MAT 127

COVERAGE OF FINAL

The midterm will consist of 10 problems, each one worth 20 points each. It will include problems on almost all topics we have covered throughout the course.

Below is a more detailed list of learning goals for each section. Keep in mind that because the final only consists of 10 problems, it is impossible to test each and every one of the learning goals listed below. All section references are to Stewart.

1. Learning goals

Sequences: (All of §8.1.)

Learning goals:

- Understand the notion of a sequence and work with different ways of describing sequences (formulas for the general term, recursive formulas, relation to functions, plots of sequences), compute several terms of a sequence given as above.
- Understand intuitively the notions of convergence and limit of a sequence.
- Identify and correctly use terminology describing properties of sequences (increasing, decreasing, monotonic, bounded above, bounded below).

Infinite series: (All of §8.2, §8.3: Integral test, p-series and comparison test only, §8.4: Alternating series test and ratio test only.)

Learning goals:

- Understand the notion of series; distinguish between sequences and series.
- Know the definition of an infinite series as the limit of the partial sum sequence.
- Recognize the geometric series and know how to identify whether it converges or diverges; in case of convergence being able to compute the sum.
- Recognize the harmonic series and know that it diverges.
- Recognize and being able to carry out simple examples of telescoping series.
- Know the statement of the divergence test, ratio test, integral test, comparison test and the alternating series test and how to use them.
- Recognize a p-series and knowing for which values it converges/diverges.
- Understand the notion of absolute convergence, and that absolute convergence implies convergence (but not vice versa). Use it to show convergence of a non-positive series by showing convergence of the corresponding positive series. Use Alternating series test to establish conditional convergence.

Power series: (All of §8.5 and §8.6.)

Learning goals:

- Identify a power series.
- Being able to find the radius and interval of convergence using the ratio test, including endpoints of the convergence interval.
- Differentiate and integrate power series to find new power series from old ones. Find radius and interval of convergence of the resulting series.

Taylor and Maclaurin series: (All of §8.7.) Learning goals:

- State the general form of Taylor and Maclaurin series and polynomials.
- Find Taylor series and Taylor polynomials with a given center, by three methods: (1) computing derivatives and using the Taylor formula, (2) using substitution and arithmetic operations on known basic Maclaurin series such as $\sin x$, $\cos x$, e^x, $\frac{1}{1-x}$, $\log(1+x)$, or (3) using differentiation or integration of known series.
- **Note: You will *not* need to memorize common Taylor series: A reference page will be provided on the second midterm.**

Differential equations, slope fields and Euler's method: (All of §7.1 and §7.2.)

Learning goals:

- Explain what a differential equation is and find the order of the given differential equation.
- Understand the notion of a solution of a differential equation; check whether a given function is a solution to the given differential equation or to the initial value problem.
- Understand slope fields as a geometric interpretation of a first order ODE and use them to derive properties solutions (plot a solution curve for a given initial value problem; determine whether the solution to an initial value problem is increasing or decreasing; determine asymptotic behavior of a solution of the initial value problem).
- Being able to sketch a slope field.
- Understand and apply Euler's method to find approximations of solutions; provide a geometric interpretation of the method.

Separable ODEs and modeling: (All of §7.3 and §7.4.)

Learning goals:

- Identify separable ODEs; apply method of separation of variables to solve differential equations; solve initial-value problems.
- Compose differential equations and initial value problems to model processes described by word problems; solve the resulting problems and give a real-life interpretation of the solution; compare the behavior of solutions to qualitative predictions obtained by vector fields; verify whether the solutions make sense in the context of the problem
- Work with exponential models for growth and decay, recognize the corresponding differential equations.

Second order ODEs: (Supplementary notes) Learning goals:

- Identify a second-order homogeneous linear differential equations with constant coefficients.
- Understand what linearity implies for solutions (the linear combination of solutions gives a solution).
- Use the standard process for solving this type of equations: write the characteristic equation for a given equation, determine two linearly independent solutions from the roots of the characteristic equation, set up the general solution, solve initial value problems.
- Perform basic operations with complex numbers: understand notation $z = a + bi$, add, subtract, and multiply complex numbers, solve quadratic equations with complex roots

2. What convergence test to use?

Being able to know **which** convergence test to use is an important skill to succeed on the final. This skill is best learned by practicing; do a lot of problems. Eventually you begin to get a sense of what convergence test is likely to work in a specific situation. (This is similar to the question "How do I know which technique to use when integrating a function?" that you likely have asked and answered in the past.) Below is one possible flow chart that is usually similar to how I personally deal with this problem when faced with a new numerical series:

- Try the divergence test: Do the terms go to zero as $n \to \infty$? If not, it diverges by the divergence test.
- Is it a p-series, i.e. of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$? If so, it converges for $p > 1$, and diverges otherwise.
- Is it a geometric series, i.e. of the form $\sum_{n=0}^{\infty} ar^n$? If so, it converges for $|r| < 1$. The sum is $\frac{a}{1-r}$, assuming that the index starts at 0.
- Is the series alternating? Try the alternating series test.

- Try the ratio test. Compute $\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. If $\rho > 1$, it diverges, if $\rho < 1$ converges and if $\rho = 1$ the ratio test is inconclusive.
- Does the series "look like" another series that you know? For example $\sum_{n=1}^{n} \frac{1}{n^2+1}$ "looks like" $\sum_{n=1}^{n} \frac{1}{n^2}$ which we recognize as a p-series with $p = 2$. Using the comparison test is likely a good idea. Remember that the terms need to be positive in order for this to be applicable.
- Is the n-th term a_n described by a function f that looks like something you can integrate? Try to use the integral test.
- Last resort: Compute the partial sums $S_N = \sum_{n=1}^{N} a_n$ and take the limit as $N \to \infty$. This will be easy in the case of a telescoping series such as $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$.

3. Reference page that will be provided on the final

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \quad |x| < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad x \in \mathbb{R}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad x \in \mathbb{R}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad |x| < 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = \binom{k}{0} + \binom{k}{1}x + \binom{k}{2}x^2\binom{k}{3}x^3 + \cdots, \quad |x| < 1$$

$$\binom{k}{0} = 1, \quad \binom{k}{n} = \frac{k(k-1)\cdot \cdots \cdot (k-n+1)}{n!}$$

The **Taylor series of $f(x)$ centered at $x = a$** is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Table of trigonometric values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N/A	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0