \log n} = \lim_{n\to\infty} \frac{\frac{e^n}{n!} + \frac{n^3}{n!}}{1 + \frac{\log n}{n!}} = \frac{0+0}{1+0} = 0.$$

- sequence is
  - increasing if

    $a_n \le a_{n+1}$, eg $a_n = n^2$.

  - decreasing if

    $a_{n+1} \le a_n$, eg. $a_n = \frac{1}{n^2}$.

  - bounded below if there is some $m > 0$:

    $m \le a_n$ for all n.

e.g. $a_n = n^2$, then $0 \le a_n$.

- **bounded above** if there is some $M > 0$:

$$a_n \le M \text{ for all } n.$$

Infinite Series (All of §8.2)

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \underbrace{\boxed{\sum_{n=1}^{N} a_n}}_{\text{partial sum } S_N} = \lim_{N\to\infty} S_N$$

- Geometric Series

$$\sum_{n=0}^{\infty} ar^n, \quad a \text{ and } r \text{ constants.}$$

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } |r| < 1}$$

ex $\displaystyle \sum_{n=0}^{\infty} \frac{1}{5^n} = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{1-1/5} = \frac{5}{4}$

- Harmonic series

$\sum_{n=1}^{\infty} \frac{1}{n}$ know that it diverges.

- Telescoping series

ex: $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$. Partial sums

$$S_N = \sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right) + \dots + \left(\cancel{\frac{1}{N}}-\frac{1}{N+1}\right)$$

$$= 1-\frac{1}{N+1}.$$

$\rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim_{N\to\infty} S_N =$

$$= \lim_{N\to\infty}\left(1-\frac{1}{N+1}\right) = 1.$$

- Convergence tests:

§8.2: Div test

§8.3: Integral test, p-series, Comparison test

§8.4: alt. series test, ratio test.

- **Div test:** $\sum_{n=1}^{\infty} a_n$

$\lim_{n\to\infty} a_n \neq 0$

$\Rightarrow \sum a_n$ divergent.

- **Ratio test:** $\sum_{n=1}^{\infty} a_n$

$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

☆ if $\rho > 1 \Rightarrow \sum a_n$ div

☆ if $\rho < 1 \Rightarrow \sum a_n$ conv

☆ if $\rho = 1 \Rightarrow$ INCONCLUSIVE

- **p-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$

conv if $p > 1$

div if $p \leq 1$

- Integral test: $\sum_{n=1}^{\infty} a_n$,

$a_n = f(n)$ f continuous, positive decreasing

$\star$ $\int_1^{\infty} f(x)\,dx$ div $\rightarrow$ $\sum a_n$ div

$\star$ $\int_1^{\infty} f(x)\,dx$ conv $\Rightarrow$ $\sum a_n$ conv

- Comparison test:

$\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ such that:

- $a_n, b_n \geq 0$ (POSITIVE TERMS)
- $a_n \leq b_n$ for all n

$\star$ $\sum b_n$ conv $\Rightarrow$ $\sum a_n$ conv

$\star$ $\sum a_n$ div $\Rightarrow$ $\sum b_n$ div

- Alt. series test: $\sum_{n=1}^{\infty} (-1)^n a_n$

if

(1) $a_n \geq a_{n+1}$

(DECREASING TERMS)

(2) $\lim_{n\to\infty} a_n = 0$

then $\sum (-1)^n a_n$ <u>Conv</u>

- Power series (§8.5 and §8.6)

$$\sum_{n=0}^{\infty} C_n (x-a)^n$$

variable

C_n and a are real numbers

Power series centered at $x=a$.

- Interval of convergence = all values of x such that $\sum C_n(x-a)^n$ <u>Converges</u>

It's always of the form

$|x-a|<R$, $|x-a|\le R$,

$-R<x-a\le R$ or $-R\le x-a<R$

where $R\ge 0$ or $R=\infty$.

$R=$ <u>radius of convergence</u>

* we always find the interval & radius of convergence by applying the <u>ratio test</u>

• $\sum_{n=0}^{\infty} c_n(x-a)^n = f(x)$ for $|x-a|<R$, then

$f'(x)$ and $\int f(x)\,dx$ exists, and

* $f'(x) = \sum_{n=1}^{\infty} c_n \cdot n(x-a)^{n-1}$ ← starts at $n=1$

* $\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \cdot \frac{(x-a)^{n+1}}{n+1}$

Many power series representations can be derived via

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad , \quad |x|<1}$$

and differentiation and integration.

<u>Convergence test Checklist</u>:

Given $\sum_{n=1}^{\infty} a_n$, how to decide what convergence test to use?

(1) Div test. $\lim_{n\to\infty} a_n \neq 0$?

(2) Is it a p-series?

(3) Is it a geometric series?

(4) Is the series alternating?
    If so, try the AST.

(5) Try the ratio test.

(6) Does the series "look like" a series you know?

eg: $\sum \frac{1}{n^2+1}$ "looks like" $\sum \frac{1}{n^2}$

Try to apply the comparison test

(7) Are the terms a_n described by a function that looks like something you can integrate? Try the integral test.

(8) Last resort: Try to compute the partial sums S_N directly.

If the series is telescoping, the partial sums will be easy to compute.