MAT 127 FINAL EXAM

PRATICE PROBLEMS

Name: ______________________________ ID: ______________

Lecture (circle yours):

L01	L02	L03
MW 3:30–4:50pm	TuTh 9:30–10:50am	TuTh 5:00–6:20pm

Instructions.

(1) Fill in your name and Stony Brook ID number and circle your lecture number at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no electronic devices.

(3) You have 165 minutes to complete this exam.

(4) Leave all answers in exact form (that is, *do not* approximate π, square roots, etc.)

(5) You must justify all your answers and show all your work. Even a correct answer without any justification will result in no credit.

For reference.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \quad |x| < 1$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad x \in \mathbb{R}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad x \in \mathbb{R}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad |x| < 1$$

$$\arctan(\sqrt{3}) = \frac{\pi}{3}, \qquad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \cos\left(\frac{2\pi}{3}\right) = \frac{1}{2}$$

The **Taylor series of** $f(x)$ **centered at** $x = a$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

1. Determine if the following series converges or diverges.

(a) (10 pts)

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

Solution. It is a geometric series with $r = \frac{1}{2}$ and therefore converges. $\square$

(b) (10 pts)

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 800}$$

Solution. We use the comparison test and note $\frac{1}{n^3+800} \leq \frac{1}{n^3}$. Therefore $\sum_{n=1}^{\infty} \frac{1}{n^3+800} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$ and the latter sum is convergent as it is a p-series with $p = 3 > 1$. $\square$

2. (a) (10 pts) Find the Taylor polynomial of degree 2, $T_2(x)$, of the function $f(x) = \ln(1+x)$ around $x = 2$.

Solution. First we have $f(2) = \ln 3$. We compute the first two derivatives of f:

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = \frac{d}{dx}(1+x)^{-1} = -(1+x)^{-2} = -\frac{1}{(1+x)^2}.$$

Then $f'(2) = \frac{1}{1+2} = \frac{1}{3}$ and $f''(2) = -\frac{1}{(1+2)^2} = -\frac{1}{9}$. Therefore

$$T_2(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 = \ln 3 + \frac{x-2}{3} - \frac{(x-2)^2}{18}.$$

□

(b) (10 pts) Find an approximation of $\arctan\left(\frac{1}{2}\right)$ as a fraction $\frac{p}{q}$ using the Maclaurin polynomial of $\arctan x$ of order 3.

Solution. From the reference page, the Maclaurin series of $\arctan x$ is

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

so the Maclaurin polynomial of order 3 is $M_2(x) = x - \frac{x^3}{3}$. We now find

$$\arctan\left(\frac{1}{2}\right) \approx M_2\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} = \frac{1}{2} - \frac{1}{24} = \frac{12}{24} - \frac{1}{24} = \frac{11}{24}.$$

□

3. Find the solution of each of the initial value problems. To receive full credits you must not leave the answer in implicit form, meaning the answer must be of the form "$y(x) = \cdots$".

(a) (10 pts) $\frac{dy}{dx} = e^{-y}\sin x$, $y(0) = 1$.

Solution.

$$\begin{aligned}\frac{dy}{dx} = e^{-y}\sin x \Leftrightarrow e^{y}dy = \sin x dx \Leftrightarrow \int e^{y}dy = \int \sin x dx \\ \Leftrightarrow e^{y} = -\cos x + C \Leftrightarrow y = \ln(-\cos x + C).\end{aligned}$$

The initial value gives

$$y(0) = \ln(-1+C) = \ln(-1+C) = 1 \Leftrightarrow -1 + C = e \Leftrightarrow C = e - 1,$$

and therefore the specific solution is given by

$$y(x) = \ln(-\cos x + e - 1).$$

□

(b) (10 pts) $y' = y^2 + 1$, $y(0) = \sqrt{3}$.

Solution.

$$\begin{aligned}y' = y^2 + 1 \Leftrightarrow \frac{dy}{dx} = y^2 + 1 \Leftrightarrow \frac{1}{y^2+1}dy = dx \\ \Leftrightarrow \int \frac{1}{y^2+1}dy = \int dx \Leftrightarrow \arctan y = x + C \Leftrightarrow y = \tan(x + C).\end{aligned}$$

The initial condition gives

$$y(0) = \tan(C) = \sqrt{3},$$

which gives $C = \frac{\pi}{3}$ so the specific solution is given by

$$y(x) = \tan\left(x + \frac{\pi}{3}\right).$$

□

4. Rewrite the following complex numbers in the standard form $a+bi$.

(a) (10 pts) $\arg(\sqrt{2}e^{\frac{\pi}{4}i}) - i^7 + \frac{4-2i}{i}$

Solution. We have $\arg\left(\sqrt{2}e^{\frac{\pi}{4}i}\right) = \frac{\pi}{4}$, and then $i^7 = i^4 \cdot i^3 = 1 \cdot i^3 = -i$. Next $\frac{4-2i}{i} = \frac{4}{i} - 2 = -4i - 2$. Therefore

$$\arg(\sqrt{2}e^{\frac{\pi}{4}i}) - i^7 + \frac{4-2i}{i} = \frac{\pi}{4} - (-i) - 4i - 2 = \left(\frac{\pi}{4} - 2\right) - 3i.$$

□

(b) (10 pts) $(-1+\sqrt{3}i)^7$.

Solution. We first rewrite $-1+\sqrt{3}i$ in polar form. The argument is $\arg(-1+\sqrt{3}i) = \pi - \arctan(\sqrt{3}) = \frac{2\pi}{3}$ and the modulus is $|-1+\sqrt{3}i| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$. Therefore $-1+\sqrt{3}i = 2e^{\frac{2\pi}{3}i}$. Then we have

$$\begin{aligned}\left(2e^{\frac{2\pi}{3}i}\right)^7 &= 2^7 e^{\frac{14\pi}{3}i} = 128e^{\left(4\pi + \frac{2\pi}{3}\right)i} = 128e^{\frac{2\pi}{3}i} = 128\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) \\ &= 128\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\end{aligned}$$

so the answer is $(-1+\sqrt{3}i)^7 = 128\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$. □

5. (20 pts) Solve the following initial-value problem

$$y'' - 12y' + 36y = 0, \quad y(0) = 1, \quad y'(0) = -1$$

Solution. The characteristic equation is given by $r^2 - 12r + 36 = 0$. The solution is a repeated root $r = 6$, because $r^2 - 12r + 36 = (r-6)^2$. Therefore the general solution is given by

$$y(x) = (C_1 + C_2 x)e^{6x}$$

The first initial condition gives

$$y(0) = (C_1 + C_2 \cdot 0)e^0 = C_1 = 1.$$

After differentiating we get

$$y'(x) = C_2 e^{6x} + (C_1 + C_2 x)6e^{6x}$$

and so

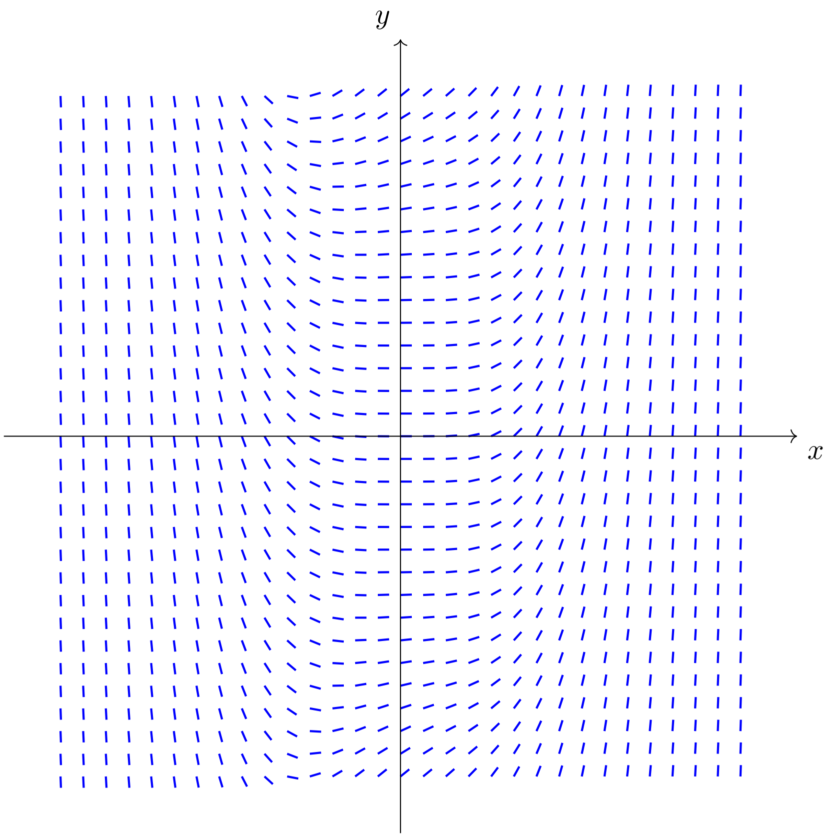
$$y'(0) = C_2 e^0 + (1 + C_2 \cdot 0)6e^0 = C_2 + 6 = -1 \Leftrightarrow C_2 = -7.$$

The specific solution to the initial-value problem is therefore given by

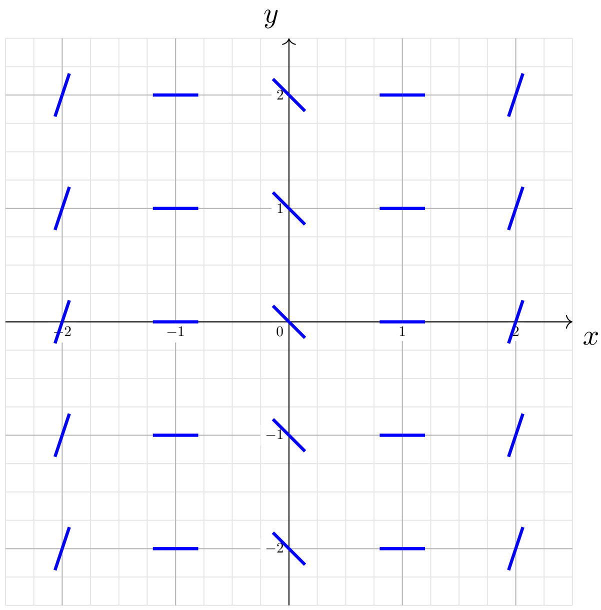
$$y(x) = (1 - 7x)e^{6x}.$$

$\square$

6. (a) (10 pts) Below is a slope field of a differential equation $y' = x^3 + \frac{y^4}{100}$. Sketch the graph of the solution passing through the origin.



(b) (10 pts) Sketch the slope field for the first order ODE $y' = x^2 - 1$ at the points (x, y) in the plane where $-2 \le x \le 2$ and $-2 \le y \le 2$ are integers.



7. (20 pts) Determine if the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

converges absolutely, converges conditionally, or if it diverges.

Solution. The sequence given by $a_n = \frac{1}{n!}$ is decreasing, since $n!$ is increasing for $n \geq 1$. Therefore the alternating series test tells us that the series is *convergent.*
To test for absolute convergence we take the absolute value of the terms and obtain the series $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n!} \right| = \sum_{n=1}^{\infty} \frac{1}{n!}$. We see that this series is convergent by using the ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

The conclusion is that the series is absolutely convergent. □

8. (a) (10 pts) Find a power series representation of the function $f(x) = x\ln(1+x^2) - 2x + +2\arctan x$.

Solution. Using reference page we get

$$\ln(1+x^2) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x^2)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n},$$

and therefore

$$\begin{aligned}
x\ln(1+x^2) - 2x + 2\arctan x &= x\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}\right) - 2x + 2\left(\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{2n+1}\right)\\
&= \left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{n}\right) - 2x + \left(\sum_{n=0}^{\infty}(-1)^{n}\frac{2x^{2n+1}}{2n+1}\right)\\
&= \left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{n}\right) - 2x + 2x + \left(\sum_{n=1}^{\infty}(-1)^{n}\frac{2x^{2n+1}}{2n+1}\right)\\
&= \sum_{n=1}^{\infty}(-1)^{n}\left(-\frac{1}{n} + \frac{2}{2n+1}\right)x^{2n+1}\\
&= \sum_{n=1}^{\infty}(-1)^{n}\frac{2n-(2n+1)}{n(2n+1)}x^{2n+1}\\
&= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{n(2n+1)}
\end{aligned}$$

□

(b) (10 pts) Find the power series representation of the integral $\int \ln(1+x^2)dx$ by integrating the Maclaurin series for $\ln(1+x^2)$.

Solution.

$$\begin{aligned}
\int \ln(1+x^2)dx &= \int \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}dx = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\int x^{2n}dx\\
&= C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{n(2n+1)}.
\end{aligned}$$

□

9. The Weber–Fechner law in psychology is a model for the rate of change of a reaction y to a stimulus of strength s. The relation is the differential equation

$$\frac{dy}{ds} = k\frac{y}{s}.$$

(a) (10 pts) Find the general solution to the differential equation, that is, express y as a function of s involving the constant k. To receive full credits you must not leave the answer in implicit form, meaning the answer must be of the form "$y(x) = \cdots$".

Solution. The equation is separable, so

$$\begin{aligned} \frac{dy}{dx} = k\frac{y}{s} \Leftrightarrow \frac{1}{y}dy = k\frac{1}{s}ds \Leftrightarrow \int \frac{1}{y}dy = \int k\frac{1}{s}ds \\ \Leftrightarrow \ln|y| = k(\ln|s| + C) = \ln|s^k| + D \\ \Leftrightarrow y = e^{\ln|s^k|+D} = Es^k. \end{aligned}$$

□

(b) (10 pts) In a completely made-up experiment, a stimulus of size $s = 1$ gave rise to a reaction of size $y = 1$ (in some appropriate units). Doubling the amount of stimulus to $s = 2$ gave rise to a reaction of size $y = \frac{3}{2}$. What does the differential equation predict the reaction to a stimulus of size s = 3 would be?

Solution. From the description we have $y(1) = 1$ and $y(2) = \frac{3}{2}$. Using these two equations we can determine the constants k and E. Namely

$$y(1) = E \cdot 1^k = E = 1.$$

Next

$$y(2) = 1 \cdot 2^k = \frac{3}{2} \Leftrightarrow k \ln 2 = \ln\frac{3}{2} \Leftrightarrow k = \frac{\ln\frac{3}{2}}{\ln 2}.$$

Therefore the reaction to a stimulus of size $s = 3$ is predicted to be

$$y(3) = 3^k = 3^{\frac{\ln\frac{3}{2}}{\ln 2}} \approx 1.90.$$

□

10. Consider the power series
$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n-1}.$$

(a) (10 pts) Find the center and radius of convergence.

Solution. The center is $x = 0$. To find the radius we use the ratio test.
$$\rho = \lim_{n\to\infty}\left|\frac{\frac{x^{2n+3}}{2(n+1)-1}}{\frac{x^{2n+1}}{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}(2n-1)}{x^{2n+1}(2n+1)}\right|$$
$$= \lim_{n\to\infty}|x^2|\frac{2n-1}{2n+1} = |x^2|\underbrace{\lim_{n\to\infty}\frac{2n-1}{2n+1}}_{=1} = |x^2|$$
This converges when $|x^2| < 1 \Leftrightarrow |x| < 1$ so the radius of convergence is 1. □

(b) (10 pts) Find the interval of convergence. Determine if the power series converges absolutely, converges conditionally or diverges at each of the endpoints.

Solution. From part (a) we have $\rho = |x^2|$, and by the ratio test, the series absolutely converges for $\rho < 1$ meaning when $|x^2| < 1 \Leftrightarrow -1 < x^2 < 1 \Leftrightarrow -1 < x^2 < 1$.
We now check the endpoints.

$x = 1$**:** We get the series $\sum_{n=0}^{\infty}\frac{1^{2n+1}}{2n-1} = \sum_{n=0}^{\infty}\frac{1}{2n-1}$ which diverges by the comparison test (compare with half times the harmonic series). Namely $\frac{1}{2n-1} > \frac{1}{2n}$ for $n > 1$ (if $n = 0$ we get division by zero) and so $\sum_{n=0}^{\infty}\frac{1}{2n-1} = -1 + \sum_{n=1}^{\infty}\frac{1}{2n-1} > -1 + \sum_{n=1}^{\infty}\frac{1}{2n}$, and this is divergent. It is half the harmonic series (or equivalently a p-series with $p = 1$).

$x = -1$**:** We get the series $\sum_{n=0}^{\infty}\frac{(-1)^{2n+1}}{2n-1} = \sum_{n=0}^{\infty}\frac{1}{2n-1} = -\sum_{n=0}^{\infty}\frac{1}{2n-1}$. This series is just the negative of the previous series, and so is divergent.

To summarize, the power series converges absolutely if $-1 < x < 1$ and diverges otherwise. □