Wild and Tame Dynamics on the Projective Plane

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Iterated Maps of the Projective Plane:

$\mathbb{P}^2$ will denote either the **complex projective plane** $\mathbb{P}^2(\mathbb{C})$,
or the **real projective plane** $\mathbb{P}^2(\mathbb{R})$.

$f : \mathbb{P}^2 \to \mathbb{P}^2$ will be a **rational map** of algebraic degree $d \geq 2$, everywhere defined (unless otherwise specified).

TWO BASIC QUESTIONS:

What kinds of attractor $A = f(A) \subset \mathbb{P}^2$ can such a map have?

What kinds of f-invariant curve $S = f(S) \subset \mathbb{P}^2$ can exist?

Trapped Attractors:

DEFINITION. A compact f-invariant set $A = f(A) \subset \mathbb{P}^2$ is called a **trapped attracting set** if it has a neighborhood N in $\mathbb{P}^2$ such that

$$\bigcap_k f^{\circ k}(N) = A\,.$$

If this condition is satisfied, then we can always choose this neighborhood N more carefully so that

$$f(\overline{N}) \;\subset\; \mathrm{interior}(N)\,.$$

N is then called a **trapping neighborhood**.

If A also contains a dense orbit, then it is called a **trapped attractor**.

Note that a trapped attractor may be a **fractal** set.

Measure Attractors

Again let $A = f(A) \subset \mathbb{P}^2$ be a compact f-invariant set.

The **attracting basin** $\mathcal{B}(A)$ is defined to be the union of all orbits which converge to A.

Note that $\mathcal{B}(A)$ need not be an open set !

DEFINITION: A will be called a **measure attractor** if:

1. The **attracting basin** $\mathcal{B}(A)$ has positive Lebesgue measure.

2. Minimality: No strictly smaller compact f-invariant set $A' \subset A$ has a basin of positive measure.

(Reference: www.scholarpedia.org/article/Attractor)

A fractal example in $\mathbb{P}^2(\mathbb{R})$

Hénon [1976] described a polynomial automorphism of $\mathbb{R}^2$:

$$(x,\ y)\ \mapsto\ (y,\ \ y^2 - 1.4 + .3x)\,.$$

He showed, at least empirically, that it has a **fractal trapped attractor.** This map extends to a quadratic rational map of $\mathbb{P}^2(\mathbb{R})$

$$(x\ :\ y\ :\ z)\ \mapsto\ (yz\ :\ y^2 - 1.4z^2 + .3xz\ :\ z^2)\,.$$

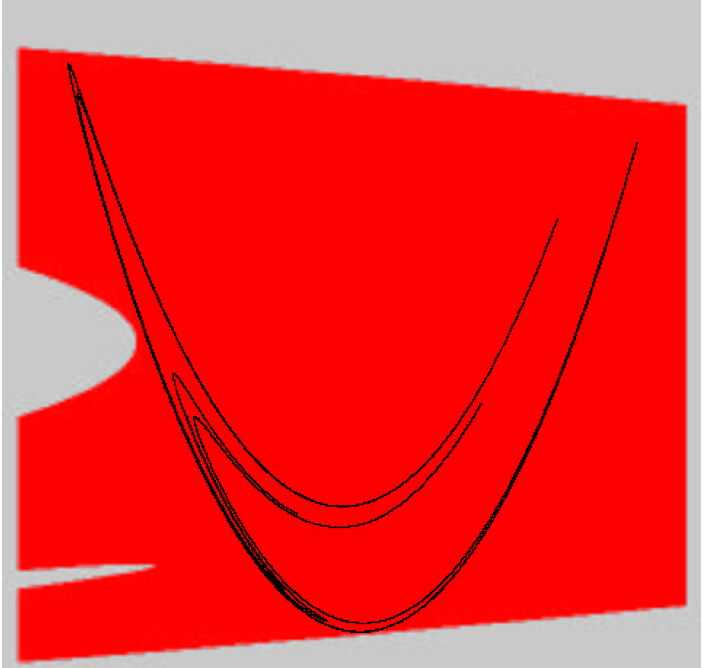
This extension is not everywhere defined — It has a point of indeterminacy at $(1\ :\ 0\ :\ 0)$. However, if we perturb so that

$$(x\ :\ y\ :\ z)\ \mapsto\ (yz\ :\ y^2 - 1.4z^2 + .3xz\ :\ z^2 + \epsilon x^2)\,,$$

then we get an everywhere defined rational map.
The corresponding map of $\mathbb{R}^2$ is:

$$(x, y)\ \mapsto\ \left(\frac{y}{1+\epsilon x^2}\,,\ \frac{y^2 - 1.4 + .3x}{1+\epsilon x^2}\right)\,.$$

The modified Hénon attractor in $\mathbb{P}^2(\mathbb{R})$



A fractal example in $\mathbb{P}^2(\mathbb{C})$

Jonsson and Weickert [2000] constructed an everywhere defined complex rational map which has a non-algebraic trapped attractor. Here is a related example. Start with a quadratic rational map of $\mathbb{P}^1(\mathbb{C})$ which has a dense orbit:

$$(x : y) \mapsto \Big(f(x,y) \ : \ g(x,y)\Big)\,.$$

Extend to a map of $\mathbb{P}^2(\mathbb{C})$ of the form

$$(x : y : z) \mapsto \Big(f(x,y) \ : \ g(x,y) : \ \epsilon z^2\Big)\,.$$

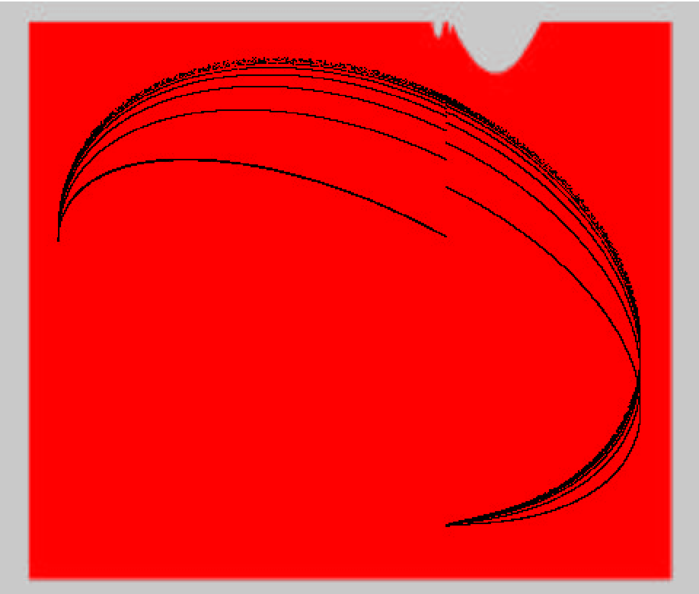
If ϵ is small, then the locus $\{z = 0\}$ will be a trapped attractor, with some trapping neighborhood N. Now perturb, so that

$$(x : y : z) \mapsto \Big(f(x,y) \ : \ g(x,y) \ : \ \epsilon z^2 + h(x,y)\Big)\,.$$

If h is small, then $A = \bigcap f^{\circ k}(N)$ will still be an attractor.

For generic h , this new attractor will be fractal.

The real slice of this fractal attractor in $\mathbb{P}^2(\mathbb{C})$



Maps with an Invariant Curve

The rest of this talk will be a report on the paper

"Elliptic Curves as Attractors in $\mathbb{P}^2$"

Bonifant, Dabija & Milnor, Experimental Math., to appear.

(also in Stony Brook IMS Preprint series;

or arXiv:math/0601015)

Invariant curves: the complex case

Let $S = f(S) \subset \mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ be an f-invariant Riemann surface. Two cases will be of special interest:

Case 1. S is conformally isomorphic to the annulus

$$\mathbb{A} = \{z \in \mathbb{C} : 1 < |z| < \text{constant}\}.$$

Then S will be called a **Herman ring**.

In this case, the restriction map $f|_S$ from S to itself necessarily corresponds to an irrational rotation

$$z \mapsto e^{2\pi i \rho} z .$$

Case 2. S is conformally isomorphic to a torus

$$\mathbb{T} = \mathbb{C}/\Lambda, \qquad \text{with} \quad \Lambda \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Then S will be called an **elliptic curve.**

In this case (choosing base point correctly), $f|_S$ corresponds to an expanding linear map $z \mapsto \alpha z \pmod{\Lambda}$.
Here the **multiplier** α is an algebraic integer with $\alpha\Lambda \subset \Lambda$.
and with $|\alpha|^2 = d > 1$.

Two Theorems

Let $f : \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ be an everywhere defined rational map with an f-invariant elliptic curve.

THEOREM 1. A complex elliptic curve $\mathcal{E}$ can never be a **trapped attractor**.

THEOREM 2. However, there exist many examples with an invariant elliptic curve as **measure attractor**.

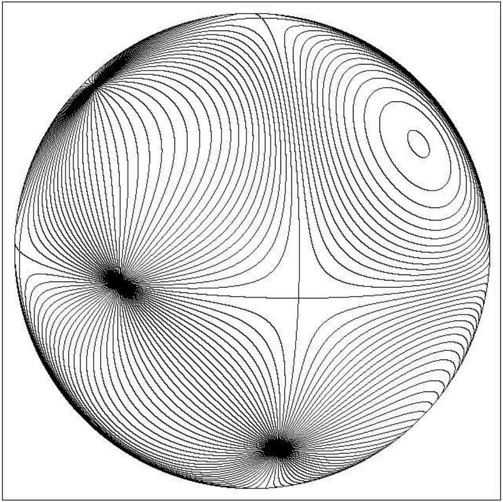
To construct examples, I will start more than 160 years ago.

O. Hesse [1844]:

described a (singular) foliation of $\mathbb{P}^2$ by curves

$$\left(x^3 + y^3 + z^3 \;:\; xyz\right) = \text{constant}$$

which includes a representative for every elliptic curve.



A. Desboves [1886]:

constructed a 4-th degree rational map

$$f_0(x:y:z) \;=\; \Big(x(y^3-z^3)\;:\;y(z^3-x^3)\;:\;z(x^3-y^3)\Big)$$

which carries each elliptic curve in the Hesse foliation into itself, with multiplier $\alpha=-2$. This map is not everywhere defined, and is not very interesting dynamically.

Bonifant and Dabija [2002] embedded f_0 in a family:

$$f_{a,b,c}(x:y:z) \;=\; \Big(x(y^3-z^3+a\Phi)\;:\;y(z^3-x^3+b\Phi)\;:\;z(x^3-y^3+c\Phi)\Big)$$

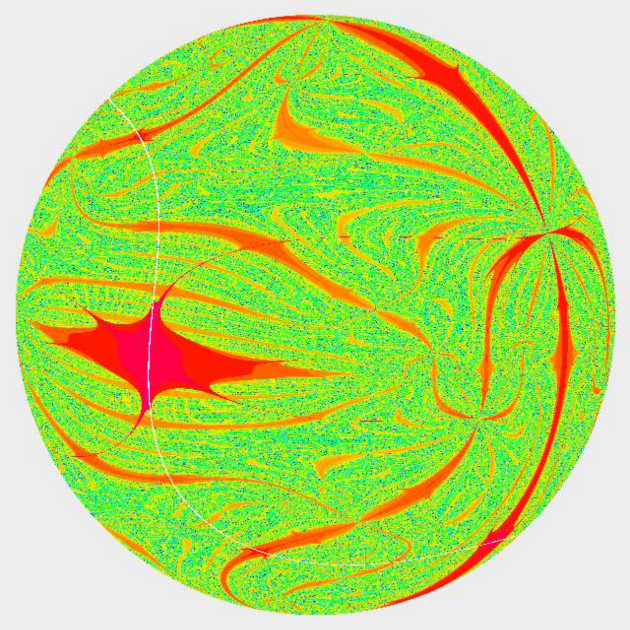
$$\text{where} \qquad \Phi \;=\; \Phi(x,y,z) \;=\; x^3+y^3+z^3\,.$$

Note that the Fermat curve $\mathcal{F}$, defined by the equation

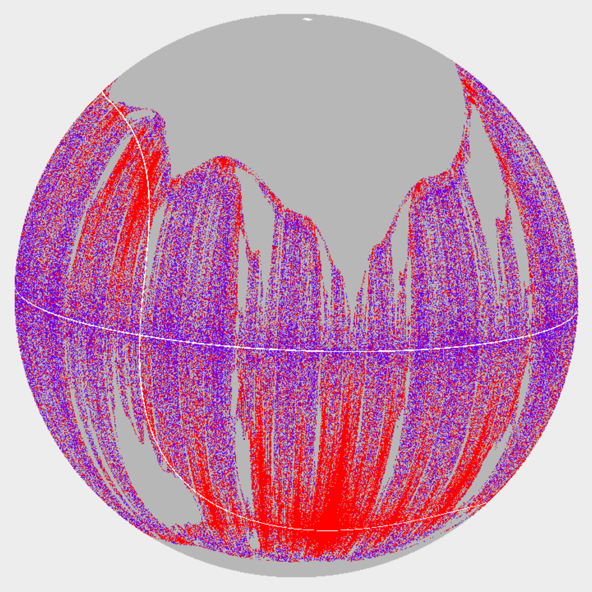
$$\Phi(x,y,z)=0\,,$$

is invariant under $f_{a,b,c}$ for every choice of a, b, c.

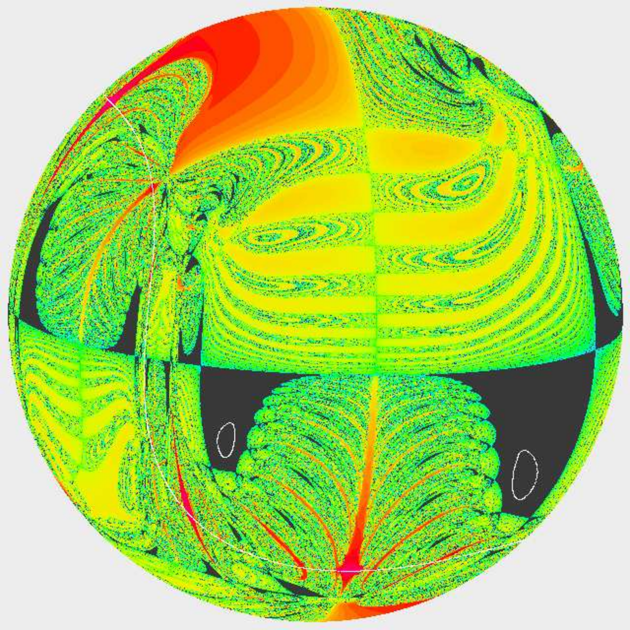
Example with $a = 1/3, \;\; b = 0, \;\; c = -1/3$



Example with $a = -1, \ b = 1/3, \ c = 1$



Example with $a = -1/5, \; b = 7/15, \; c = 17/15$



From Circle in $\mathbb{P}^2(\mathbb{R})$ to Ring in $\mathbb{P}^2(\mathbb{C})$

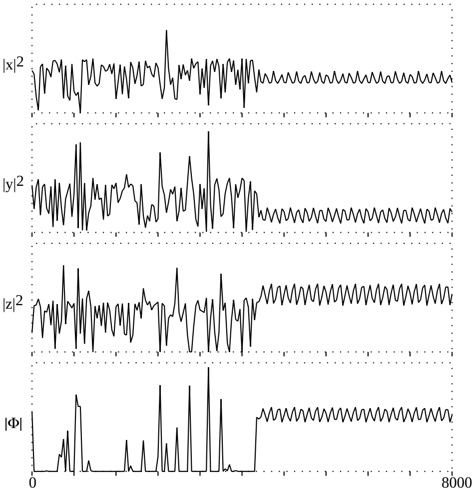
We conjecture that the two small white circles in the last figure form the real part of a cycle of two Herman rings in $\mathbb{P}^2(\mathbb{C})$.

Arguments of Herman and Yoccoz imply the following:

THEOREM. If $\Gamma \subset \mathbb{P}^2(\mathbb{R})$ is an f-invariant real analytic circle with Diophantine rotation number, then Γ is contained in an f-invariant Herman ring $S \subset \mathbb{P}^2(\mathbb{C})$.

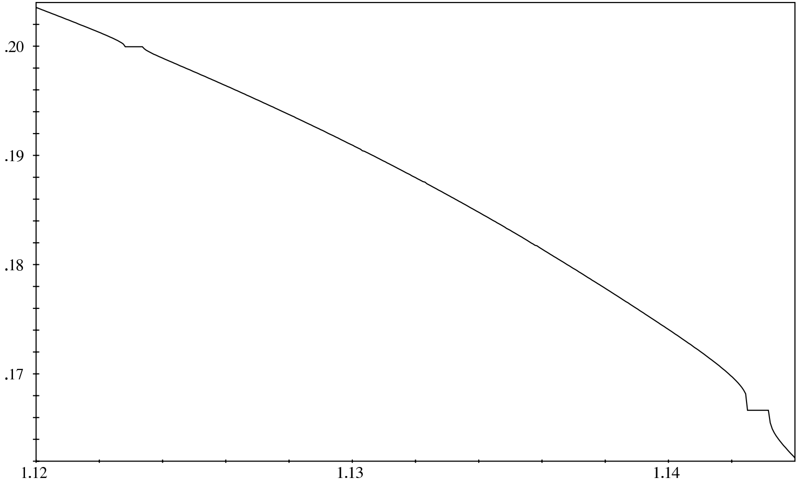
Furthermore, if Γ is a trapped attractor in $\mathbb{P}^2(\mathbb{R})$, then a neighborhood of Γ in S is "locally attracting" in $\mathbb{P}^2(\mathbb{C})$.

Studying a Typical Orbit in $\mathbb{P}^2(\mathbb{C})$



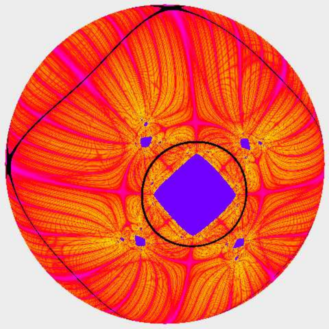
The plot suggests that this particular orbit converges to a Herman ring after some 4000 iterations.

The rotation number as a function of c.



This suggests that invariant circles, with varying rotation number, persist for a substantial region in real parameter space.

The Cassini Quartic: A singular real example.



The real slice of an immersed torus in $\mathbb{P}^2(\mathbb{C})$ with equation

$$x^2y^2 - (x^2+y^2)z^2 + kz^4 = 0 \qquad \text{where} \qquad k = 1/8\,.$$

The outer black curve is a trapped attractor under a 4-th degree rational map.

The Transverse Lyapunov Exponent

Let $S = f(S) \subset \mathbb{P}^2(\mathbb{C})$ be an f-invariant Riemann surface.

Form the “normal” complex line bundle with fibers

$$T_p(\mathbb{P}^2)/T_p(S) \qquad \text{for} \qquad p \in S\,.$$

Then f induces a holomorphic bundle map f_* from this line bundle to itself.

Let μ be an ergodic f-invariant probability measure on S.
Choose a norm on the line bundle, and hence on f_*.

DEFINITION. The average

$$\mathrm{Lyap}(f,\,\mu) \;=\; \int_S \log \|f_*\|\, d\mu$$

is called the **transverse Lyapunov exponent**.

If $\mathrm{Lyap} < 0$, then we can expect attraction !
While if $\mathrm{Lyap} > 0$, we expect repulsion ??

The Siegel disk case

First suppose that S is conformally isomorphic to the unit disk $\mathbb{D}$, and that $f|_S$ corresponds to an irrational rotation

$$z \mapsto e^{2\pi i\rho} z\,.$$

Then each concentric circle $|z| = r > 0$ has a unique ergodic probability measure μ_r.

THEOREM (Jensen [1899]). If we use $\log(r)$ as variable, then the function

$$\log(r) \mapsto \mathrm{Lyap}(f, \mu_r)$$

is convex and piecewise linear. Its slope $d\,\mathrm{Lyap}(f, \mu_r)/d\log(r)$ is equal to the number of zeros of f_* in the disk $\{z : |z| < r\}$.

(Jensen considered a holomorphic map $g : \mathbb{D} \to \mathbb{C}$ and studied the average of $\log|g|$ on a family of concentric circles. But the proof is much the same.)

The Herman ring case

is completely analogous; but with an extra free constant.

If S is isomorphic to $\mathbb{A} = \{z \ : \ 1 < |z| < \mathrm{constant}\}$, then the map $\log r \mapsto \mathrm{Lyap}(f, \mu_r)$ is convex and piecewise linear. The derivative $d\,\mathrm{Lyap}(f, \mu_r)/d \log r$ is equal to the number of zeros of f_* with $1 < |z| < r$ plus a constant.

If there is a non-empty subset of S where $\mathrm{Lyap} < 0$, then it will be called the **attracting region** in S.

In both the Siegel and Hermann ring cases, this attracting region is connected, and nearby orbits are attracted to it.

But there is no trapping neighborhood!

The attracting region can evaporate under the smallest perturbation.

Elliptic Curves

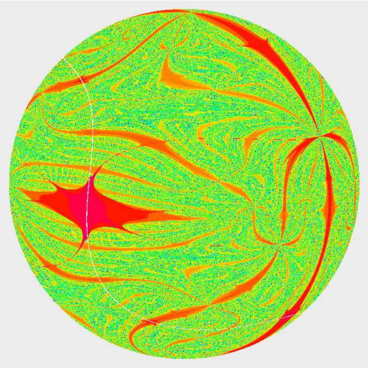
An f-invariant elliptic curve $\mathcal{E}$ has a canonical smooth ergodic measure. Hence it has a uniquely defined transverse Lyapunov exponent $\mathrm{Lyap}(f, \mathcal{E}) \in \mathbb{R}$.

This transverse exponent can be effectively computed, using the theory of elliptic functions.

For specified $\mathcal{E}$, it is only necessary to know the $d^2 + d + 1$ zeros of f_* on $\mathcal{E}$, and to know the norm $\|f_*(p)\|$ at one fixed point which is not a zero of f_*.

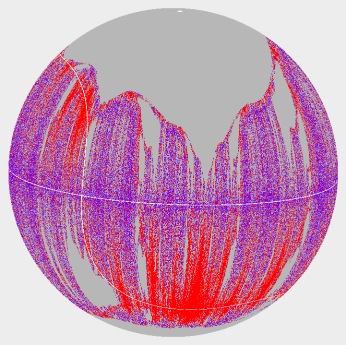
Whenever $\mathrm{Lyap}(f, \mathcal{E}) < 0$, it follows that $\mathcal{E}$ is a measure attractor.

First Example (with $\mathcal{E}$ equal to the Fermat curve $\mathcal{F}$).



$$\mathrm{Lyap}_{\mathbb{R}} = -1.456\cdots, \qquad \mathrm{Lyap}_{\mathbb{C}} = -.549\cdots.$$

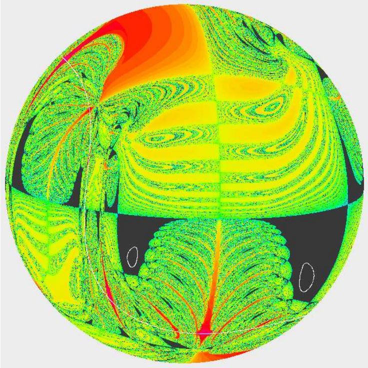
Second Example (with $\mathcal{E} = \mathcal{F}$).



$$\mathrm{Lyap}_{\mathbb{R}}(\mathcal{F}) = -.5700\cdots, \qquad \mathrm{Lyap}_{\mathbb{C}}(\mathcal{F}) = -.1315\cdots,$$

$$\mathrm{Lyap}_{\mathbb{R}}(\mathbb{P}^1) = -.0251\cdots, \qquad \mathrm{Lyap}_{\mathbb{C}}(\mathbb{P}^1) = -.0092\cdots.$$

Third Example (with $\mathcal{E} = \mathcal{F}$).



$$\mathrm{Lyap}_{\mathbb{R}}(\mathcal{F}) = -.509\cdots, \qquad \mathrm{Lyap}_{\mathbb{C}}(\mathcal{F}) = +.402\cdots.$$

THE END

HAPPY BIRTHDAY BILL !