DIFFERENTIAL TOPOLOGY 46 YEARS LATER

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1. Introduction

In the 1965 Hedrick Lectures,[2] I described the state of Differential Topology, a field which was then young but growing very rapidly. During the intervening years, many problems in differential and geometric topology which had seemed totally impossible have been solved, often using drastically new tools. The following is a brief survey, describing some of the highlights of these many developments.

2. Major Developments

The first big breakthrough, by [Kirby-Siebenmann, 1969, 1977], was an obstruction theory for the problem of triangulating a given topological manifold as a PL (= Piecewise-Linear) manifold. If B_{Top} and B_{PL} are the stable classifying spaces (as described in the lectures), they showed that the relative homotopy group $\pi_j(B_{\mathrm{Top}},\, B_{\mathrm{PL}})$ is cyclic of order two for $j = 4$, and zero otherwise. Given an n-dimensional topological manifold M^n, it follows that there is an obstruction $\mathbf{o} \in H^4(M^n; \mathbb{Z}/2)$ to triangulating M^n as a PL-manifold. In dimensions $n \geq 5$ this is the only obstruction. Given such a triangulation, there is a similar obstruction in $H^3(M^n; \mathbb{Z}/2)$ to its uniqueness up to a PL-isomorphism which is topologically isotopic to the identity. In particular, they proved the following.

Theorem 2.1. *If a topological manifold M^n without boundary satisfies*

$$H^3(M^n; \mathbb{Z}/2) = H^4(M^n; \mathbb{Z}/2) = 0 \quad \textit{with} \quad n \geq 5\,,$$

then it possesses a PL-manifold structure which is unique up to PL-isomorphism.

(For manifolds with boundary one needs $n > 5$.) The corresponding theorem for all manifolds of dimension $n \leq 3$ had been proved much earlier by [Moise, 1952]. However, we will see that the corresponding statement in dimension 4 is false.

An analogous obstruction theory for the problem of passing from a PL-structure to a smooth structure had previously been introduced by [Munkres 1960, 1964a, 1964b] and [Hirsch, 1963]. (See also [Hirsch-Mazur, 1974].) Furthermore Cerf had filled in a crucial step by proving that the space of orientation preserving diffeomorphisms of the 3-sphere is connected. (See the Cartan Seminar Lectures of 1962/63, as well as [Cerf, 1968].) Combined with other known results, this led to the following.

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[2] These lectures have recently been digitized by MSRI. For a temporary version, see http://www.math.sunysb.edu/Videos/jack/Differential_Topology . Thanks to Dusa McDuff for unearthing the original tapes. (With regard to Wilder's introduction, compare [Milnor, 1999].)

Theorem 2.2. *Every PL-manifold of dimension $n \le 7$ possesses a compatible differentiable structure; and this structure is unique up to diffeomorphism whenever $n < 7$.*

For further details see §4.

The next big breakthrough was the classification of simply-connected closed topological 4-manifolds by [Freedman, 1982]. He proved, using wildly non-differentiable methods, that such a manifold is uniquely determined by

(1) the isomorphism class of the symmetric bilinear form $H^2 \otimes H^2 \to H^4 \cong \mathbb{Z}$, where $H^k = H^k(M^4; \mathbb{Z})$, together with
(2) the Kirby-Siebenmann invariant $\mathbf{o} \in H^4(M^4; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

These can be prescribed arbitrarily, except for two restrictions: The bilinear form must have determinant ± 1; and in the "even case" where $x \cup x \equiv 0 \pmod{2\, H^4}$ for every $x \in H^2$, the Kirby-Siebenmann class must be congruent to (1/8)-th of the signature. As an example, the Poincaré Hypothesis for 4-dimensional topological manifolds is an immediate consequence. For if M^4 is a homotopy sphere, then both H^2 and the obstruction class must be zero.

One year later, [Donaldson, 1983] used gauge theoretic methods to show that many of these topological manifolds can not possess any smooth structure (and hence by Theorem 2 cannot be triangulated as PL-manifolds). More explicitly, if M^4 is smooth and simply-connected with positive definite bilinear form, he showed that this form must be diagonalizable. In other words, M^4 must be homeomorphic to a connected sum of copies of the complex projective plane. There are many positive definite bilinear forms with determinant ± 1 (and with signature divisible by 16 in the even case) which are not diagonalizable. (See for example [Milnor-Husemoller, 1973].) Each of these corresponds to a topological manifold M^4 with no smooth structure, but such that $M^4 \times \mathbb{R}$ does have a smooth structure which is unique up to diffeomorphism.

The combination of Freedman's topological results and Donaldson's analytic results quickly led to rather amazing consequences. For example, it followed that there are uncountably many non-isomorphic smooth or PL structures on $\mathbb{R}^4$. (Compare [Gompf, 1993].) All other dimensions are better behaved: For $n > 4$, [Stallings, 1962] showed that the topological space $\mathbb{R}^n$ has a unique PL-structure up to PL-isomorphism. Using the Moise result for $n < 4$ together with the Munkres-Hirsch-Mazur obstruction theory, it follows that the differentiable structure of $\mathbb{R}^n$ is unique up to diffeomorphism for $n \neq 4$.

A satisfactory theory of 3-dimensional manifolds took longer. The first milestone was the Geometrization Conjecture by [Thurston,1982, 1986], which set the goal for what a theory of 3-manifolds should look like. This conjecture was finally verified by [Perelman, 2002, 2003a, 2003b]. (Compare the expositions of [Morgan-Tian 2007], and [Kleiner-Lott, 2008].) The 3-dimensional Poincaré Hypothesis followed as a special case.

3. The Poincaré Hypothesis: three versions

First consider the purely topological version.

Theorem 3.1. *The Topological Poincaré Hypothesis is true in all dimensions.*

That is, every closed topological manifold with the homotopy type of an n-sphere is actually homeomorphic to the n-sphere. For $n > 4$ this was proved by [Newman, 1966] and by [Connell, 1967], both making use of the "engulfing method" of [Stallings, 1960]. For $n = 4$ it is of course due to Freedman. For $n = 3$ it is due to Perelman, using [Moise, 1952] to pass from the topological to the PL case, and then using the Munkres-Hirsch-Mazur obstruction theory to pass from PL to smooth. □

Theorem 3.2. *The Piecewise-Linear Poincaré Hypothesis is true for n-dimensional manifolds except possibly when $n = 4$.*

That is, any closed PL manifold of dimension $n \neq 4$ with the homotopy type of an n-sphere is PL-homeomorphic to the n-sphere. For $n > 4$ this was proved by [Smale, 1962]; while for $n = 3$ it follows from Perelman's work, together with the Munkres-Hirsch-Mazur obstruction theory. □

The Differentiable Poincaré Hypothesis is more complicated, being true in some dimensions and false in others, while remaining totally mysterious in dimension 4. We can formulate the question more precisely by noting that the set of all oriented diffeomorphism classes of closed smooth homotopy (or topological) n-spheres forms a commutative monoid $\mathscr{S}_n$ under the connected sum operation. In fact this monoid is actually a finite abelian group except possibly when $n = 4$. Much of the following outline is based on [Kervaire-Milnor, 1963], which showed in principle how to compute these groups[3] in terms of the stable homotopy groups of spheres for $n > 4$. Unfortunately, many proofs were put off to Part II of this paper, which was never completed. However, the missing arguments have been supplied elsewhere; see especially [Levine, 1985].

Using Perelman's result for $n = 3$, the group $\mathscr{S}_n$ can be described as follows for small n. (Here for example $2 \cdot 8$ stands for the group $\mathbb{Z}/2 \oplus \mathbb{Z}/8$.)

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\mathscr{S}_n$	1	1	1	?	1	1	28	2	$2{\cdot}4$	6	992	1	3	2	$2{\cdot}8128$	2	$2{\cdot}8$	$2{\cdot}8$

Thus the Differentiable Poincaré Hypothesis is true in dimensions 1, 2, 3, 5, 6, and 12, but unknown in dimension 4. Conjecturally it is false in all other dimensions:

Conjecture 3.3. *The group $\mathscr{S}_n$ is non-trivial for all $n > 6$, $n \neq 12$.*

(Any precise computation for large n is impossible at the present time, since not enough is known about the stable homotopy groups of spheres. However, it seems likely that enough is known to prove this conjecture.)

Denote the stable homotopy groups of spheres by

$$\Pi_n = \pi_{n+q}(S^q) \qquad \text{for} \quad q > n+1\,,$$

[3] The Kervaire-Milnor paper worked rather with the group Θ_n of homotopy spheres up to h-cobordism. This makes a difference only for $n = 4$, since it is follows from the h-cobordism theorem of [Smale, 1962] that $\mathscr{S}_n \xrightarrow{\cong} \Theta_n$ for $n \neq 4$. However the difference is important in the 4-dimensional case, since Θ_4 is trivial but the semigroup $\mathscr{S}_4$ is completely mysterious.

and let $J_n \subset \Pi_n$ be the image of the stable Whitehead homorphism $J : \pi_n(\mathbf{SO}) \to \Pi_n$. (See [Whitehead, 1942].) This subgroup J_n is cyclic of order[4]

$$|J_n| \;=\; \begin{cases} \text{denominator}\left(\frac{B_k}{4k}\right) & \text{for } n = 4k-1\,, \\ 2 & \text{for } n \equiv 0,\,1 \pmod 8\,, \text{and} \\ 1 & \text{for } n \equiv 2,\,4,\,5,\,6 (\text{mod } 8)\,, \end{cases}$$

where the B_k are Bernoulli numbers; for example:

$$B_1 = \frac{1}{6},\; B_2 = \frac{1}{30},\; B_3 = \frac{1}{42},\; B_4 = \frac{1}{30},\; B_5 = \frac{5}{66},\; B_6 = \frac{691}{2730}\,.$$

(Compare [Milnor-Stasheff, 1974, Appendix B].)

According to Pontrjagin and Thom, the stable n-stem Π_n can also be described as the group of all framed cobordism classes of framed manifolds. (Here one considers manifolds smoothly embedded in a high dimensional Euclidean space, and a *framing* means a choice of trivialization for the normal bundle.) Every homotopy sphere is stably parallelizable, and hence possesses such a framing. If we change the framing, then the corresponding class in Π_n will be changed by an element of the subgroup J_n. Thus there is an exact sequence

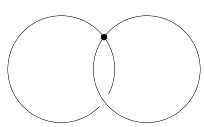
$$0 \;\to\; \mathscr{S}_n^{\mathbf{bp}} \;\to\; \mathscr{S}_n \;\to\; \Pi_n/J_n\,, \tag{1}$$

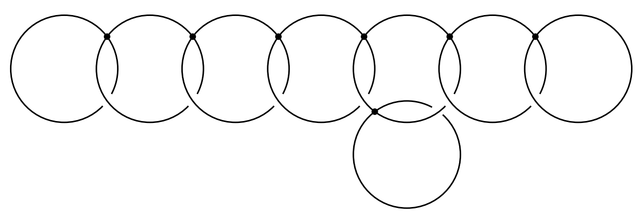
where $\mathscr{S}_n^{\mathbf{bp}} \subset \mathscr{S}_n$ stands for the subgroup represented by homotopy spheres which bound parallelizable manifolds. This subgroup is the part of $\mathscr{S}_n$ which is best understood. It can be partially described as follows.

Theorem 3.4. *For $n \neq 4$ the group $\mathscr{S}_n^{\mathbf{bp}}$ is finite cyclic with an explicitly known generator. In fact this group is:*

- *trivial when n is even,*
- *either trivial or cyclic of order two when $n = 4k-3$, and*
- *cyclic of order* $2^{2k-2}(2^{2k-1}-1)\,\text{numerator}\left(\frac{4B_k}{k}\right)$ *when $n = 4k-1 > 3$.*

(This last number depends on the computation of $|J_{4k-1}|$ as described above.) In the odd cases, setting $n = 2q-1$, an explicit generator for the $\mathscr{S}_{2q-1}^{\mathbf{bp}}$ can be constructed using one basic building block, namely the tangent disk-bundle of the q-sphere, together with one of the following two diagrams.





Here each circle represents one of our $2q$-dimensional building blocks, which is a $2q$-dimensional parallelizable manifold with boundary, and each dot represents a plumbing construction in which two of these manifolds are pasted across each other

[4]This computation of $|J_{4k-1}|$ is a special case of the ***Adams Conjecture*** ([Adams, 1963, 1965]). The proof was completed by [Mahowald, 1970]; and the full Adams Conjecture was proved by [Quillen, 1971], [Sullivan, 1974], and by [Becker-Gottlieb, 1975]. Adams also showed that J_n is always a direct summand of Π_n.

so that their central q-spheres intersect transversally with intersection number $+1$. The result will be a smooth parallelizable manifold with corners. After smoothing these corners we obtain a smooth manifold X^{2q} with smooth boundary.

For q odd, use the left diagram, and for q even use the right diagram. In either case, if $q \neq 2$, the resulting smooth boundary ∂X^{2q} will be a homotopy sphere representing the required generator of $\mathscr{S}^{\mathbf{bp}}_{2q-1}$. (The case $q = 2$ is exceptional since ∂X^4 has only the homology of the 3-sphere. In all other cases where $\mathscr{S}^{\mathbf{bp}}_{2q-1}$ is trivial, the boundary will be diffeomorphic to the standard $(2q-1)$-sphere.)

The exact sequence (1) can be complemented by the following information.

Theorem 3.5. *For $n \not\equiv 2 \pmod 4$, $n \neq 4$, every element of Π_n can be represented by a homotopy sphere. Hence the exact sequence* (1) *takes the more precise form*

$$0 \;\to\; \mathscr{S}^{\mathbf{bp}}_n \;\to\; \mathscr{S}_n \;\to\; \Pi_n/J_n \;\to\; 0\,. \tag{2}$$

However, for $n = 4k-2$ it rather extends to an exact sequence

$$0 = \mathscr{S}^{\mathbf{bp}}_{4k-2} \;\to\; \mathscr{S}_{4k-2} \;\to\; \Pi_{4k-2}/J_{4k-2} \;\xrightarrow{\Phi_k}\; \mathbb{Z}/2 \;\to\; \mathscr{S}^{\mathbf{bp}}_{4k-3} \;\to\; 0\,. \tag{3}$$

Furthermore, [Brumfiel, 1968, 1970] showed that the exact sequence (2) is split exact, except possibly in the case where n has the form 2^k-3. (In fact it could fail to split only in the cases $n = 2^k - 3 \geq 125$. See the discussion below.)

The **Kervaire homomomorphism** Φ_k in (3) was introduced in [Kervaire, 1960]. (The image $\Phi_k(\theta) \in \mathbb{Z}/2$ is called the ***Kervaire invariant*** of the homotopy class θ.) Thus there are two possibilities:

- If $\Phi_k = 0$, then $\mathscr{S}^{\mathbf{bp}}_{4k-3} \cong \mathbb{Z}/2$, generated by the manifold ∂X^{4k-2} described above, and every element of Π_{4k-2} can be represented by a homotopy sphere.
- If $\Phi_k \neq 0$, then $\mathscr{S}^{\mathbf{bp}}_{4k-3} = 0$. This means that the boundary of X_{4k-2} is diffeomorphic to the standard S^{4k-3}. We can glue a $4k-2$ ball onto this boundary to obtain a framed $(4k-2)$-manifold which is not framed cobordant to any homotopy sphere. In this case the kernel of Φ_k forms a subgroup of index two in Π_{4k-2}/J_{4k-2} consisting of those framed cobordism classes which can be represented by homotopy spheres.

The question as to just when $\Phi_k = 0$ was the last major unsolved problem in understanding the group of homotopy spheres. It has recently been solved in all but one case:

Theorem 3.6 (**Hill, Hopkins, and Ravenel**). *The Kervaire homorphism Φ_k is non-zero for $k = 1, 2, 4, 8, 16$, and possibly for $k = 32$, but is zero in all other cases.*

In fact [Browder, 1969] showed that Φ_k can be non-zero only if n is a power of two, and [Barratt-Jones-Mahowald, 1984] completed the verification that Φ_k is indeed non-zero for $k = 1, 2, 4, 8, 16$. Finally, [Hill-Hopkins-Ravenel, 2010] have shown shown that $\Phi_k = 0$ whenever $k > 32$. Thus only the case $k = 32$, with $4k - 2 = 126$, remains unsettled. □

In particular, for $n \neq 4, 125, 126$, if the order $|\Pi_n|$ is known, then we can compute the number $|\mathscr{S}_n|$ of exotic n-spheres precisely. In fact, except for a few sparse exceptions, the group $\mathscr{S}_n$ can be described completely whenever the structure of Π_n is known.

4. Further Details

In conclusion, here is an argument that was postponed above.

Outline Proof of Theorem 2.2. It is not difficult to check that the group $\pi_0\big(\mathrm{Diff}^+(S^n)\big)$ consisting of all smooth isotopy classes of orientation preserving diffeomorphisms of the unit n-sphere is abelian. Define Γ_n to be the quotient of $\pi_0\big(\mathrm{Diff}^+(S^{n-1})\big)$ by the subgroup consisting of those isotopy classes which extend over the closed unit n-disk. There is a natural embedding $\Gamma_n \subset \mathscr{S}_n$ which sends each $(f) \in \Gamma_n$ to the "twisted n-sphere" obtained by gluing the boundaries of two n-disks together under f. It followed from [Smale, 1962] that $\Gamma_n = \mathscr{S}_n$ for $n \geq 5$, and from [Smale, 1959] that $\Gamma_3 = 0$. Since it is easy to check that $\Gamma_1 = 0$ and $\Gamma_2 = 0$, we have

$$\Gamma_n \;=\; \mathscr{S}_n \qquad \text{for every} \qquad n \neq 4\,.$$

On the other hand, Cerf proved[5] that $\pi_0\big(\mathrm{Diff}^+(S^3)\big) = 0$ and hence that $\Gamma_4 = 0$ (although $\mathscr{S}_4$ is completely unknown). Using results about $\mathscr{S}_n$ as described above, it follows that $\Gamma_n = 0$ for $n < 7$, and that Γ_n is finite abelian for all n.

The Munkres-Hirsch-Mazur obstructions to the existence of a smooth structure on a given PL-manifold M^n lie in the groups $H^k(M^n;\Gamma_{k-1})$; while obstructions to its uniqueness lie in $H^k(M^n;\,\Gamma_k)$. (Unlike most of the constructions discussed above, this works even in dimension four.) Evidently Theorem 2.2 follows. $\square$

For further historical discussion see [Milnor, 1999, 2007, 2009].

References

Adams, J. F. [1963] *On the groups $J(X)$ I,* Topology **2**: 181–195.

Adams, J. F. [1965] *On the groups $J(X)$ II,* Topology **3**: 137–171.

Antonelli, P. L., Burghelea, D. and Kahn, P. J. [1972] *The non-finite homotopy type of some diffeomorphism groups,* Topology **11**, 1–49.

Barratt, M. G., Jones, J. D. S. and Mahowald, M. E. [1984] *Relations amongst Toda brackets and the Kervaire invariant in dimension* 62*,* J. London Math. Soc. **30** 3: 533–550.

Becker, J. C. and Gottlieb, D. H. [1975] *The transfer map and fiber bundles,* Topology **14**: 1–12.

Browder, W. [1969] *The Kervaire invariant of framed manifolds and its generalization,* Ann. of Math. **90**: 157–186.

Brumfiel, G. [1968] *On the homotopy groups of BPL and PL/O,* Ann. of Math. **88**, 291–311.

Brumfiel, G. [1970] *On the homotopy groups of BPL and PL/O. III,* Michigan Math. J. **17**, 217–224.

Cerf, J. [1968] "Sur les difféomorphismes de la sphère de dimension trois ($\Gamma_4 = 0$)." Lecture Notes in Mathematics, **53** Springer-Verlag, Berlin-New York xii+133 pp.

[5][Hatcher, 1983] later proved the sharper result that the inclusion $\mathrm{SO}(4) \to \mathrm{Diff}^+(S^3)$ is a homotopy equivalence. On the other hand, for $n \geq 7$, [Antonelli, Burghelia and Kahn, 1972] showed that $\mathrm{Diff}^+(S^n)$ does not have the homotopy type of any finite complex. (For earlier results, see [Novikov, 1963].) For $n = 6$ this group is not connected since $\Gamma_7 \neq 0$, but I am not aware of any results about $\mathrm{Diff}^+(S^n)$ for $n = 4$, 5.

Connell, E. H. [1967] *A topological H-cobordism theorem for $n \geq 5$,* Illinois J. Math. **11**, 300–309.

Donaldson, S. K. [1983] *An application of gauge theory to four-dimensional topology,* J. Differential Geom. **18**: 279–315.

Freedman, M. H. [1982] *The topology of four-dimensional manifolds,* J. Differential Geom. **17**: 357–453.

Gompf, R. [1993] *An exotic menagerie,* J. Differential Geom. **37**: 199–223.

Hatcher, A. E. [1983] *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$*,* Ann. of Math. **117**, 553–607.

Hill, M. A., Hopkins, M. J. and Ravenel, D. C. [2010] *On the non-existence of elements of Kervaire invariant one,* arXiv 0908.3724v2 [math.AT] (November).

Hirsch, M. W. [1963] *Obstruction theories for smoothing manifolds and maps,* Bull. Amer. Math. Soc. **69**: 352–356.

Hirsch, M. W. and Mazur, B. [1974] "Smoothings of piecewise linear manifolds." Annals of Mathematics Studies, No. 80. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. ix+134 pp.

Kervaire, M. A. [1960] *A manifold which does not admit any differentiable structure.* Comment. Math. Helv. **34**: 257–270.

Kervaire, M. A. and Milnor, J. [1963] *Groups of homotopy spheres, I,* Ann. of Math. **77**: 504–537.

Kirby, R. C. and Siebenmann, L. C. [1969] *On the triangulation of manifolds and the Hauptvermutung,* Bull. Amer. Math. Soc. **75**: 742–749.

Kirby R. C. and Siebenmann, L. C. [1977] "Foundational essays on topological manifolds, smoothings, and triangulations. With notes by John Milnor and Michael Atiyah." Annals of Mathematics Studies, **88**. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, vii+355 pp.

Kleiner, B. and Lott, J. [2008] *Notes on Perelman's papers,* Geom. Topol. **12**: 2587-2855.

Levine, J. P. [1985] *Lectures on groups of homotopy spheres,* In: "Algebraic and Geometric Topology" 62–95, Lecture Notes in Math. **1126** Springer.

Mahowald, M. [1970] *The order of the image of the J-homomorphism,* Proc. Advanced Study Inst. on Algebraic Topology (Aarhus, 1970), **II**: 376–384 Mat. Inst., Aarhus Univ.

Milnor, J. [1999] *Growing up in the old Fine Hall.* In: "Prospects in Mathematics," 1–11, Amer. Math. Soc., Providence, RI.

Milnor, J. [2007] "Collected Papers III, Differential Topology," Amer. Math. Soc., Providence, RI.

Milnor, J. [2009] "Collected Papers IV, Homotopy, Homology, and Manifolds," Amer. Math. Soc., Providence, RI.

Milnor J. and Husemooler, D. [1973] "Symmetric Bilinear Forms." Ergebnisse der Mathematik und ihrer Grenzgebiete, **73**, Springer-Verlag, viii+147 pp.

Milnor, J. and Stasheff, J. [1974] "Characteristic classes," Annals of Math. Studies, **76**. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, vii+331 pp.

Moise, E. E. [1952] *Affine structures in* 3*-manifolds. V. The triangulation theorem and Hauptvermutung,* Ann. of Math. **56**: 96–114.

Morgan, J. and Tian, G. [2007] "Ricci flow and the Poincaré conjecture," Clay Mathematics Monographs, **3**. Amer. Math. Soc., Providence, RI; Clay Mathematics Institute, Cambridge, MA, xii+521 pp.

Munkres, J. [1960] *Obstructions to the smoothing of piecewise-differentiable homeomorphisms,* Ann. of Math. **72**: 521–554.

Munkres, J. [1964a] *Obstructions to imposing differentiable structures.* Illinois J. Math. **8**, 361–376.

Munkres, J. [1964b] *Obstructions to extending diffeomorphisms.* Proc. Amer. Math. Soc. **15**, 297–299.

Newman, M. H. A. [1966] *The engulfing theorem for topological manifolds,* Ann. of Math. **84**: 555–571.

Novikov, S. P. [1963] *Homotopy properties of the group of diffeomorphisms of the sphere.* (Russian) Dokl. Akad. Nauk SSSR **148**, 32–35.

Perelman, G. [2002] *The entropy formula for the Ricci flow and its geometric applications,* arXiv: math.DG/0211159 .

Perelman, G. [2003a] *Ricci flow with surgery on three-manifolds,* arXiv:math.DG/0303109 .

Perelman, G. [2003b] *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds,* arXiv:math.DG/0307245 .

Quillen, D. [1971] *The Adams conjecture,* Topology **10**: 67–80.

Smale, S. [1959] *Diffeomorphisms of the* 2*-sphere.* Proc. Amer. Math. Soc. **10**, 621-626.

Smale, S. [1962] *On the structure of manifolds,* Amer. J. Math. **84**: 387–399

Stallings, J. [1960] *Polyhedral homotopy-spheres.* Bull. Amer. Math. Soc. **66**: 485–488.

Stallings, J. [1962] *The piecewise-linear structure of Euclidean space,* Proc. Cambridge Philos. Soc. **58**: 481–488.

Sullivan, D. [1974] *Genetics of homotopy theory and the Adams Conjecture,* Annals of Math. **100**: 1–79.

Thurston, W. [1982] *Hyperbolic geometry and* 3*-manifolds,* In:"Low-dimensional topology," (Bangor, 1979), 9–25, London Math. Soc. Lecture Note Ser., **48**, Cambridge Univ. Press, Cambridge-New York.

Thurston, W. [1986] *Hyperbolic structures on* 3*-manifolds. I. Deformation of acylindrical manifolds,* Ann. of Math. (2) **124**: 203–246.

Whitehead, G. W. [1942] *On the homotopy groups of spheres and rotation groups.* Ann. of Math. **43**: 634–640.