MAT 127 Lec 1, M Jan 22

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§ Admin stuff

Class time: MW 4:00-5:20 pm

Grading:
- Homework 20%
- Midterms 20% each
- Final 40%

Text: Stewart 5^{th} edition (WebAssign)

Grader: Filip Samuelsen (Math 3-106)

Me: Johan Asplund (Math 3-116)

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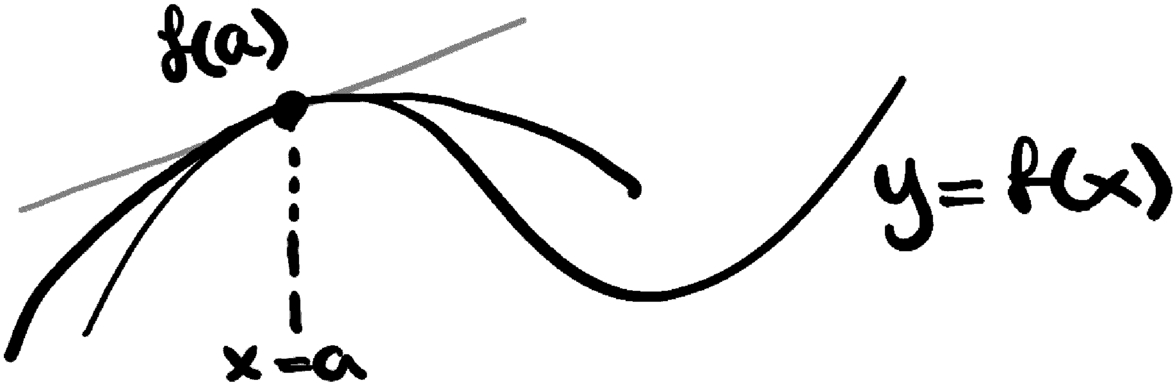
HW:
- Weekly. due: M 12 pm
via WebAssign (Access via Brightspace.)
- Short in-class quiz at the end of (almost) every W class. You will get it back in Gradescope.

§ Overview of course

Main topics:

1. Infinite Series (Taylor series)
2. Differential equations.

① Calc A: Derivatives



Linear approx at $x = a$.

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

Taylor series: Approximate $f(x)$ better by involving more derivatives!

$$\boxed{f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$$

"Parabolic approx at $x=a$"

Use infinite number of terms to not only get an approx, but equality!

- Tools to deal w/ infinite sums

② ordinary differential equations

Ex Find function $f(x)$ such that $f'(x) = f(x)$.

We see that $f(x) = e^x$ works.

What about $f'(x) \cdot f(x) = 2$?

- Develop tools to solve these.
- Applications

§ Sequences (Stewart 8.1)

An ordered list of "things".

usually numbers

Ex:

① $1, 2, 3, 4, \dots$

② $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \dots$

③ $14, 18, 23, 28, 34, 42, 50, 59, \dots$

Def:

- An (infinite) sequence is an ordered list of numbers.

$$a_1, a_2, a_3, \dots = \{a_n\}_{n=1}^{\infty}$$

1st term, 2nd term, 3rd term ...

- n is called the index.
- a_n is the n^{th} term

Ex ① $a_n = n$ for all $n \geq 1$

② $a_n = (-1)^n \cdot \frac{1}{n}$ for all $n \geq 1$

③ $a_n = n^{th}$ numbered stop on NYC uptown 1 train.

④ $\frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25}, \frac{7}{36}, \ldots$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$

Numerators increase by 1, so n^{th} numerator is $n+2$.

Denominators are squares.

$2^2 = 4$. $3^2 = 9$ etc.

so $\boxed{a_n = \frac{n+2}{(n+1)^2}}$.

Sometimes it's hard or difficult to find a defining equation for a_n.

Ex: Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

+ + +

$2=1+1$ $3=2+1$ $5=3+2$

In general:

$\boxed{a_n = a_{n-1} + a_{n-2}}$ for all $n \geq 3$

Recurrence relation

Ex: Consider the sequence given by $a_n = \frac{n}{n+1}$.

First few terms:

$a_1 = \frac{1}{1+1} = \frac{1}{2}$

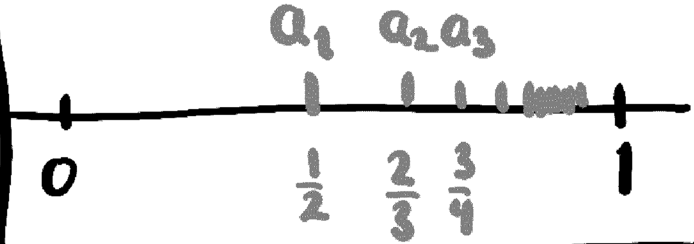
$a_2 = \frac{2}{2+1} = \frac{2}{3}$

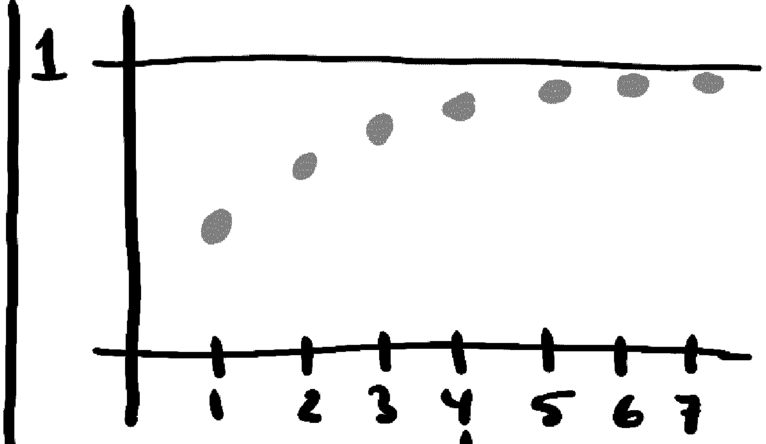
$a_3 = \frac{3}{4}$

$a_4 = \frac{4}{5}$

$a_5 = \frac{5}{6}$

on real line:





Plot points
$(1,a_1), (2,a_2), \ldots$

Points seem to go closer and closer to 1.

<u>Note</u>: n^{th} term is $\frac{n}{n+1}$ & it can never be ≥ 1 because denominator > numerator.

<u>How close is a_n to 1?</u>

$$1-a_n = 1-\frac{n}{n+1} = \frac{n+1}{n+1}-\frac{n}{n+1} = \frac{1}{n+1}.$$

when n is big, this difference is very small: $\frac{1}{10{,}001} = 1 - a_{10{,}000}$

We say that $\frac{n}{n+1}$ approaches 1 as n becomes large.

$$\boxed{\lim_{n\to\infty} \frac{n}{n+1} = 1}$$

Def A sequence $\{a_n\}_{n=1}^{\infty}$ has the limit L, written as

$$\boxed{\lim_{n\to\infty} a_n = L} \text{ or } \boxed{a_n \to L \text{ as } n \to \infty}$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists we say that $\{a_n\}$ converges, or that it is convergent. Otherwise $\{a_n\}$ diverges, or is divergent.

Note that this is similar to limits of functions $\lim_{x\to\infty} f(x)$.

Thm: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ where n is an integer, then $\lim_{n\to\infty} a_n = L$.

Ex: We know $\lim_{x\to\infty} \frac{1}{x^r} = 0$ for $r > 0$. Therefore, with $f(x) = \frac{1}{x^r}$ the sequence $a_n = f(n) = \frac{1}{n^r}$ converges to 0 as $n \to \infty$

$$\boxed{\lim_{n\to\infty} \frac{1}{n^r} = 0 \quad \text{for } r>0}$$

<u>Limit laws</u>

$\{a_n\}$, $\{b_n\}$ two sequences and
c = constant.

1. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

2. $\lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n$

3. $\lim_{n\to\infty} c = c$

4. $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$.

6. $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ $p>0$.
and $a_n > 0$.

Ex: $a_n = \frac{n^2+3}{27n^2-\pi+n}$. Use same technique as for functions: Since its the quotient of two polynomials we divide both the numerator and denominator by the highest power of n.

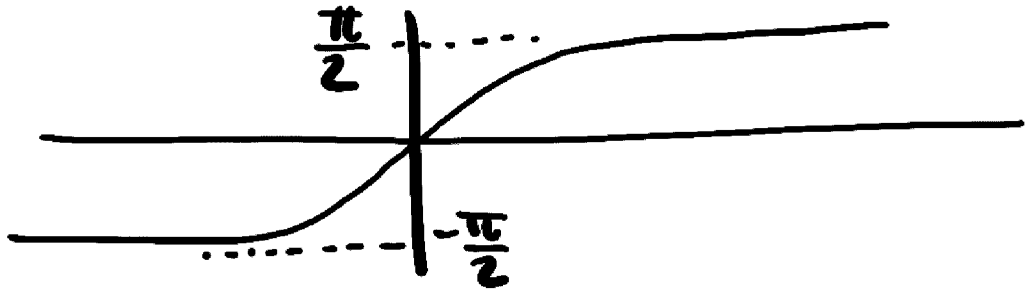
$$\lim_{n\to\infty} \frac{n^2+3}{27n^2-\pi+n} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{3}{n^2}}{\frac{27n^2}{n^2}-\frac{\pi}{n^2}+\frac{n}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{3}{n^2}}{27-\frac{\pi}{n^2}+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{3}{n^2}}{\lim_{n\to\infty} 27 - \lim_{n\to\infty}\frac{\pi}{n^2} + \lim_{n\to\infty}\frac{1}{n}}$$

$$= \frac{1+0}{27-0+0} = \frac{1}{27}.$$

Ex: $a_n = \sqrt{\arctan(n)}$.

$a_n = f(n)$, $f(x) = \arctan(x)$.



Remember $\lim\limits_{x\to\infty} \arctan(x) = \frac{\pi}{2}$

So

$$\lim_{n\to\infty} \sqrt{\arctan(n)} = \lim_{n\to\infty} (\arctan(n))^{1/2}$$

$$= \left(\lim_{n\to\infty} \arctan(n)\right)^{1/2} = \left(\frac{\pi}{2}\right)^{1/2} = \sqrt{\frac{\pi}{2}}.$$

Ex: Find $\lim\limits_{n\to\infty} \frac{\ln n}{n}$.

We have $a_n = \frac{\ln n}{n} = f(n)$

for the real-valued function

$f(x) = \frac{\ln x}{x}$.

Consider $\lim_{x \to \infty} \frac{\ln x}{x}$ and notice

$$\underbrace{\lim_{x \to \infty} \ln x = \infty}_{\text{numerator}}, \quad \underbrace{\lim_{x \to \infty} x = \infty}_{\text{denominator}}$$

Remember l'Hôpital's rule from Calculus A:

$\lim_{x \to \infty} f(x)$ is of the indeterminate form $\frac{\infty}{\infty}$ so

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} x}$$

$$= \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = 0$$

Hence $\boxed{\lim_{n \to \infty} \frac{\ln n}{n} = 0}$.