MAT 127 FINAL EXAM

PRATICE PROBLEMS

Name: ______________________________ ID: ______________

Lecture (circle yours): L01 (MW 3:30–4:50pm) L02 (TuTh 9:30–10:50am) L03 (TuTh 5:00–6:20pm)

Instructions.

(1) Fill in your name and Stony Brook ID number and circle your lecture number at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no electronic devices.

(3) You have 165 minutes to complete this exam.

(4) Leave all answers in exact form (that is, *do not* approximate π, square roots, etc.)

(5) You must justify all your answers and show all your work. Even a correct answer without any justification will result in no credit.

For reference.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \quad |x| < 1$$

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad x \in \mathbb{R}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad x \in \mathbb{R}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad |x| < 1$$

$$\arctan(\sqrt{3}) = \frac{\pi}{3}, \qquad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \cos\left(\frac{2\pi}{3}\right) = \frac{1}{2}$$

The **Taylor series of** $f(x)$ **centered at** $x = a$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

1. Determine if the following series converges or diverges.

(a) (10 pts)

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

(b) (10 pts)

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 800}$$

2. (a) (10 pts) Find the Taylor polynomial of degree 2, $T_2(x)$, of the function $f(x) = \ln(1 + x)$ around $x = 2$.

(b) (10 pts) Find an approximation of $\arctan\left(\frac{1}{2}\right)$ as a fraction $\frac{p}{q}$ using the Maclaurin polynomial of $\arctan x$ of order 3.

3. Find the solution of each of the initial value problems. To receive full credits you must not leave the answer in implicit form, meaning the answer must be of the form "$y(x) = \cdots$".

(a) (10 pts) $\frac{dy}{dx} = e^{-y} \sin x$, $y(0) = 1$.

(b) (10 pts) $y' = y^2 + 1$, $y(0) = \sqrt{3}$.

4. Rewrite the following complex numbers in the standard form $a + bi$.

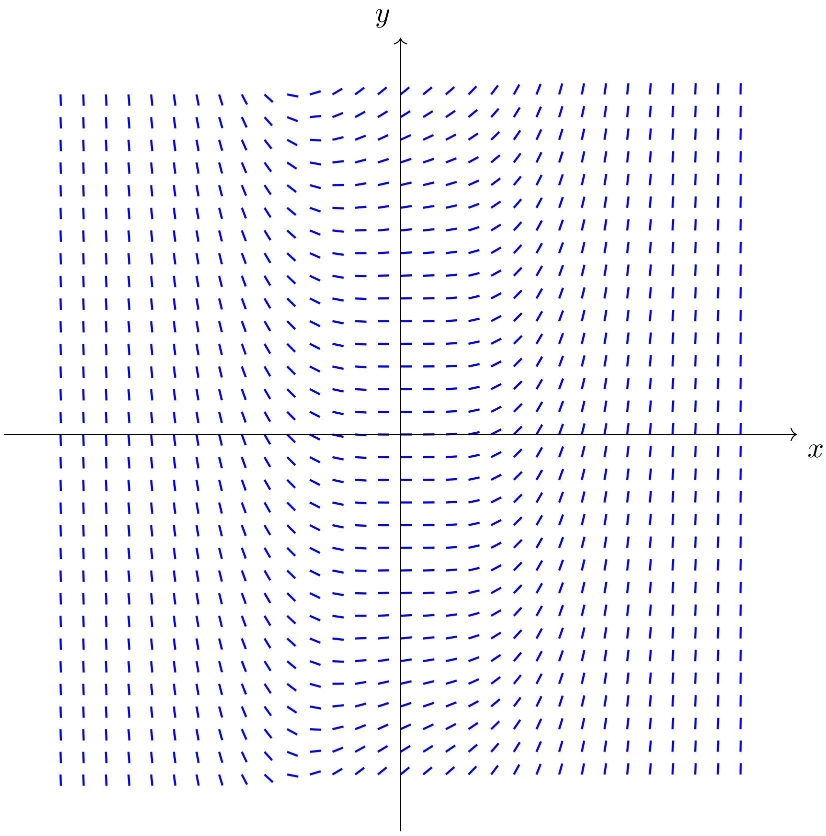
(a) (10 pts) $\arg(\sqrt{2}e^{\frac{\pi}{4}i}) - i^7 + \frac{4-2i}{i}$

(b) (10 pts) $(-1 + \sqrt{3}i)^7$.

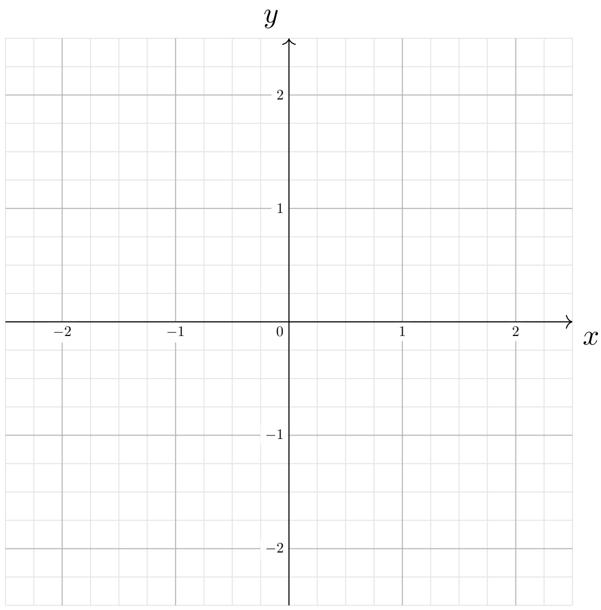
5. (20 pts) Solve the following initial-value problem

$$y'' - 12y' + 36y = 0, \quad y(0) = 1, \quad y'(0) = -1$$

6. (a) (10 pts) Below is a slope field of a differential equation $y' = x^3 + \frac{y^4}{100}$. Sketch the graph of the solution passing through the origin.



(b) (10 pts) Sketch the slope field for the first order ODE $y' = x^2 - 1$ at the points (x, y) in the plane where $-2 \le x \le 2$ and $-2 \le y \le 2$ are integers.



7. (20 pts) Determine if the series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$
converges absolutely, converges conditionally, or if it diverges.

8. (a) (10 pts) Find a power series representation of the function $f(x) = x\ln(1+x^2) - 2x +$ $+2\arctan x$.

(b) (10 pts) Find the power series representation of the integral $\int \ln(1+x^2)dx$ by integrating the Maclaurin series for $\ln(1+x^2)$.

9. The Weber–Fechner law in psychology is a model for the rate of change of a reaction y to a stimulus of strength s. The relation is the differential equation

$$\frac{dy}{ds} = k\frac{y}{s}.$$

(a) (10 pts) Find the general solution to the differential equation, that is, express y as a function of s involving the constant k. To receive full credits you must not leave the answer in implicit form, meaning the answer must be of the form "$y(x) = \cdots$".

(b) (10 pts) In a completely made-up experiment, a stimulus of size $s = 1$ gave rise to a reaction of size $y = 1$ (in some appropriate units). Doubling the amount of stimulus to $s = 2$ gave rise to a reaction of size $y = \frac{3}{2}$. What does the differential equation predict the reaction to a stimulus of size s = 3 would be?

10. Consider the power series

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n-1}.$$

(a) (10 pts) Find the center and radius of convergence.

(b) (10 pts) Find the interval of convergence. Determine if the power series converges absolutely, converges conditionally or diverges at each of the endpoints.