MAT 127 MIDTERM II

PRACTICE PROBLEMS

Reference page.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \quad |x| < 1$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad x \in \mathbb{R}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad x \in \mathbb{R}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad |x| < 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = \binom{k}{0} + \binom{k}{1} x + \binom{k}{2} x^2 \binom{k}{3} x^3 + \cdots, \quad |x| < 1$$

$$\binom{k}{0} = 1, \quad \binom{k}{n} = \frac{k(k-1) \cdot \cdots \cdot (k-n+1)}{n!}$$

The **Taylor series of** $f(x)$ **centered at** $x = a$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

1. (a) (10 pts) Calculate the degree 4 Taylor polynomial $T_4(x)$ of $f(x) = \cos(2x)$ centered around $x = \pi$.

(b) (10 pts) Find a power series representation of the integral

$$\int e^{-x^2}\, dx.$$

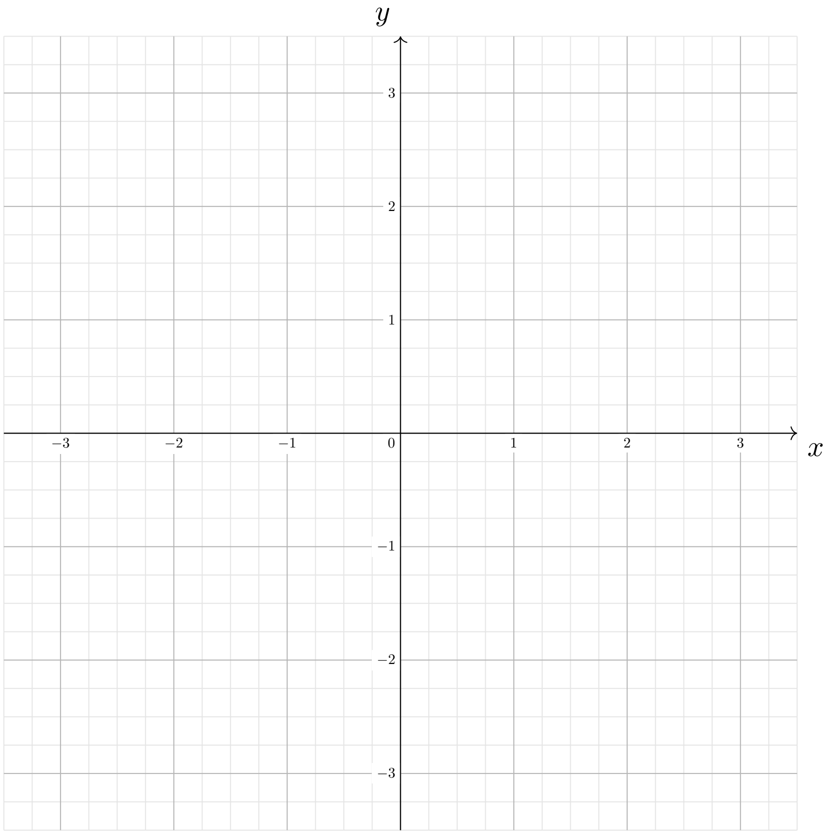
2. (a) (10 pts) Verify that $y = Ce^{-x} + x^2 - 2x$ for any value of the constant C is a solution to the second order ODE $y'' + y' = 2x$.

(b) (10 pts) Find a solution to the initial-value problem

$$y'' + y' = 2x, \quad y(0) = 1.$$

(You may use the result from part (a), even if you did not solve part (a).)

3. (a) (10 pts) Sketch the slope field for the first order ODE $y' = xy - 1$.



(b) (10 pts) Use Euler's method with step size 1 to estimate $y(3)$ where $y(x)$ is the solution to the initial-value problem

$$y' = x + y + 1, \quad y(0) = 0.$$

4. Find the general solution to the following separable first order ODEs

(a) (10 pts) $3y^2y' = \frac{1}{x}$

(b) (10 pts) $\frac{dy}{dx} = -xe^{-y}$

5. After drinking a cup of coffee containing 100 mg of caffeine, the amount of caffeine in a person's body t hours after drinking the cup is described by the differential equation

$$\frac{dC}{dt} = -\frac{7}{50}C(t).$$

(a) (10 pts) Solve the initial-value problem

$$\frac{dC}{dt} = -\frac{7}{50}C(t), \quad C(0) = 100.$$

(b) (10 pts) Solve the equation $C(t) = 50$. (Leave the answer in its exact form.) This is approximately the half-life of caffeine.