§6. Counting Periodic Points.

The numbers of periodic points of various periods form a powerful tool for exploring dynamic complexity; and the many different ways of counting periodic points form a surprisingly rich area of mathematics. (Compare [Fuller], [Atiyah and Bott], [Milnor and Thurston], [Ruelle 1991, 1993], [Boyland 1994], [T. Ueda].) This section will describe three out of the many possible ways of counting.

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§6A. Zeta Functions. For this subsection, $f : X \to X$ can be an arbitrary function. No topology or other structure will be required. Let $\#\mathbf{Fix}(f)$ be the number of fixed points of f, and more generally let $\#\mathbf{Per}_k(f)$ be the number of periodic points of period exactly k. These numbers may be finite of infinite, but we will be primarily interested in the case where

$$0 \;\le\; \#\mathbf{Per}_k(f) \;<\; \infty\;,$$

or equivalently $0 \le \#\mathbf{Fix}(f^{\circ k}) < \infty$, for all k. There are only two basic restrictions on these numbers (compare Problem 6-b):

(1) *The number* $\#\mathbf{Per}_k(f)$ *is always divisible by* k*.*

(2) *We have*

$$\#\mathbf{Fix}(f^{\circ k}) \;=\; \sum_{d|k} \#\mathbf{Per}_d(f)\;,$$

where the summation extends over all divisors $1 \le d \le k$ *of* k*.*

The proofs are obvious. For example $f^{\circ k}(x) = x$ if and only if the point x is periodic with period dividing k.

It is often convenient to put all of these numbers together into one object by considering formal infinite series. For example, if we introduce a formal indeterminate t and work in the ring $[[t]]$ of all *formal power series* $a_0 + a_1 t + a_2 t^2 + \cdots$ with integer coefficients, then we can describe the entire infinite sequence of integers $\#\mathbf{Fix}(f^{\circ k})$ by the single formal series

$$\sum_{k \ge 1} \#\mathbf{Fix}(f^{\circ k})\, t^k \quad \in \quad [[t]]\;. \tag{6 : 1}$$

(The word "formal" means that we do not make any convergence assumption: The numbers $\#\mathbf{Fix}(f^{\circ k})$ are allowed to grow arbitrarily rapidly as $k \to \infty$.) An extremely useful variant of this construction, in a chain of ideas which goes back in one form or another through Artin and Mazur and Weil, to Riemann and Euler, is to work with formal series with rational coefficients, and to consider the *zeta function*

$$Z_f(t) \quad = \quad \exp \sum_{k=1}^{\infty} \#\mathbf{Fix}(f^{\circ k})\, t^k/k \quad \in \quad [[t]]\;. \tag{6 : 2}$$

Here some explanation is needed. The *exponential* of a formal power series

$$S \;=\; s_1 t + s_2 t^2 + s_3 t^3 + \cdots$$

with rational coefficients, and with constant term zero, is simply defined to be the formal power series

$$\exp(S) \quad = \quad 1 \,+\, S \,+\, S^2/2! \,+\, S^3/3! \,+\, \cdots\;,$$

again with rational coefficients, and with constant term $+1$. (We add and multiply such expressions in the obvious way.) Since there are many denominators in this formula, it is surprising that Z_f turns out to be a series with integer coefficients. However, we can establish this in two easy steps, as follows.

Special Case. Consider a set X which consists of a single periodic orbit, of period $p \geq 1$. If k is a multiple of p then every point of X is fixed by $f^{\circ k}$, but if k is not a multiple of p then $f^{\circ k}$ has no fixed points at all. Hence

$$Z_f(t) \quad = \quad \exp\left(p\left(t^p/p + t^{2p}/(2p) + t^{3p}/(3p) + \cdots\right)\right) \quad = \quad \exp\left(t^p + t^{2p}/2 + t^{3p}/3 + \cdots\right) .$$

Recalling the formula

$$\log\left(\frac{1}{1-u}\right) \quad = \quad u + u^2/2 + u^3/3 + \cdots$$

and setting $u = t^p$, we see that

$$Z_f(t) \quad = \quad 1/(1-t^p) \quad = \quad 1 \,+\, t^p \,+\, t^{2p} \,+\, t^{3p} \,+\, \cdots .$$

General Case. Express X as the disjoint union of finite or infinitely many orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots$ of periods $p_1, p_2, \ldots$, together with a complementary subset X' which has no periodic points at all. Then clearly

$$\#\mathbf{Fix}(f^{\circ k}) \quad = \quad \sum_i \#\mathbf{Fix}(f^{\circ k}|\mathcal{O}_i) .$$

Since the exponential function takes sums to products, this proves the following.

Lemma 6.1 *The zeta function can be expressed as a product*

$$Z_f \quad = \quad \prod_{\mathcal{O}} \frac{1}{1-t^{p(\mathcal{O})}} \quad = \quad \prod_{\mathcal{O}} \left(1 + t^{p(\mathcal{O})} + t^{2p(\mathcal{O})} + \cdots\right) .$$

Here $\mathcal{O}$ ranges over all periodic orbits of f and $p(\mathcal{O})$ is the period of $\mathcal{O}$. In particular, Z_f is always a power series with non-negative integer coefficients.

We can recover the more naive power series (6 : 1) from Z_f as follows. It will be convenient to define the "*logarithmic derivative*" of Z_f to be

$$\frac{d\log Z_f(t)}{d\log t} \quad = \quad \frac{t\,Z_f'(t)}{Z_f(t)} .$$

Lemma 6.2. *The logarithmic derivative of this zeta function is given by*

$$t\,Z_f'/Z_f \quad = \quad \sum_{k=1}^{\infty} \#\mathbf{Fix}(f^{\circ k})\,t^k .$$

Proof. By definition

$$\log(Z_f) \quad = \quad \sum_{k\geq 1} \#\mathbf{Fix}(f^{\circ k})\ t^k/k .$$

Differentiating with respect to t, and then multiplying by t, we obtain the required expression. $\square$

One virtue of this formalism is that, in many cases, the zeta function turns out to be a relatively simple function. Here is an example. Let $S_d : [0,1] \to [0,1]$ be the *sawtooth map*, with slope alternately $+d$ and $-d$ on the segments $[i/d\,,\,(i+1)/d]$ of length $1/d$), where $d \ge 2$. (Compare Figure 14, Problem 2-b.) The graph of S_d is made up out of d straight line segments, each of which crosses the diagonal exactly once. Thus $\#\mathbf{Fix}(S_d) = d$. On the other hand,

$$S_d \circ S_m \;=\; S_{dm}\;.$$

It follows easilythat $S_d^{\circ k} = S_{d^k}$, hence $\#\mathbf{Fix}(S_d^{\circ k}) = d^k$. We can now compute

$$Z_{S_d}(t) \;=\; \exp \sum d^k t^k/k \;=\; \exp\left(\log \frac{1}{1-dt}\right) \;=\; \frac{1}{1-dt}\;, \qquad (6:3)$$

or in other words

$$Z_{S_d}(t) \;=\; 1 + dt + (dt)^2 + \cdots\;.$$

Exactly the same formula holds for the degree d Chebyshef map $Ч_d$, which can be defined by the equation $\mathrm{Re}((x+iy)^d) = Ч_d(x)$ when $x^2+y^2=1$, since $Ч_d$ is topologically conjugate to the sawtooth map. (Problems 2-a,b.) It is interesting to note that the fixed point equation $Ч_d^{\circ k}(x) = x$ is a polynomial equation of degree d^k . Since $\#\mathbf{Fix}(Ч_d) = d^k$, we see that all complex roots of this equation are real and distinct, belonging to the interval $[-1\,,\,1]$.

Similarly, for any real or complex polynomial map of degree $d \ge 2$, we have

$$\#\mathbf{Fix}(f^{\circ k}) \;\le\; d^k\;,$$

since a polynomial equation of degree d^k can have at most d^k distinct roots. In the complex case, if none of the equations $f^{\circ k}(z) = z$ have multiple roots, we get the precise formula $Z_f = 1/(1-dt)$.

Although we have introduced the symbol t as a formal indeterminate, sometimes it is useful to think of t as an honest complex variable. Then we can ask whether the power series $Z_f(t)$ converges for small values of $|t|$. As an example, for the power series $1/(1-dt)$ of formula (6 : 3), we evidently have convergence for $|t| < 1/d$.

Definition. Let r be the radius of convergence of the power series Z_f , or equivalently of $t\,Z_f'/Z_f$. (Compare Problem 6-c.) It will be convenient to introduce the number $h_{\rm per} = \log(1/r)$. By a standard result on complex power series, we can write

$$h_{\rm per} \;=\; h_{\rm per}(f) \;\;=\;\; \limsup_{k\to\infty}\, \frac{1}{k}\log\left(\#\mathbf{Fix}(f^{\circ k})\right)\;.$$

We will call $h_{\rm per}$ the "rate of exponential growth" for the numbers of periodic orbits.

If we exclude the case where f has no periodic orbits at all, then evidently $0 \le r \le 1$, or equivalently

$$0 \le h_{\rm per}(f) \le \infty\;.$$

As an example, for the sawtooth map we have $h_{\rm per}(S_d) = \log(d)$, and for any polynomial map of degree $d \ge 2$ we have $h_{\rm per}(f) \le \log(d)$. (In the linear case, we must only exclude polynomial such as $f(x) = \pm x$ with some $f^{\circ k}(x)$ identically equal to x .)

The case $h_{\rm per}(f) > 0$, where at least some of the numbers $\#\mathbf{Fix}(f^{\circ k})$ grow exponentially, always seems to be associated with extremely complicated dynamics; while the case

$h_{\rm per} = 0$ is often associated with less complicated dynamics. (But not always: For example, the Cartesian product of an irrational rotation of the circle with a completely arbitrary map has no periodic orbits at all (Example 4 in §4D). The case $h_{\rm per}(f) = \infty$ corresponds to the possibility that the numbers $\#\mathbf{Fix}(f^{\circ k})$ grow more than exponentially with k, so that the radius of convergence is zero. Following is a basic result.

> **Theorem of Artin and Mazur.** *If M is a compact smooth manifold without boundary, then for a C^r-dense set of maps $f : M \to M$ we have $h_{\rm per}(f) < \infty$.*

In fact we have $h_{\rm per}(f) < \infty$ for many familiar maps, for example for one dimensional polynomial maps, and for Smale's Axiom A maps. In fact it has sometimes been conjectured that $h_{\rm per}(f) < \infty$ for a C^r-generic map. However, this has been disproved in all dimensions ≥ 2 by [Kaloshin]. Thus the case of more than exponential growth actually occurs reasonably often. Kaloshin's paper also contains a proof of the Artin-Mazur result which is much easier than the original proof.

We will mention these numbers $h_{\rm per}(f)$ again in §7.

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§6B. The Lefschetz Fixed Point Index.

Instead of counting the raw number of fixed points, we can count each one with an appropriate weight, called the *Lefschetz index*, in order to obtain a quantity which is more stable under perturbation. To motivate this construction, consider the three graphs in Figure 28. The map on the left has two fixed points. As we deform the map, these come together, and then disappear.[1] Following Lefschetz, we will assign the attracting fixed point on the left the index $L = +1$ and the repelling fixed point next to it the index $L = -1$. When these two points come together (middle picture), we obtain a one-sided attracting fixed point, with index $L = 0$. This fixed point immediately disappears, and we are left (in the right hand picture) with a map having no fixed points at all in this region. Thus the sum of the Lefschetz indices of the fixed points remains constant under such a deformation.

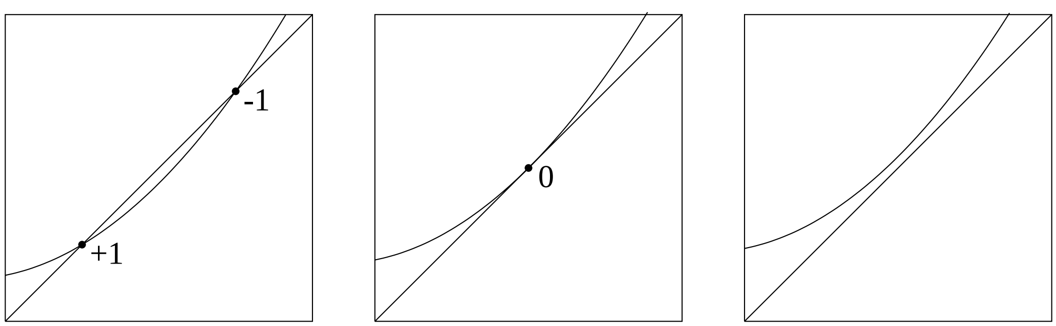

Figure 28. A "saddle-node" bifurcation, showing the Lefschetz indices of the fixed points.

This section will sketch the main ideas of the definition, without giving full details. (For complete treatments, see for example [Dold] or [Brown].) First some generalities about intersection numbers.

[1] This pair creation or annihilation process is called a *saddle-node birfurcation*, in analogy with the corresponding 2-dimensional situation where a "saddle" fixed point, repelling along one axis and attracting along the other, and an attracting or repelling "node" can come together and annihilate each other.

By definition, two bases $(e_1, \ldots, e_m)$ and $(e'_1, \ldots, e'_m)$ for an m-dimensional real vector space determine the *same orientation* or the *opposite orientation* according as the matrix a_{ij} which expresses one basis in terms of the other,

$$e'_j \;=\; \sum_i e_i a_{ij} \;,$$

has positive or negative determinant. Recall that a smooth manifold M^m is *oriented* if we have prescribed a collection of smooth coordinate charts such that, on the overlap between any two such charts, say with coordinates $(u_1, \ldots, u_m)$ and $(v_1, \ldots, v_m)$ respectively, the derivative matrix $[\partial v_i/\partial u_j]$ has positive determinant. A completely equivalent condition would be that we have prescribed a preferred orientation for the tangent space $T_x M^m$ at each point $x \in M^m$ so that this orientation varies continuously with the point.

Now consider two oriented manifolds M^m and N^n which are smoothly embedded as submanifolds of a smooth oriented manifold P^{m+n}. The two are said to intersect *transversally* at a point $x \in M^m \cap N^n$ if the tangent space $T_x P^{m+n}$ is spanned by its two subspaces $T_x M^m$ and $T_x N^n$ The inclusion maps then induce a vector space isomorphism

$$T_x M^m \oplus T_x N^n \;\xrightarrow{\cong}\; T_x P^{m+n} \;.$$

In this case, we can define the *intersection multiplicity*

$$\cap\#(M^m, N^n, x) \;=\; \pm 1$$

to be either $+1$ or -1 according as an positively oriented basis for $T_x M^m$ followed by a positively oriented basis for $T_x N^n$ does or does not yield a positively oriented basis for $T_x P^{m+n}$. If there are only finitely many intersections between M^m and N^n, and if all of them are transverse, then we can define the *intersection number* $M^m \bullet N^n$ to be the sum of intersection multiplicities

$$M^m \bullet N^n \;=\; \sum_x \cap\#(M^m, N^n, x) \qquad \text{where} \quad x \quad \text{ranges over} \quad M^m \cap N^n \;.$$

Note that

$$N^n \bullet M^m \;=\; (-1)^{mn} M^m \bullet N^n \;\in \quad . \tag{6 : 4}$$

Now consider a smooth map $f : M \to M$, where M is a smooth oriented manifold. The *diagonal*

$$\Delta_M \;=\; \{(x, x)\;;\; x \in M\}$$

is a smooth submanifold of $M \times M$, and its image under the map $(x, y) \mapsto (x, f(y))$ from $M \times M$ to itself is another smooth submanifold

$$G_f \;=\; \Big\{\Big(x\,,\, f(x)\Big)\;;\; x \in M\Big\} \;\subset\; M \times M$$

called the *graph* of f.

The given orientation for M determines an orientation for each of these manifolds; where the orientation for $M \times M$ can be described by the requirement that each $M \times \{y\}$ intersects each $\{x\} \times M$ with intersection number $+1$. Note that a point (x, x) of the diagonal also belongs to the graph G_f if and only if $x = f(x)$ is a fixed point of f.

Definition. If the intersection between G_f and Δ_M at (x, x) is transverse, then x

is called a *simple* fixed point of f , and the intersection multiplicity

$$\cap\#\Big(G_f\,,\,\Delta_M\,,\,(x,x)\Big) \;=\; \pm 1$$

is called the *Lefschetz index* $\Lambda(f\,,\,x)$ of the fixed point x . If there are only finitely many such intersections, all transverse, then we define the *Lefschetz number* $\Lambda(f)$ to be the intersection number of G_f and Δ_M , or equivalently the sum of the Lefschetz indices:

$$\Lambda(f) \;=\; G_f \bullet \Delta_M \;=\; \sum_{x=f(x)} \Lambda(f\,,\,x)\;. \tag{6:5}$$

In terms of a smooth local coordinate system near x , we can compute the Lefschetz index as follows. Suppose that the fixed point $x = f(x)$ has coordinates $(u_1\,,\,\ldots\,,\,u_n)$ $= (0\,,\,\ldots\,,\,0)$, and suppose that f induces the correspondence

$$(u_1\,,\,\ldots\,,\,u_m) \;\mapsto\; (u'_1\,,\,\ldots\,,\,u'_m)$$

near the origin. Let $A = [a_{ij}]$ be the $m \times m$ matrix of partial derivatives $a_{ij} = \partial u'_i/\partial u_j$, evaluated at the origin, and let I be the $m \times m$ identity matrix.

Lemma 6.3. *The intersection between G_f and Δ_M is transverse at (x,x) if and only if the matrix $I - A$ is non-singular. If this condition is satisfied, then the index $\Lambda(f\,,\,x)$ is equal to the sign of the determinant of $I - A$.*

Proof. Let $e_1\,,\,\ldots\,,\,e_m$ be the standard basis for T_xM associated with this local coordinate system, so that the derivative map Df_x of f at x carries the vecctor e_j to $e'_j = \sum_i e_i a_{ij}$. Then the vectors

$$e_1 \oplus 0\,,\,\ldots\,,\,e_m \oplus 0\,,\; 0 \oplus e_1\,,\,\ldots\,,\,0 \oplus e_n$$

form an oriented basis for the tangent space $T_{(x,x)}\Big(M \times M\Big) \cong T_xM \oplus T_xM$. Similarly, the vectors $e_j \oplus e_j$ form an oriented basis for the tangent space of Δ_M at (x,x) and the vectors $e_j \oplus e'_j$ form an oriented basis for the tangent space of G_f at (x,x) . Expressing the basis $\Big(\{e_j \oplus e'_j\}\,,\,\{e_j \oplus e_j\}\Big)$ in terms of the basis $\Big(\{e_j \oplus 0\}\,,\,\{0 \oplus e_j\}\Big)$, and then subtracting the last m rows of the resulting matrix from the first m rows, it follows that the Lefschetz index $\Lambda\Big(G_f\,,\,\Delta_M\,,\,(x,x)\Big)$ is equal to the sign of the determinant

$$\det\begin{bmatrix} I & I \\ A & I \end{bmatrix} \;=\; \det\begin{bmatrix} I-A & 0 \\ A & I \end{bmatrix} \;=\; \det\Big(I - A\Big)\;,$$

as asserted. $\Box$

In terms of the eigenvalues $\lambda_1\,,\,\ldots\,,\,\lambda_m$ of the matrix A of derivatives, we can write

$$\det(I - A) \;=\; \prod_j (1 - \lambda_j)\;. \tag{6:6}$$

Corollary 6.4. *The point $x = f(x)$ is a simple fixed point of f if and only if no eigenvalue of the derivative matrix A is equal to $+1$. The Lefschetz index at a simple fixed point can be computed as $(-1)^k$ where k is the number of eigenvalues λ_j which are real, and satisfy $\lambda_j > 1$.*

Proof. We need only look at those eigenvalues which are real, since each pair of conjugate complex eigenvalues makes a contribution

$$(1-\lambda)(1-\overline{\lambda}) \;=\; |1-\lambda|^2 \;>\; 0$$

to the product $(6:6)$. The conclusion follows immediately. $\square$

As examples, in the strongly attracting case where $|\lambda_j|<1$ for all j, it follows that x is a simple fixed point of index $+1$. In the 1-dimensional case, for a repelling fixed point with $|f'(x)|>1$, the index is either -1 or $+1$ according as $f'(x)$ is positive or negative.

More generally, for any *isolated* fixed point $x=f(x)$, we can define an integer valued Lefschetz index. The idea of the construction is suggested by Figure 28, where a non-transverse intersection in the middle picture can be replaced by some collection of nearby transverse intersections by a small perturbation of the smooth map f. We simply define $\Lambda(f,x)$ to be the sum of the Lefschetz indices of the resulting transverse intersections.

Lemma 6.5. *If f is C^1-smooth with an isolated fixed point at x_0, then by an arbitrarily small perturbation of f this fixed point can be replaced by a cluster of nearby simple fixed points. Furthermore, the sum of the Lefschetz indices of these nearby fixed points is independent of the choice of perturbation.*

By definition, this sum is called the Lefschetz index $\Lambda(f\,,\,x_0)$. A proof of 6.5 will be sketched in §6E.

Here is an example. Consider the polynomial map $f(z)=z+z^n$ from to itself. The fixed point equation $f(z)=z$ reduces to $z^n=0$. Thus f has one and only one fixed point, at zero. But if we consider the perturbed map $f_\epsilon(z)=z+z^n-\epsilon$, then we obtain n distinct fixed points at the n solutions to the equation $z^n=\epsilon$. These n points may be either attracting or repelling, but it is not difficult to check that each one has Lefschetz index $+1$. (Compare Problem 6-d.) It follows that $\Lambda(f,0)=n$.

Lefschetz's main result was a computation of $\Lambda(f)$, the sum of the fixed point indices, in terms of homology. Let $H_k(M)=H_k(M\,;\;\;)$ be the k-dimensional homology group of M with rational coefficients. Then any continuous map $f:M\to M'$ induces a linear map of homology groups. It will be convenient to use the somewhat non-standard notation

$$H_k(f):H_k(M)\;\to\;H_k(M')$$

for this linear map. If $M=M'$ is compact, then $H_k(M)$ is a finite dimensional rational vector space, and $H_k(f)$ has a well defined trace, which is always an integer.

Theorem 6.6 (Lefschetz Fixed Point Theorem). *If M is a compact m-dimensional manifold, and if $f:M\to M$ has only finitely many fixed points, then*

$$\Lambda(f) \;=\; \sum_{x=f(x)} \Lambda(f,x) \qquad \textit{is equal to} \qquad \sum_{k=0}^{m}(-1)^k\operatorname{trace}(H_k(f))\;.$$

Thus the Lefschetz number $\Lambda(f)$ can be computed in terms of homology theory. A proof will be outlined in §6E.

It will be convenient to introduce the notation $H_{\rm even}$ for the direct sum $\bigoplus_k H_{2k}$ and $H_{\rm odd}$ for $\bigoplus_k H_{2k+1}$. We can then write simply

$$\Lambda(f) \;=\; {\rm trace}\Big(H_{\rm even}(f)\Big) \;-\; {\rm trace}\Big(H_{\rm odd}(f)\Big)\;. \tag{6 : 7}$$

Corollary 6.7. *If this difference* ${\rm trace}(H_{\rm even}(f))-{\rm trace}(H_{\rm odd}(f))$ *is non-zero, then* f *must have at least one fixed point.*

The proof is clear. □

Corollary 6.8. *If* f *can be continuously deformed to the identity map of* M *, then* $\Lambda(f)$ *is equal to the Euler characteristic*

$$\chi(M) \;=\; {\rm rank}\, H_{\rm even}(M) \;-\; {\rm rank}\, H_{\rm odd}(M)\;.$$

This follows, since each $H_k(f)$ is the identity map of $H_k(M)$, so that its trace is equal to the k-th Betti number ${\rm rank}\, H_k(M)$. □

As an example, for the sphere S^m we have $H_0(S^m)\cong H_m(S^m)\cong$, while the other homology groups are zero. Since $H_0(f)$ is the identity map while $H_m(f)$ is multiplication by the ***degree*** $d(f)$, we get the formula

$$\Lambda(f) \;=\; 1 \,+\, (-1)^m d(f)\;.$$

If $f({\bf x})\ne -{\bf x}$ for all ${\bf x}$ then it is easy to check that f is homotopic to the identity map of S^m , hence $\Lambda(f)=\chi(S^m)=1+(-1)^m$. On the other hand, for the antipodal map $f({\bf x})=-{\bf x}$ there are no fixed points, so $\Lambda(f)=0$, and the degree $d(f)$ must equal $(-1)^{m+1}$. (This can be related to the fact that the antipodal map extends to a linear map of the Euclidean space $^{m+1}\supset S^m$ with determinant $(-1)^{m+1}$.)

Here is a special case. A generic polynomial map $p: \;\to\;$ of degree $d>1$ has d simple fixed points, and it is not hard to check that each one has index $+1$. If we extend over the Riemann sphere $\cup\infty\cong S^2$, then there is an additional attracting fixed point at infinity, so that we obtain $\Lambda(p)=1+d$, as expected.

Using this Lefschetz formula (6 : 7), we can also study the growth of the Lefschetz numbers $\Lambda(f^{\circ k})$ as $k\to\infty$. Let $\lambda_1\,,\,\ldots\,,\,\lambda_p$ be the eigenvalues for the linear transformation $H_{\rm even}(f)$, and let $\eta_1\,,\,\ldots\,,\,\eta_q$ be the eigenvalues for $H_{\rm odd}(f)$.

Corollary 6.9. *The Lefschetz number of* $f^{\circ k}$ *is given by*

$$\Lambda(f^{\circ k}) \;=\; (\lambda_1^k + \cdots + \lambda_p^k) \;-\; (\eta_1^k + \cdots + \eta_q^k)\;.$$

Again the proof is straightforward. □

As an application, if $|\lambda_1|>1$, and if $|\lambda_1|>|\eta_j|$ for all j so that there can be no cancellation, then it follows that the Lefschetz numbers of $f^{\circ k}$ grow exponentially as k tends to infinity. However, in other cases there can definitely be cancellation. As an example, consider the cartesian product map $f\times r: M\times S^1\to M\times S^1$, where r is an irrational rotation of the circle. Then $f\times r$ has no periodic points, even though the trace of $H_{\rm even}\Big((f\times r)^{\circ k}\Big)$ may grow very rapidly as $k\to\infty$.

For more discussion of the topology associated with the Lefschetz formula, see §6E.

§6C. How Periodic Orbits Bifurcate. This subsection, and also the next one, will be based on [Chow, Mallet-Parret and Yorke]. However, let me begin with the following question, asked by Shub and Sullivan. Consider a map $f : M \to M$. Suppose that we can compute the Lefschetz number for every iterate $f^{\circ k}$, and find that the sequence of numbers $|\Lambda(f^{\circ k})|$ is unbounded as $k \to \infty$. Can we conclude that f has infinitely many periodic points? If we do not assume any differentiability, then Shub and Sullivan provide a counterexample as follows. Map the Riemann sphere $\mathbb{C} \cup \infty$ to itself by the continuous function

$$f(z) \;=\; 2z^2/|z| \;, \qquad \text{with} \qquad |f(z)| \;=\; 2|z| \;.$$

This has degree two, so 6.6 implies that $\Lambda(f^{\circ k}) = 1 + 2^k$. These numbers are unbounded, and yet $f^{\circ k}$ has only two fixed points, namely an attracting fixed point at infinity and a repelling fixed point at the origin. (The fixed point indices are $\Lambda(f^{\circ k}, \infty) = 1$ and $\Lambda(f^{\circ k}, 0) = 2^k$.)

In the differentiable case they show that this kind of behavior cannot occur. More precisely, if f is C^1-smooth throughout a neighborhood of $\mathbf{0} \in \mathbb{R}^n$, and if every $f^{\circ k}$ has $\mathbf{0}$ as an isolated fixed point, then they show that the sequence of integers $\Lambda(f^{\circ k}, \mathbf{0})$ is uniformly bounded. Chow, Mallet-Paret and Yorke sharpen this statement substantially by showing that this sequence of numbers $L_k = \Lambda(f^{\circ k}, \mathbf{0})$ is strictly periodic. That is, they specify an integer $p \ge 1$, which is completely determined by the eigenvalues of the first derivative map $Df_{\mathbf{0}}$, so that

$$L_k \;=\; L_{k+p} \qquad \text{for every} \qquad k \ge 1 \;.$$

Further, they show that the average $(L_1 + \cdots + L_p)/p$ is always an integer. (See 6.15 below.) Their proof is based on a careful study of what happens to a periodic orbit of f under a small perturbation of f.

To begin the analysis, suppose that f is C^1-smooth throughout a neighborhood of $\mathbf{0} \in \mathbb{R}^n$. The first derivative of f at $\mathbf{0}$ will be thought of as a linear transformation $Df_{\mathbf{0}} : \mathbb{R}^n \to \mathbb{R}^n$.

Definition. An integer $p \ge 1$ will be called a *virtual period* for $(f, \mathbf{0})$ if there is some vector $\mathbf{v} \in \mathbb{R}^n$ which is periodic, with period exactly p, under this linear mapping $Df_{\mathbf{0}} : \mathbb{R}^n \to \mathbb{R}^n$.

These virtual periods can easily be computed as follows.

Lemma 6.10. *A number $k \ge 1$ is a virtual period for $(f, \mathbf{0})$ if and only either:*

(*a*) $k = 1$ *,*

(*b*) $Df_{\mathbf{0}}$ *has a primitive k-th root of unity as eigenvector, or*

(*c*) k *can be expressed as a least common multiple of such numbers.*

In particular, it follows that there is a largest virtual period, and that every other virtual period is a divisor of it.

The proof is not difficult, and will be left to the reader. □

The following Lemma helps to justify this term 'virtual period'. Let X be a compact region in $\mathbb{R}^n$, and let $C^1(X, \mathbb{R}^n)$ be the space of all C^1-maps from X to $\mathbb{R}^n$, with the C^1-topology. We suppose that $f \in C^1(X, \mathbb{R}^n)$, and that $f(\mathbf{0}) = \mathbf{0}$ is an interior point of X.

Virtual Period Lemma 6.11. *An integer $p \geq 1$ is a virtual period for $(f, \mathbf{0})$ if and only if, for every neighborhood $\mathcal{F}$ of f in the space $C^1(X, \mathbb{R}^n)$ and every neighborhood N of $\mathbf{0}$ in X, there exists a map $g \in \mathcal{F}$ having a periodic orbit $\{x_1, \ldots, x_p\}$ which is contained in N and has period exactly p.*

Proof. First suppose that, for every neighborhood $\mathcal{F}$ of f and every neighborhood N of $\mathbf{0}$, there exists a map $g \in \mathcal{F}$ having a periodic orbit $\{x_1, \ldots, x_p\}$ which is contained in N and has period exactly p. Then we must prove that p is a virtual period, ie., that the linear map $Df_{\mathbf{0}} : \mathbb{R}^n \to \mathbb{R}^n$ has a point of period exactly p. Let $q \geq 1$ be the largest divisor of p which is a virtual period. If $q < p$, then we will obtain a contradiction, as follows.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map $Df_{\mathbf{0}}^{\circ q}$, and set $p = qr$ with $r > 1$. Note first that the linear operator $I + A + A^2 + \cdots + A^{r-1}$ on $\mathbb{R}^n$ must be non-singular. For otherwise, if $(I + A + A^2 + \cdots + A^{r-1})\,\mathbf{v} = \mathbf{0}$ with $\mathbf{v} \neq \mathbf{0}$, then it would follow that $(I - A^r)\,\mathbf{v} = \mathbf{0}$, where $A^r = Df^{\circ p}(\mathbf{0})$. Thus the period of $\mathbf{v}$ under $Df_{\mathbf{0}}$ would be a divisor of p, and a virtual period for $(f, \mathbf{0})$, and hence a divisor of q by 6.10(c). Therefore $A\,\mathbf{v} = \mathbf{v}$, hence $(I + A + A^2 + \cdots + A^{r-1})\,\mathbf{v} = r\,\mathbf{v} \neq \mathbf{0}$, which is a contradiction.

Since $I + A + A^2 + \cdots + A^{r-1})$ is non-singular, it follows that we can choose a constant $\eta > 0$ so that

$$\|\,(I + A + A^2 + \cdots + A^{r-1})\,\mathbf{v}\| \;\geq\; 3\,\eta\,\|\mathbf{v}\|$$

for every $\mathbf{v} \in \mathbb{R}^n$. Now choose the neighborhood N of $\mathbf{0}$ so small that, for any $\mathbf{x} \in N$ the linear operator $A_{\mathbf{x}} = Df^{\circ q}(\mathbf{x})$ satisfies a corresponding inequality

$$\|\,(I + A_{\mathbf{x}} + A_{\mathbf{x}}^2 + \cdots + A_{\mathbf{x}}^{r-1})\,\mathbf{v}\| \;\geq\; 2\,\eta\,\|\mathbf{v}\|\;.$$

Then choose the neighborhood $\mathcal{F}$ of f small enough so that, for every $\mathbf{x} \in N$ and every $g \in \mathcal{F}$, the operator $B = Dg^{\circ q}(\mathbf{x})$ satisfies

$$\|\,(I + B + B^2 + \cdots + B^{r-1})\,\mathbf{v}\| \;\geq\; \eta\,\|\mathbf{v}\|\;.$$

The first order Taylor expansion of $g^{\circ q}$ at $\mathbf{x}$ has the form

$$g^{\circ q}(\mathbf{x} + \mathbf{h}) \;=\; (\mathbf{x} + \mathbf{c}) + B\,\mathbf{h} + R(\mathbf{h}) \tag{6 : 8}$$

with remainder term $R(\mathbf{h})$, where by choosing N and $\mathcal{F}$ small enough, we may assume that the ratio $\|R(\mathbf{h})\|/\|\mathbf{h}\|$ is uniformly less than any preassigned constant, for $\mathbf{x}$ and $\mathbf{x} + \mathbf{h}$ in N. In particular, taking $\mathbf{h} = \mathbf{c}$, since $g^{\circ q}(\mathbf{x}) = \mathbf{x} + \mathbf{c}$, we get

$$g^{\circ 2q}(\mathbf{x}) \;=\; \mathbf{x} + (I + B)\,\mathbf{c} + R(\mathbf{c})\;.$$

Now applying (6 : 8) repeatedly to both sides of this last equation, we find by induction that

$$g^{\circ rq}(\mathbf{x}) \;=\; \mathbf{x} + (I + B + \cdots + B^{q-1})\,\mathbf{c} + R_q(\mathbf{c})\;,$$

where again, by choosing N and $\mathcal{F}$ small enough, we may assume that the ratio $\|R_q(\mathbf{c})\|/\|\mathbf{c}\|$ is uniformly less than any preassigned constant. In particular, we may assume that this ratio is strictly less than η. If $g^{\circ q}(\mathbf{x}) \neq \mathbf{x}$, so that $\mathbf{c} \neq \mathbf{0}$, it then follows that $g^{\circ p}(\mathbf{x}) = g^{\circ rq}(\mathbf{x}) \neq \mathbf{x}$. This completes the proof that g has no point of period p in N.

Conversely, suppose that $Df_{\mathbf{0}}$ does have a point of period p in R^n. Let

$$f(\mathbf{x}) \;=\; Df_{\mathbf{0}}(\mathbf{x}) + R(\mathbf{x})$$

be the first order Taylor expansion of f about $\mathbf{0}$. Choose a smooth function $\psi(t)$ which vanishes for $t \le 1$ and takes the constant value $\psi(t) = 1$ for $t \ge 2$. Then the function

$$f_\delta(\mathbf{x}) \;=\; Df_{\mathbf{0}}(\mathbf{x}) + \psi(\|x\|/\delta)\, R(\mathbf{x})$$

converges to f in the C^1-topology as $\delta \to 0$, and has points of period p arbitrarily close to the origin. $\square$

As an immediate corollary, we have the following statement. Choose some large integer k, and suppose that $\mathbf{0}$ is isolated as a fixed point of $f^{\circ m}$ for every $m < k$. Let B_ϵ denote the closed ball of radius ϵ centered at the origin. Then we can choose $\epsilon > 0$ so that, for every $m < k$, the map $f^{\circ m}$ is defined throughout the ball B_ϵ, with no fixed points other than $\mathbf{0}$ in B_ϵ, and with image $f^{\circ m}(B_\epsilon)$ contained in the interior of X.

> **Orbit Fragmentation Lemma 6.12.** *With these hypotheses, there exists a neighborhood $\mathcal{F}$ of f in $C^1(X\,,\ {}^n)$ so that, for every $g \in \mathcal{F}$ and every $m < k$:*
>
> (1) *If g has a period m orbit $\mathcal{O}$ which intersects B_ϵ, then m is a virtual period for $(f\,,\mathbf{0})$, and the entire orbit $\mathcal{O}$ is contained in the interior of B_ϵ.*
>
> (2) *If $g^{\circ m}$ has only finitely many fixed points in B_ϵ, then the sum of the Lefschetz indices $\Lambda(g^{\circ m}\,,\mathbf{x}_i)$ of these fixed points is equal to $\Lambda(f^{\circ m}\,,\mathbf{0})$.*

Proof. Applying 6.11 to each $m < k$ which is not a virtual period, and taking the intersection of the resulting neighborhoods, we find neighborhoods N of $\mathbf{0}$ and $\mathcal{F}$ of f so that no $g \in \mathcal{F}$ has an orbit of period $m < k$ contained in N unless m is a virtual period. Furthermore, we may choose N small enough so that $f^{\circ m}(\bar{N})$ is contained in the interior of B_ϵ for every $m < k$, and then choose $\mathcal{F}$ small enough so that $g^{\circ m}(\bar{N})$ is also contained in the interior of B_ϵ for every $g \in \mathcal{F}$ and every $m < k$. Since $\|f^{\circ m}(\mathbf{x}) - \mathbf{x}\|$ is bounded away from zero for $\mathbf{x} \in B_\epsilon \smallsetminus N$, we may also choose $\mathcal{F}$ small enough so that $g^{\circ m}(\mathbf{x}) \ne \mathbf{x}$ for $\mathbf{x} \in B_\epsilon \smallsetminus N$. It follows that every orbit of g of period $< k$ which intersects B_ϵ must be contained in N, and hence contained in the interior of B_ϵ. This proves the first half of 6.12. The second half then follows easily from 6.5, provided that the neighborhood $\mathcal{F}$ is sufficiently small. $\square$

Remarks. Here k can be arbitrarily large, but if we increase k then we must decrease the size of the neighborhoods B_ϵ and $\mathcal{F}$. As an example, the identity map $f(z) = z$ on the unit disk can be approximated arbitrarily closely by maps $g(z) = e^{2\pi i/p}\, z$ having periodic orbits of high period arbitrarily close to 0. For that matter, even f itself may have periodic orbits of high period which are contained in an arbitrarily small neighborhood of zero. This happens for the map given in polar coordinates by $(r\,,\theta) \mapsto (r\,,\theta + r^2)$, or for a "Cremer polynomial" of the form $f(z) = e^{2\pi i \xi}\, z + z^2$ where ξ is irrational but well approximable by rationals. (See for example [Milnor 1999].)

If g is a generic perturbation of f, then the hypothesis that g has only finitely many orbits of period $< k$ in B_ϵ is always satisfied.

Definition. A periodic orbit $\mathcal{O} = \{x_1\,,\,\ldots\,,\,x_q\}$ of a smooth n-dimensional mapping f is called *hyperbolic* if its multipliers (the eigenvalues $\lambda_1\,,\,\ldots\,,\,\lambda_n$ of the first derivative map $Df^{\circ q}(x_i)$) lie off of the unit circle, $|\lambda_j| \neq 1$. Note that a hyperbolic period q point is necessarily isolated as a fixed point of $f^{\circ q}$.

Theorem 6.13 (Kupka-Smale). *For a generic map g in the complete metric space $C^1(X, \mathbb{R}^n)$, every periodic orbit is hyperbolic, hence there are only finitely many orbits of period $< k$.*

More generally, that same statement holds in $C^r(M, M)$ for any compact smooth manifold and any $r \geq 1$. Proofs may be found for example in [Peixoto] or [Palis and de Melo].

For a periodic orbit which is nearly hyperbolic, we can give a very substantial strengthening of 6.12. Again it suffices to consider the special case of a map which is C^1-smooth throughout a neighborhood of $\mathbf{0} \in \mathbb{R}^n$, with $f(0) = 0$.

Lemma 6.14. *Suppose that the first derivative $Df_{\mathbf{0}}$ is non-singular, with at most one eigenvalue equal to ± 1 , but all other eigenvalues off the unit circle. Then there exists a neighborhood N of f so that any map g which is sufficiently close to f in the C^1-topology can have no periodic orbits of period greater than two which are completely contained in N .*

(The hypothesis that $Df_{\mathbf{0}}$ is non-singular is probably not necessary.)

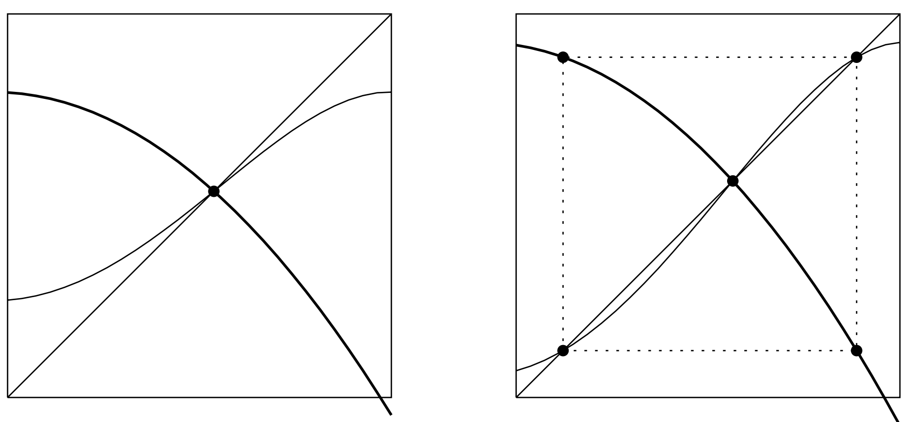

Figure 29. A period doubling bifurcation. On the left, part of the graph of $f(x) = 2.9\,x(1-x)$ and of $f \circ f$, near an attracting fixed point of multiplier $\lambda = -0.9$. On the right, corresponding graphs for $f(x) = 3.1\,x(1-x)$. The fixed point has become repelling, with multiplier -1.1 , and an attracting period two orbit has split off from it.

As an example, this hypothesis on eigenvalues is always satisfied at a non-critical point in the 1-dimensional case. It is also satisfied in the 2-dimensional case if f is strictly *area reducing*, that is if we can choose local coordinate systems so that the Jacobian determinant satisfies $|\det Df(\mathbf{x})| < 1$ everywhere. For then the multipliers for any periodic orbit must satisfy $|\lambda_1 \lambda_2| < 1$.

More precisely, if the exceptional eigenvalue is $+1$, then the proof will show that only fixed points can occur. The standard example of this is the saddle-node bifurcation, as shown in Figure 28. If this eigenvalue is -1 , then there there will always be exactly one fixed point. Any other periodic orbits which appear must have period two. The standard example here

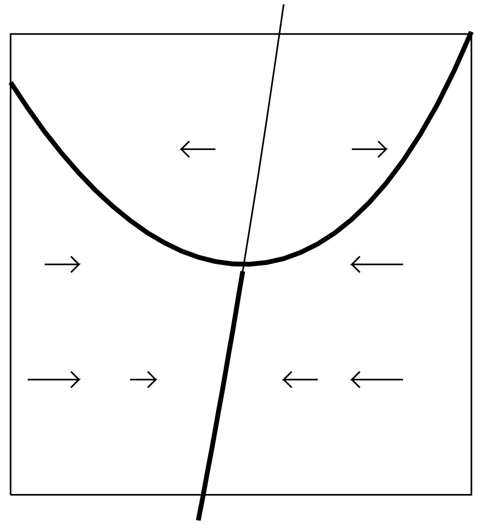

Figure 30. Part of the (x, c)-plane, showing orbits of period one and two for the family of maps $f_c(x) = cx(1-x)$ as a function of the parameter c. As c increases, the attracting fixed point bifurcates into an attracting period two orbit together with a repelling fixed point. (Figure 29.) The arrows show the direction of displacement from x to $f_c \circ f_c(x)$ for fixed c.

is the *period doubling bifurcation*, as illustrated in Figures 29 and 30. Roughly speaking, the saddle-node and period doubling bifurcations are the only ones which can occur under the hypothesis of 6.14.

Proof of 6.14. Replacing f by $f \circ f$ if necessary, let us suppose that the matrix M of first derivatives has the form

$$M \;=\; Df_{\mathbf{0}} \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & S \end{bmatrix} ,$$

and correspondingly write $\mathbf{x} \in \mathbb{R}^n$ as $\mathbf{x} = (x_1, \mathbf{x}_u, \mathbf{x}_s)$, where $\mathbf{x}_u$ is the component in the unstable directions and $\mathbf{x}_s$ is the component in the stable directions, so that (choosing a norm appropriately) we have

$$\|U\mathbf{x}_u\| \;\geq\; \lambda\|\mathbf{x}_u\| , \qquad \|S\mathbf{x}_s\| \;\leq\; \lambda^{-1}\|\mathbf{x}_s\| ,$$

with $\lambda > 1$. (Here $\mathbf{x}$ should be thought of as a column vector, but we write it horizontally for convenience.) Given $\epsilon > 0$, we can choose a neighborhood $\mathcal{F}$ of f and a compact neighborhood N of $\mathbf{0}$ which are small enough so that, whenever $g \in \mathcal{F}$, $\mathbf{x} \in N$ and $\mathbf{x} + \Delta\mathbf{x} \in N$, we have

$$g(\mathbf{x} + \Delta\mathbf{x}) \;=\; g(\mathbf{x}) + M\,\Delta\mathbf{x} + R , \qquad (6:9)$$

with remainder term R satisfying

$$\|R\| \;\leq\; \epsilon\,\|\Delta\mathbf{x}\| .$$

Assume also that f and g restricted to N are diffeomorphisms. Let $K \subset N$ be the compact set consisting of all points whose forward and backward orbits under g are completely contained in N. Evidently any periodic orbit for g in N is contained in K.

Suppose in particular that we choose $\epsilon \ll \lambda - 1$. If both $\mathbf{x}$ and $\mathbf{x}+\Delta\mathbf{x}$ belong to K, then it follows that the unstable component of $\Delta\mathbf{x}$ is small compared with the other two components:

$$\|\Delta\mathbf{x}_u\| \;\le\; (|\Delta x_1| + \|\Delta\mathbf{x}_s\|)/2\ .$$

For otherwise, choosing $\mathbf{x}$ and $\mathbf{x}+\Delta\mathbf{x}$ so that the difference $\|\Delta\mathbf{x}_u\| - (|\Delta x_1| + \|\Delta\mathbf{x}_s\|)/2$ is maximized, a brief computation using (6 : 9) would should that this difference takes a larger value for the pair $g(\mathbf{x})$ and $g(\mathbf{x}+\Delta\mathbf{x})$, which is impossible. Similarly, using g^{-1} in place of g , we see that

$$\|\Delta\mathbf{x}_s\| \;\le\; (|\Delta x_1| + \|\Delta\mathbf{x}_u\|)/2\ .$$

Adding these two inequalities, it follows that

$$\|\Delta\mathbf{x}_u\| + \|\Delta\mathbf{x}_s\| \;\le\; 4\,|\Delta x_1|\ .$$

In particular, this shows that both $\mathbf{x}_u$ and $\mathbf{x}_s$ can be expressed as functions of the single coordinate x_1 , if $\mathbf{x} = (x_1\,,\,\mathbf{x}_u\,,\,\mathbf{x}_s)$ belongs to K . Furthermore, our error estimate for the remainder term R can be replaced by

$$\|R\| \;\le\; 6\,\epsilon\,|\Delta x_1|\ .$$

Thus we can ignore the stable and unstable components and simply write

$$g(\mathbf{x}+\Delta\mathbf{x})_1 \;=\; g(\mathbf{x})_1 + \Delta\mathbf{x}_1 + R_1 \quad \text{with} \quad |R_1| \le 6\,\epsilon\,|\Delta x_1|\ .$$

Taking $6\,\epsilon < 1$, this proves that the initial component of g is a strictly monotone function, for $\mathbf{x}\in K$:

If $\Delta x_1 > 0$ *, with* $\mathbf{x}$ *and* $\mathbf{x}+\Delta\mathbf{x}$ *in* K *, then* $g(\mathbf{x}+\Delta\mathbf{x})_1 > g(\mathbf{x})_1$.

As an immediate consequence, we see that g can not have any periodic orbits of period greater than one in N .

Further details of the proof are straightforward, and will be left to the reader. □

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§**6D. The Orbit Index: Cascades of Bifurcations.** Again consider a map f which is defined and C^1-smooth in a neighborhood of the fixed point $\mathbf{0}\in {}^{n}$. Let $\ell \ge 1$ be the largest virtual period for $(f,\mathbf{0})$ (compare Lemma 6.10), and let $L_k = \Lambda(f^{\circ k}\,,\,\mathbf{0})$ be the Lefschetz index of the k-th iterate of f at the origin.

Periodicity Theorem 6.15. *With these hypotheses, the sequence of integers* $L_k = \Lambda(f^{\circ k}\,,\,\mathbf{0})$ *is strictly periodic, with* $L_k = L_{k+2\ell}$ *for all* $k \ge 1$ *. Furthermore, the average* $\phi(f\,,\,\mathbf{0}) = (L_1 + \cdots + L_{2\ell})/(2\ell)$ *of this periodic sequence of fixed point indices is always an integer.*

Definition. Following Chow, Mallet-Parret and Yorke, this average $\phi(f,\mathbf{0})$ is called the *orbit index* of the periodic orbit $\{\mathbf{0}\}$. More generally, if $\mathcal{O} = \{x_1\,,\,\ldots\,,\,x_q\}$ is an orbit of period q for a C^1-smooth map f , then the integer $\phi(f^{\circ q}\,,\,x_i)$ is called the *orbit index* $\phi(f\,,\,\mathcal{O})$ for the periodic orbit $\mathcal{O}$.

As an example, consider a fixed point $f(0) = 0$ of an interval map. Let $\lambda = f'(0)$ be the multiplier (= first derivative). If $\lambda \ne \pm 1$ then the only virtual period is $\ell = 1$, so the sequence $\{\Lambda(f^{\circ k}\,,\,0)\}$ is periodic of period either 1 or 2. In fact, it follows from (6 : 6) in

§6B that $\Lambda(f^{\circ k}\,,\,0) = \mathrm{sgn}(1-\lambda^k)$, so that the orbit index is given by

$$\phi(f,0) \;=\; \begin{cases} 0 & \text{if} \quad \lambda < -1\,, \\ \Lambda(f,0) = \;\;1 & \text{if} \quad |\lambda| < 1\,, \\ \Lambda(f,0) = -1 & \text{if} \quad \lambda > 1\,. \end{cases} \tag{6 : 10}$$

(Compare 6.15.) This formula may seem strange; however, its utility becomes clear as we look at bifurcations of periodic orbits. One common example is the saddle node bifurcation, as illustrated in Figure 28. For the fixed points in this Figure, the orbit index coincides with the Lefschetz index, hence the sum of orbit indices remains zero as the two fixed points appear or disappear. A more interesting example is the *period doubling bifurcation*, as shown in Figures 29, 30. Before this bifurcation, the only periodic orbit within the illustrated region is an attracting fixed point, with $\lambda = -1+\epsilon$ hence $\phi = 1$. After the bifurcation, this fixed point has become repelling, with $\lambda = -1-\epsilon$, hence $\phi = 0$. However, a new attracting orbit $\mathcal{O}$ of period two has appeared, with $\phi(f\,,\,\mathcal{O}) = 1$. Hence the sum of the orbit indices of the periodic orbits in the Figure remains unchanged.

Unfortunately, this conservation of the total orbit index under bifurcation depends on Lemma 6.14, and hence only works in low dimensional cases. Let us call a period q orbit $f(\mathcal{O}) = \mathcal{O}$ *nearly-hyperbolic* if its multipliers λ_i are non-zero, with at most one multiplier on the unit circle. Recall then that for a small neighborhood N of $\mathcal{O}$ and for any g sufficiently C^1-close to f , all periodic orbits of g within N have period q or (when one eigenvalue is -1) $2q$. For a generic choice of g , it follows from 6.13 that g has only finitely many periodic orbits contained in N .

Local Conservation Law 6.16. *Let $\mathcal{O}$ be a nearly-hyperbolic orbit of period q for the map f , and let N be a small compact neighborhood of $\mathcal{O}$. Suppose that f has no other periodic orbits contained in N . Then if g has only finitely many periodic orbits contained in N , and is sufficiently C^1-close to f , it follows that the sum of the orbit indices for these finitely many orbits of g is equal to the orbit index $\phi(f,\mathcal{O})$.*

Remarks. The condition that $Df^{\circ q}(x_i)$ has at most one eigenvalue on the unit circle is awkward, but essential. As an example, identifying with ^{2} , the map $f(z) = z^2+z$ in two real variables has multipliers $\lambda_1 = \lambda_2 = 1$, and has orbit index $\phi(f,0) = 2$. It can be approximated arbitrarily closely by maps $g(z) = z^2 + e^{2\pi i/p}\,z$ having two nearby fixed points, with orbit indices $\phi(g,0) = 2$ and $\phi(g\,,\,1-e^{2\pi i/p}) = 1$. Thus, in this case, the sum of the orbit indices in a neighborhood is not conserved.

These constructions apply equally well to periodic orbits for a flow, since any smooth flow can be converted to a smooth mapping via the Poincaré first return map. (Compare Figure 1. The multipliers for a periodic orbit of the Poincaré map are also known as *Floquet multipliers* for the associated periodic orbit of the flow. See for example [Guckenheimer and Holmes].) Suppose that $\mathbf{x}(t) = \mathbf{x}(t+c)$ is a periodic solution to a smooth differential equation $d\mathbf{x}/dt = \mathbf{v}(x)$, with period $c > 0$, which is isolated among periodic solutions of period $< k$ for every k . Then there is a well defined orbit index, and the total orbit index in a neighborhood satisfies an analogous Local Conservation Law in the nearly hyperbolic case. As an example, the period doubling bifurcation of Figure 29, with $f'(x) < 0$, can occur as the first return map for a differential equation on the Möbius band. Under smooth perturbation, a periodic orbit of multiplier -1 , and period say c , splits into a repelling orbit of period $\approx c$, together with an attracting orbit of period $\approx 2c$ which wraps around it twice. (For topological reasons, no such bifurcation can occur on an orientable surface.)

To begin the proof of 6.15 and 6.16, if $\mathbf{0}$ is a hyperbolic fixed point of f, then the Lefschetz indices $\Lambda(f^{\circ k}, \mathbf{0})$ can easily be computed as follows. Let $r \ge 0$ be the number of eigenvalues of $Df_{\mathbf{0}}$ which are real, repelling and positive, $\lambda_j > 1$, and let $s \ge 0$ be the number which are real, repelling and negative, $\lambda_j < -1$.

Lemma 6.17. *The Lefschetz index $\Lambda(f^{\circ k}, \mathbf{0})$ is equal to $(-1)^{r+(k-1)s}$. If s is even, this is independent of k, and*

$$\phi(f, \mathbf{0}) \;=\; \Lambda(f, \mathbf{0}) \;=\; (-1)^r \;.$$

On the other hand, if s is odd, then the numbers $\Lambda(f^{\circ k}, \mathbf{0})$ alternate in sign, with average $\phi(f, \mathbf{0}) = 0$.

The proof, based on 6.4, is straightforward and will be left as an exercise. □

Proof of the Periodicity Theorem 6.15. Let ℓ be the largest virtual period for $(f, \mathbf{0})$. We must show that the numbers $L_m = \Lambda(f^{\circ m}, \mathbf{0})$ satisfy $L_m = L_{m+2\ell}$, with an integer as average. Choose $k > m + 2\ell$, choose neighborhoods B_ϵ of $\mathbf{0}$ and $\mathcal{F}$ of f as in 6.12, and choose a hyperbolic map $g \in \mathcal{F}$. Since L_m is equal to the sum of $\Lambda(g^{\circ m}, \mathbf{x}_i)$, where $\mathbf{x}_i$ ranges over all fixed points of $g^{\circ m}|B_\epsilon$, it suffices to prove a corresponding property for g. In fact let $\mathcal{O}$ be any periodic orbit for g in B_ϵ with period $p < k$. Then evidently, for any integer m which is a multiple of p, we have

$$\sum_{\mathbf{x} \in \mathcal{O}} \Lambda(g^{\circ m}, \mathbf{x}) \;=\; \pm p \;.$$

Here the signs either alternate or are fixed, as m ranges over multiples of p. Hence these contributions to the total Lefschetz index for $g^{\circ m}|B_\epsilon$ either have period p with average ± 1, or have period $2p$ with average zero. Since each such p is a divisor of ℓ, the conclusion follows. □

Proof of 6.16. In the almost-hyperbolic case, g has only finitely many orbits in a neighborhood, so all of then are accounted for by the discussion above. The conclusion follows easily. □

A naive reading of this "Local Conservation Law" 6.16 for the orbit index might suggest that the sum

$$\phi(f) \;=\; \sum \phi(f, \mathcal{O}) \;,$$

extended over all periodic orbits of f, should be invariant under deformation of f whenever this sum makes sense, at least in the 1-dimensional case. However this is far from true. (Compare [Yorke and Alligood].) As an example, consider the family of real quadratic mappings. In order to avoid discussion of fixed point indices at an end point of the interval, let us work on the entire real line, with the family of maps

$$f_c(x) \;=\; x^2 - c \qquad (6:11)$$

from to . For $c < -0.25$ we have $f_c(x) > x$, and all orbits diverge to $+\infty$. Hence $\phi(f_c) = 0$. On the other hand, for $c = 2$ the map f_2 is linearly conjugate to the Chebyshef map $q(x) = \text{Ч}_2(x)$, as studied in §2A. There are infinitely many periodic orbits $\mathcal{O}$, all repelling. Of these, infinitely many have multiplier $-2^p < -1$, so that $\phi(\mathcal{O}) = 0$ by formula (6 : 8). These are sometimes called "*flip orbits*". (In higher dimensions, a hyperbolic orbit is called a "flip orbit" if the number of real eigenvalues in the interval $(-\infty\,,\,-1)$ is odd.) However, infinitely many of these orbits are repelling with positive multiplier, so that $\phi(\mathcal{O}) = -1$. Thus the total orbit index $\phi(f_2)$ is equal to $-\infty$.

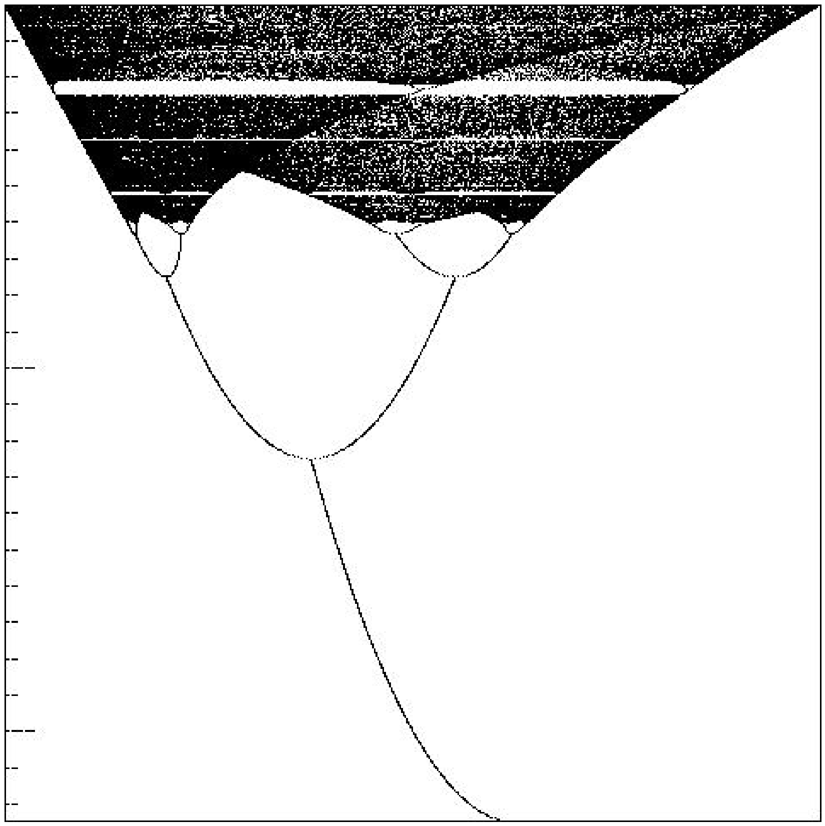

Figure 31. Plot in the (x, c)-plane, showing the generic attractor for the map $x \mapsto x^2 - c$ for each height $c \in [-0.25\,,\,2]$.

How can the total orbit index change under deformation, when it seems to be so tightly controlled by the Conservation Law 6.16? To answer this question, let us look more closely at the family of maps (6 : 11), as shown in Figure 31. As the parameter c increases from $-\infty$, no periodic orbits appear until the saddle-node bifurcation at $c = -0.25$. For $-0.25 \le c \le 0.75$ there is one attracting fixed point with $\phi = +1$, and one positive derivative repelling fixed point with $\phi = -1$. Thus the total orbit index

$$\phi(f_c) \;=\; -1 + 1$$

remains zero. At $c = 0.75$ there is a period doubling bifurcation; thus for $0.75 < c \le 1.25$ the attracting fixed point has been replaced by a flip repelling fixed point with $\phi = 0$ together with an attracting period two orbit with $\phi = +1$. The total index

$$\phi(f_c) \;=\; -1 + 0 + 1$$

is still zero. At $c = 1.25$ there is another period doubling bifurcation, replacing the attracting period two orbit by a flip repelling period two orbit with $\phi = 0$ together with an attracting period four orbit with $\phi = +1$. As c increases, the period of the attracting orbit doubles over and over, but the total orbit index

$$\phi(f_c) \;=\; -1+0+0+\cdots+0+1$$

remains equal to zero, until we get to the limiting value $c_F = 1.4011\cdots$. For this limiting *Feigenbaum map* f_{c_F}, the period of the attracting orbit has diverged to infinity. There no longer is any attracting periodic orbit, and the total orbit index is given by the infinite series

$$\phi(f_{c_F}) \;=\; -1+0+0+0+\;\cdots$$

with sum -1. *Thus we lose control of the the total index as the period of the attracting orbit, which contributes* $+1$ *to this total, tends to infinity.* In this limiting case, it is interesting to note that the finite periodic attractor for our mapping is replaced by a fractal Cantor set attractor.

Now as c increases past c_F, say to $c_F+\epsilon$, the situation becomes much worse. There are still infinitely many flip repelling orbits with $\phi = 0$, and at most one atracting orbit with $\phi = 1$. But infinitely many positive multiplier repelling orbits with $\phi = -1$ have suddenly appeared. *Thus the total orbit index* $\phi(f_c)$ *jumps discontinuously to* $-\infty$.

How can this happen? The answer is that the scenario described above in the interval between $c = -0.25$ and $c = c_F$ is repeated infinitely often in the interval between c_F and $c_F+\epsilon$. A pair of periodic orbits, with orbits indices $+1$ and -1 respectively, are created by a saddle-node bifurcation. (The periods of these orbits are extremely large for c close to c_F.) Then the attracting orbit, with index $+1$, undergoes a period doubling bifurcation repeatedly, and disappears as its period tends to infinity, so that we are left with a net contribution of -1 to the total orbit index.

6E. Further Topological Details. To conclude this section I will give more details about the topology associated with the Lefschetz Fixed Point Theorem. First we must show that the index of an isolated fixed point is well defined. The following is a more precise statement of Lemma 6.5.

Let M be a smooth manifold with metric $\mathbf{d}(x,y)$, and let $f: M\to M$ be a smooth map with isolated fixed point $x_0 = f(x_0)$.

Lemma 6.18. *If* $D\subset M$ *is a small closed disk around* x_0, *then for any* $\epsilon>0$ *there exist maps* $F: D\to M$ *which have only simple fixed points, and which satisfy* $\mathbf{d}(f(x),F(x))<\epsilon$ *for every* $x\in D$. *Furthermore, if* ϵ *is sufficiently small, then the sum*

$$\sum_{y=F(y)\in D}\Lambda(F\,,\,y) \tag{6 : 12}$$

is independent of the choice of such a map F.

Once this lemma has been proved, we can define this expression (6 : 12) to be the Lefschetz index $\Lambda(f,x_0)$ in the case of an isolated but non-simple fixed point.

Outline Proof of 6.18. We will use some ideas from elementary differential topology. See for example [Milnor 1965, §5] for a more detailed presentation of such methods.

Since the statement is purely local, it will suffice to work in local coordinates $\mathbf{u} = (u_1\,,\,\ldots\,,\,u_m)$. That is, we can replace the global map $f: M\to M$ by a map $f(\mathbf{u}) = \mathbf{u}'$

which is defined on some neighborhood U of the origin in $\mathbb{R}^m$ and which takes values in $\mathbb{R}^m$. Similarly, we can work with the Euclidean metric. Let

$$g(\mathbf{u}) \;=\; \mathbf{u} - f(\mathbf{u})\;,$$

so that g is also a smooth map from U to $\mathbb{R}^m$, with $g^{-1}(\mathbf{0})$ equal to the set of fixed points of f. Note that the matrix of first derivatives of g at the origin is just the matrix $I - A$ of 6.3. According to Sard's Theorem, almost every point $\hat{\mathbf{u}} \in \mathbb{R}^m$ is a *regular value* of g. By definition, this means that for every point $\mathbf{u} \in g^{-1}(\hat{\mathbf{u}})$ the first derivative map for g carries the tangent space $T_{\mathbf{u}}M$ onto the tangent space $T_{\hat{\mathbf{u}}}M$.

Now consider the perturbed map $F(\mathbf{u}) = f(\mathbf{u}) + \hat{\mathbf{u}}$, where $\hat{\mathbf{u}}$ is any regular value. Here $\hat{\mathbf{u}}$ can be arbitrarily close to the origin, so F can be arbitrarily close to f. But the fixed points $\mathbf{u} = F(\mathbf{u}) = f(\mathbf{u}) + \hat{\mathbf{u}}$ of the function F all belong to $g^{-1}(\hat{\mathbf{u}})$, and hence are simple.

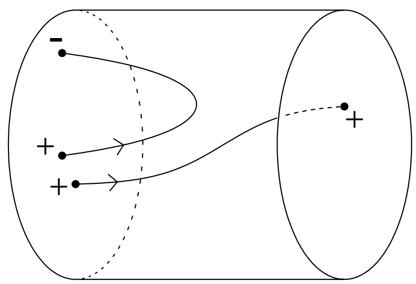

Figure 32. The curves $G^{-1}(\hat{\mathbf{u}}) \subset [0,1] \times D$.

In order to guarantee that the sum (6 : 12) is independent of choices, first choose a small closed disk D about the origin so that $D \subset U$, and so that f has no other fixed points within D. Let 2ϵ be the Euclidean distance between the image $g(\partial D)$ and the origin. Then $\epsilon > 0$ since g has no other zeros within D. Let $\mathcal{N} \subset C^1(D, \mathbb{R}^m)$ be the convex set consisting of all C^1-smooth maps $F : D \to \mathbb{R}^m$ which satisfy $\|F(\mathbf{u}) - f(\mathbf{u})\| < \epsilon$ for all $\mathbf{u} \in D$. Given F_0 and F_1 in $\mathcal{N}$, we can set $G_j(\mathbf{u}) = \mathbf{u} - F_j(\mathbf{u})$, and define a smooth map

$$G : [0,1] \times D \to \mathbb{R}^m \qquad \text{by the formula} \qquad G(t, \mathbf{u}) \;=\; t\, G_1(\mathbf{u}) + (1-t)\, G_0(\mathbf{u})\,.$$

If $\hat{\mathbf{u}}$ is a regular value of G with $\|\hat{\mathbf{u}}\| < \epsilon$, then the set $G^{-1}(\hat{\mathbf{u}})$ is a union of disjoint smooth oriented curves in $D \times [0,1]$, and these curves do not meet the boundary cylinder $[0,1] \times \partial D$. If $\hat{\mathbf{u}}$ is also a regular value for G_0 and G_1, then these curves intersect $0 \times D$ and $1 \times D$ transversally. The intersection points correspond to simple fixed points of $F_0 + \hat{\mathbf{u}}$ or $F_1 + \hat{\mathbf{u}}$, with Lefschetz index $+1$ or -1 according as the curve crosses this boundary disk from left to right or from right to left. It follows easily that the sum of indices for $F_0 + \hat{\mathbf{u}}$ is the same as the sum of indices for $F_1 + \hat{\mathbf{u}}$. Finally, since F_j has only simple fixed points for $j = 0, 1$, it follows that the fixed point indices for each F_j are the same as those for $F_j + \hat{\mathbf{u}}$, provided that $\|\hat{\mathbf{u}}\|$ is sufficiently small. $\square$

Next I will outline an intuitive proof of the Fixed Point Theorem 6.6. First some further properties of the intersection number. Suppose that we are given smooth oriented submanifolds $P, Q \subset M$ which intersect transversally in finitely many points so that the intersection number $P \bullet Q$ is well defined, and also manifolds $P', Q' \subset M'$ so that $P' \bullet Q'$ is well defined. Then we can also form the intersection number $(P \times P') \bullet (Q \times Q')$ in

$M \times M'$. In fact the following formula is easily verified

$$(P \times P') \bullet (Q \times Q') \;=\; (-1)^{p'q}\,(P \bullet Q)\,(P' \bullet Q')\ , \tag{6 : 13}$$

where $p' = \dim(P')$ and $q = \dim(Q)$.

Now consider the diagonal submanifold Δ_M in $M \times M$. Again suppose that P and Q are oriented submanifolds of M which intersect transversally. Then

$$(P \times Q) \bullet \Delta_M \;=\; (-1)^q\, P \bullet Q\ . \tag{6 : 14}$$

(Equivalently we could write $\Delta_M \bullet (P \times Q) = (-1)^p\, P \bullet Q$.) In fact it is clear that the intersections of $P \times Q$ with the diagonal correspond precisely to the intersections of P with Q , and it is not difficult to compute the sign.

We will need to generalize these ideas by considering not only intersections of submanifolds but also intersections of rational homology classes. In fact if M is compact and oriented, and if $\xi \in H_p(M)$ and $\eta \in H_q(M)$ where $p + q = m$ is equal to the dimension of M , then there is a well defined intersection number

$$\xi \bullet \eta \;=\; (-1)^{pq}\, \eta \bullet \xi \quad \in \quad ,$$

which depends bilinearly on ξ and η . It is convenient to extend the definition by setting $\xi \bullet \eta = 0$ whenever the dimension sum $p + q$ is not equal to m . In the special case where $\xi = (P)$ and $\eta = (Q)$ are the homology classes associated with compact oriented submanifolds P and Q which intersect transversally, as above, this homology intersection number $\xi \bullet \eta = (P) \bullet (Q)$ can be identified with the manifold intersection number $P \bullet Q \in$.

(One way of constructing such an intersection theory would be to use singular homology theory based on smooth maps from the standard simplexes of various dimensions into M . Generically, two such singular simplexes of complementary dimension intersect transversally, so that intersection multiplicities are defined.)

We will need the analogues of formulas (6 : 13) and (6 : 14) for such homology intersection numbers. Also we will need two standard results. Suppose that we choose a basis for each finite dimensional vector space $H_k(M)$. Considering these all together, we will describe the resulting homology classes, say $\xi_1, \ldots, \xi_b$, briefly as a *basis* for the direct sum $H_\oplus(M) = \bigoplus_{i=0}^m H_i(M)$.

> **Poincaré Duality Theorem.** If M is compact and oriented, then given any basis $\xi_1, \ldots, \xi_b$ for $H_\oplus(M)$, there exists a dual basis $\eta_1, \ldots, \eta_b$ for $H_\oplus(M)$ so that
>
> $$\xi_i \bullet \eta_j \;=\; \begin{cases} +1 & \text{if } \ i = j\,, \\ 0 & \text{otherwise.} \end{cases}$$

The rational homology of a product space $M \times M'$ can be described as follows.

> **Künneth Theorem.** there is a well defined bilinear product which assigns to each $\xi \in H_p(M)$ and each $\eta \in H_q(M')$ an element $\xi \times \eta \in H_{p+q}(M \times M')$. Furthermore, if $\xi_1, \ldots, \xi_b$ is a basis for $H_\oplus(M)$ and $\eta_1, \ldots, \eta_{b'}$ is a basis for $H_\oplus(M')$, then the products $\xi_i \times \eta_j$ form a basis for $H_\oplus(M \times M')$.

Combining these statements, we will prove the following.

Lemma 6.19. The Diagonal Homology Class. *Again let* M *be compact and oriented. If* $\xi_1, \ldots, \xi_b$ *is a basis for* $H_\oplus(M)$ *and* $\eta_1, \ldots, \eta_b$ *is a dual basis, then*

$$(\Delta_M) \;=\; \sum_i \xi_i \times \eta_i \quad \in \quad H_m(M \times M)\ . \tag{6 : 15}$$

Proof. In view of Poincaré duality and the Künneth Theorem, it suffices to check that

$$(\eta_h \times \xi_j) \bullet (\Delta_M) \;=\; (\eta_h \times \xi_j) \bullet \sum (\xi_i \times \eta_i)$$

for every basis element $\eta_h \times \xi_j$. But it is easy to check that both sides are zero unless $h = j$, so we are reduced to checking the sign in the identity

$$(\eta_j \times \xi_j) \bullet (\Delta_M) \;=\; (\eta_j \times \xi_j) \bullet (\xi_j \times \eta_j)\ .$$

This is a straightforward exercise using the analogues of (6 : 13) and (6 : 14). □

Proof of the Lefschetz Theorem 6.6. If $\{\xi_i\}$ has dual basis $\{\eta_i\}$, then evidently $\{\eta_i\}$ has a dual basis of the form $\{\pm\xi_i\}$. Thus we can rewrite (6 : 15) as $(\Delta_M) = \sum \pm\eta_i \times \xi_i$, or more precisely

$$(\Delta_M) \;=\; \sum_i (-1)^{\dim \xi_i \dim \eta_i}\ \eta_i \times \xi_i\ .$$

To compute the homology class (G_f) associated with a graph submanifold, we take take the image of (Δ_M) under the homomorphism from $H_m(M \times M)$ to itself induced by $(x, y) \mapsto (x\,,\ f(y))$. This yields

$$(G_f) \;=\; \sum_i (-1)^{\dim \xi_i \dim \eta_i}\ \eta_i \times \xi'_i\ ,$$

where ξ'_i is the image of the basis element ξ_i under the homomorphism from $H_\oplus(M)$ to itself induced by f .

We can now compute the Lefschetz number

$$\Lambda(f) \;=\; (G_f) \bullet (\Delta_M) \;=\; \sum_{i,j} (-1)^{\dim \xi_i \dim \eta_i}\ (\eta_i \times \xi'_i) \bullet (\xi_j \times \eta_j)\ .$$

Using the analogue of (6 : 13), this is easily evaluated as $\sum_i (-1)^{\dim \xi_i}\ \ \xi'_i \bullet \eta_i$. If we set $\xi'_i \;=\; \sum_j a_{ij}\xi_j$, then it follows that

$$\Lambda(f) \;=\; \sum_i (-1)^{\dim \xi_i}\ a_{ii}\ ,$$

which is clearly equivalent to the required formula of 6.6. □

§6F. Problems for the Reader.

Problem 6-a. If σ_d is the one or two-sided shift on d symbols (§2D, Problem 2-f), show that $Z_{\sigma_d}(t) = 1/(1 - dt)$. For the d-th power map $P_d(z) = z^d$ on the unit circle S^1 , show that

$$Z_{P_d}(t) \;=\; \frac{1 - t}{1 - dt}\ .$$

Problem 6-b. Let f be any self map such that each iterate has only finitely many fixed points. Using the inequalities

$$\#\mathbf{Per}_k(f) \;\le\; \#\mathbf{Fix}(f^{\circ k}) \;\le\; \sum_{j=1}^{k} \#\mathbf{Per}_j(f)\ ,$$

show that

$$h_{\rm per}(f) = \limsup_{k\to\infty} \frac{\log^+ \#\mathbf{Fix}(f^{\circ k})}{k} \qquad \text{is equal to} \qquad \limsup_{k\to\infty} \frac{\log^+ \#\mathbf{Per}_k(f)}{k}\ .$$

(Here $\log^+(x)$ is defined to be $\log(x)$ for $x \ge 1$, but is defined to be zero for $0 \le x \le 1$.) Using the fact that both $Z_f(t)$ and $t\,Z_f'(t)/Z_f(t)$ are power series with non-negative real coefficients, show that both have radius of convergence r equal to $\exp(-h_{\rm per}(f))$, assuming that there is at least one periodic orbit.

Problem 6-c. Show that

$$h_{\rm per}(f) \;\le\; h_{\rm per}(f^{\circ q}) \;\le\; q\,h_{\rm per}(f)\ .$$

Give examples where $h_{\rm per}(f) < h_{\rm per}(f^{\circ q}) = q\,h_{\rm per}(f)$. On the other hand, for a map of the 2-dimensional closed disk show that the sequence of integers $\#\mathbf{Per}_k(f)/k \ge 0$ can be completely arbitrary, except for the requirement that $\#\mathbf{Per}_1(f) \ge 1$. Find examples with $h_{\rm per}(f) = h_{\rm per}(f^{\circ q}) < q\,h_{\rm per}(f)$.

Problem 6-d. Consider a complex $n\times n$ matrix, written as $A + i\,B$ with A and B real. Show that the complex linear transformation $X + i\,Y \;\mapsto\; (X+i\,Y)(A+i\,B)$ corresponds to a real transformation

$$[\,X \quad Y\,] \;\mapsto\; [\,X \quad Y\,]\begin{bmatrix} A & B \\ -B & A\end{bmatrix}$$

in $2n$ variables. If $A+i\,B$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, show that the matrix $\begin{bmatrix} A & B \\ -B & A\end{bmatrix}$ has eigenvalues λ_j and $\overline{\lambda}_j$. Conclude that the Lefschetz number of a simple complex fixed point is always $+1$.

Problem 6-e. If f is a linear torus map corresponding to a matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$, show that $\Lambda(f^{\circ k}) = \det(I - A^k) \;=\; (1-\lambda_1^k)\cdots(1-\lambda_n^k)$. If no eigenvalue is a root of unity, show that $f^{\circ k}$ has exactly $|\Lambda(f^{\circ k})|$ fixed points, all simple of index ± 1.

Problem 6-f. Using an argument similar to the proof of 6.18, define the intersection multiplicity $\cap\#(P, Q, x_0)$ between two oriented submanifolds of complementary dimension at an isolated but not transverse intersection point.

Problem 6-g. With notation as in 6.9, show that the "Lefschetz zeta function"

$$Z_f^{\rm Lef}(t) \;=\; \exp\sum_{k\ge1} \Lambda(f^{\circ k})\,t^k/k \qquad \text{(with logarithmic derivative} \quad \sum \Lambda(f^{\circ k})\,t^k\ )$$

can be evaluated as the quotient of two characteristic polynomials,

$$Z_f^{\rm Lef}(t) \;=\; \frac{(1-\eta_1\,t)\cdots(1-\eta_q\,t)}{(1-\lambda_1\,t)\cdots(1-\lambda_p\,t)} \;=\; \frac{\det\Big(I_{\rm odd} - t\,H_{\rm odd}(f)\Big)}{\det\Big(I_{\rm even} - t\,H_{\rm even}(f)\Big)}\ .$$