MAT 127 MIDTERM I

PRACTICE PROBLEMS

1. Determine whether the following sequences converge or diverge.

(a) (10 pts)

$$a_n = \frac{2n^3 + n - 1}{4n^3 + n^2 + n - 1}$$

Solution. Since a_n is described by the function $f(n) = \frac{2n^3+n-1}{4n^3+n^2+n-1}$, we compute the limit by dividing the numerator and denominator by n^3 (the dominant factor)

$$\lim_{n\to\infty} \frac{2n^3 + n - 1}{4n^3 + n^2 + n - 1} = \lim_{n\to\infty} \frac{2 + \frac{1}{n^2} - \frac{1}{n^3}}{4 + \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3}} = \frac{2 + 0 - 0}{4 + 0 + 0 - 0} = \frac{1}{2},$$

and therefore the sequence **converges to** $\frac{1}{2}$. □

(b) (10 pts)

$$a_n = \frac{2^n + \sin(n)}{n!}$$

Solution. Since a_n is described by the function $f(n) = \frac{2^n+\sin(n)}{n!}$, we first split the quotient up into two parts: $f(n) = \frac{2^n}{n!} + \frac{\sin(n)}{n!}$.

$$\lim_{n\to\infty} \frac{2^n + \sin(n)}{n!} = \lim_{n\to\infty} \frac{2^n}{n!} + \frac{\sin(n)}{n!}.$$

The first term $\frac{2^n}{n!} \to 0$ as $n \to \infty$, because the factorial will be much larger than 2^n as n is very large. For the second term, we know that $-1 \leq \sin(n) \leq 1$, and the $n!$ in the denominator will make the entire quotient go to zero $\frac{\sin(n)}{n!} \to 0$ as $n \to \infty$. The more rigorous way of arguing is to use the squeeze theorem because $-\frac{1}{n!} \leq \frac{\sin(n)}{n!} \leq \frac{1}{n!}$. Therefore $\lim_{n\to\infty} \frac{2^n}{n!} + \frac{\sin(n)}{n!} = 0$, and so $\{a_n\}_{n=1}^{\infty}$ **converges to** 0. □

2. Determine whether the following series converge or diverge.

(a) (10 pts)

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{7^n}{6^{n+1}}\right)$$

Solution. We split the sum up into two separate terms

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{7^n}{6^{n+1}}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{7^n}{6^{n+1}}.$$

The first term is a p-series with $p=2>1$, so it is convergent. The second term is a geometric series with $a=\frac{1}{6}$ and $r=\frac{7}{6}>1$:

$$\sum_{n=1}^{\infty}\frac{7^n}{6^{n+1}}=\sum_{n=1}^{\infty}\frac{7^n}{6\cdot 6^n}=\frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{7}{6}\right)^n=\infty.$$

The second term is therefore divergent. The sum of a convergent and a divergent series is divergent, and therefore $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{7^n}{6^{n+1}}\right)$ is **divergent**. □

(b) (10 pts)

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}$$

Solution. Since it is alternating, we can use the alternating series test. We have $a_n=\frac{1}{2n-1}$. We see that a_n is decreasing because $a_n=\frac{1}{2n-1}\geq\frac{1}{2n+1}=a_{n+1}$. We also have $\lim_{n\to\infty}\frac{1}{2n-1}=0$. The alternating series test then gives that it converges. □

3. (20 pts) Is the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ convergent or divergent? [*Hint: Use the integral test and note the starting point of the series.*]

Solution. As per the suggestion we use the integral test. We first have that the function $f(x) = \frac{1}{x \log x}$ is positive for all $x \geq 2$. We next need to show that f is decreasing. Namely, we compute the derivative

$$f'(x) = \frac{d}{dx}(x \log x)^{-1} = -(x \log x)^{-2} \cdot \left(\log x + x\frac{1}{x}\right) = -\frac{\log x + 1}{(x \log x)^2}$$

Both the numerator and the denominator are always positive for $x \geq 2$, so the minus sign in front gives that $f'(x) < 0$ for all $x \geq 2$. The integral test is therefore applicable. We compute the indefinite integral

$$\int_2^{\infty} \frac{1}{x \log x} dx = \begin{bmatrix} u = \log x \\ du = \frac{1}{x} dx \\ x = 2 \Rightarrow u = \log 2 \\ x \to \infty \Rightarrow u \to \infty \end{bmatrix} = \int_{u=\log 2}^{\infty} \frac{1}{u} du = [\log |u|]_{u=\log 2}^{\infty}$$

$$= \left(\lim_{u \to \infty} \log |u|\right) - \log \log 2 = \infty.$$

Since this indefinite integral diverges, we conclude that $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ **diverges** by the integral test. $\square$

4. Find the radius and interval of convergence of the following power series.

(a) (10 pts)

$$\sum_{n=1}^{\infty} \frac{(-1)^n(3x)^n}{n!}$$

Solution. We use the ratio test, with the terms $a_n = \frac{(-1)^n(3x)^n}{n!}$.

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(3x)^{n+1}}{(n+1)!}}{\frac{(-1)^n(3x)^n}{n!}}\right| = \lim_{n\to\infty}|3x|\cdot\frac{n!}{(n+1)!} = \lim_{n\to\infty}|3x|\cdot\frac{1}{n+1} = 0.$$

This limit is always equal to zero, for any value of x. Therefore the ratio test we conclude that the series **converges** for all x. The interval of convergence is $(-\infty,\infty)$ and the radius of convergence is $R=\infty$. □

(b) (10 pts)

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$$

Solution. We use the ratio test, with the terms $a_n = \frac{(x-1)^n}{\sqrt{n}}$.

$$\rho = \lim_{n\to\infty}\left|\frac{\frac{(x-1)^{n+1}}{\sqrt{n+1}}}{\frac{(x-1)^n}{\sqrt{n}}}\right| = \lim_{n\to\infty}|x-1|\cdot\frac{\sqrt{n}}{\sqrt{n+1}} = |x-1|\cdot\sqrt{\lim_{n\to\infty}\frac{n}{n+1}} = |x-1|.$$

By the ratio test, the series **converges** when $\rho = |x-1| < 1$, which is equivalent to $-1 < x-1 < 1 \Leftrightarrow 0 < x < 2$. The ratio test also gives that the series **diverges** when $\rho = |x-1| > 1$ which is equivalent to $x<0$ or $x>2$. We also need to check the endpoints of the convergence interval.

$x=0$**:** Plugging in the value $x=0$ gives $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$. This series is alternating and of the form $\sum_{n=1}^{\infty}(-1)^n a_n$, where $a_n = \frac{1}{\sqrt{n}}$. We have that a_n is a decreasing sequence with $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$. Therefore the alternating series test tells us that $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ is **convergent**.

$x=2$**:** Plugging in the value $x=0$ gives $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$. This is a p-series with $p=\frac{1}{2}\le 1$, and hence **divergent**.

To summarize, the interval of convergence is $0\le x<2$, and the radius of convergence is $R=1$.

□

5. Find a power series representation of the following functions. Also determine the interval of convergence.

(a) (10 pts) $f(x) = \dfrac{2}{3+x^2}$

Solution. We know that the function $\frac{1}{1-x}$ has the power series representation $\sum_{n=0}^{\infty} x^n$ for $|x| < 1$ (this is the geometric series). We rewrite the function as

$$f(x) = \frac{2}{3+x^2} = 2 \cdot \frac{1}{3\left(1 - \left(-\frac{x^2}{3}\right)\right)} = \frac{2}{3} \cdot \frac{1}{1 - \left(-\frac{x^2}{3}\right)}.$$

Then we get

$$f(x) = \frac{2}{3} \sum_{n=0}^{\infty} \left(-\frac{x^2}{3}\right)^n = \frac{2}{3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3^n}.$$

The interval of convergence is $\left|-\frac{x^2}{3}\right| < 1 \Leftrightarrow |x| < \sqrt{3}$. □

(b) (10 pts) $g(x) = \frac{2}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right)$ [*Hint: What is the derivative of $g(x)$?*]

Solution. By the hint, we note that $g'(x) = \frac{2}{3} \frac{1}{1+\left(\frac{x}{\sqrt{3}}\right)^3} = \frac{2}{3+x^2}$. We know from part (a) that

$$g'(x) = \frac{2}{3+x^2} = \frac{2}{3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3^n}$$

We integrate both sides to obtain

$$g(x) + C = \int \frac{2}{3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3^n} dx = \frac{2}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} \frac{x^{2n+1}}{2n+1}.$$

We determine the value of C by plugging in $x = 0$ in this equation, noting that $g(0) = 0$:

$$C = \frac{2}{3} \sum_{n=0}^{\infty} (-1) \frac{(-1)^n}{3^n} \frac{0^{2n+1}}{2n+1} = 0.$$

Therefore

$$\frac{2}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) = \frac{2}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} \frac{x^{2n+1}}{2n+1}.$$

The interval of convergence is the same as that of the power series representation of $f(x)$ of part (a), namely $|x| < \sqrt{3}$. □