§5. Attraction and Repulsion. (revised September 2001.)

We continue to study the iterates of a map $f : X \to X$, concentrating on properties which are invariant under topological conjugacy.

§5A. Attracting Sets. Let $f : X \to X$, and let A be a closed subset of X. We will say that a point $x \in X$ is *attracted* to A, or that x belongs to the *basin of attraction* $B(A)$, if, for every neighborhood U of A, there exists an n_0 so that $f^{\circ n}(x) \in U$ for $n \ge n_0$. If X is compact, then a completely equivalent condition would be that $\omega(x, f) \subset A$.

Definition. A closed non-vacuous subset $A \subset X$ will be called a (locally) *attracting set* if the following three conditions are satisfied. (See for example [Anosov and Arnold, p. 198].)

(a) The set A is *forward invariant*, that is $f(A) = A$.

(b) *Liapunov stability*. For any neighborhood U of A there exists a smaller neighborhood V so that $f^{\circ n}(V) \subset U$ for every $n \ge 0$.

(c) There exists a neighborhood U_0 of A so that every $x \in U_0$ is attracted to A.

Note that these three conditions are independent of each other. For the map

$$f(\tau) = \tau + \sin^2(\pi\tau)/10$$

from the circle $\mathbb{R}/\mathbb{Z}$ to itself, as shown in Figure 21 (§4B), the one point set $A = \{0 \bmod \mathbb{Z}\}$ is closed, forward invariant, and satisfies $\omega(\tau, f) = A$ for every $\tau \in \mathbb{R}/\mathbb{Z}$. However, this set A is not Liapunov stable: Points such as $\tau = 10^{-10}$, which are extremely close to A, get pushed far away from A under iteration of the mapping. On the other hand, for the map

$$g(x) = \begin{cases} x\,(1 - \sin^2(\pi/x)) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \qquad (5:1)$$

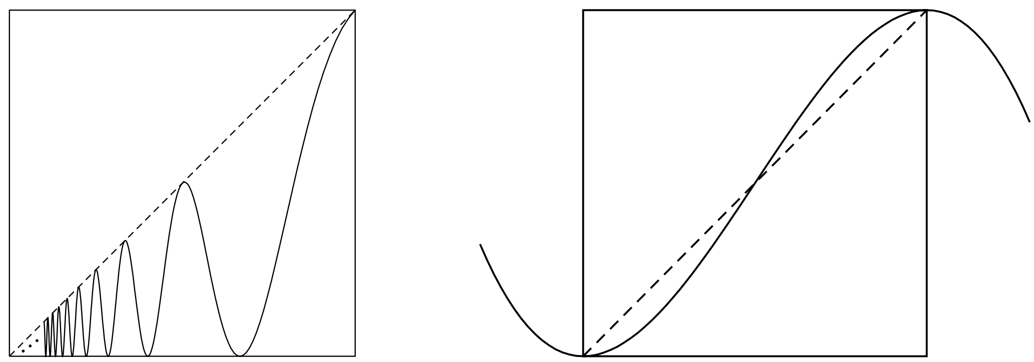

Figure 23. Graphs of the map $g : [0,1] \to [0,1]$ of $(5:1)$, and of the map $f(x) = x^2(3 - 2x)$ from the real line to itself.

from the interval $[0,1]$ to itself, as shown on the left in Figure 23, the set $A = \{0\}$ is Liapunov stable, but is not attracting since there are fixed points $x = 1/n$ arbitrarily close to A . (However, each interval $[0\,,\,1/n]$ is attracting.) As a final example, for the map $f(x)=x/2$ on the real line, the set $A=[-1\,,\,1]$ satisfies conditions (b) and (c), but is not forward invariant.

Note that an attracting set, in this sense, need not be minimal. As an example, for the map $f(x)=x^2(3-2x)$, shown on the right of Figure 23, the unit interval $[0,1]$ is an attracting set. However, each of the two endpoints $\{0\}$ and $\{1\}$ is also an attracting set. In fact the orbit of every point other than $1/2$ in a neighborhood of $[0,1]$ converges to either 0 or 1 .

Remark. If the set A is attracting, note that its basin of attraction $B=B(A)$ (consisting of all points $x\in X$ which are attracted to A) is an open set containing A . In fact, with U_0 as in (c) above, we have

$$A\ \subset\ B\ =\ U_0\cup f^{-1}(U_0)\cup f^{-2}(U_0)\cup\cdots\ .$$

Note that this set B is always *backward invariant*, $B=f^{-1}(B)$. (This implies the weaker statement that B is *subinvariant*, $f(B)\subset B$.) If X is locally connected, then the union of all connected components of B which intersect A is an open set called the *immediate basin* of A .

For the rest of this section, let us assume that the space X is locally compact and metric. We will be particularly interested in forward invariant sets $A\subset X$ which are compact. When A is compact, there are many alternative conditions which are equivalent to the statement that A is attracting. The first of these is due to Auslander, Bhatia, and Seibert, who called such a set an "asymptotically stable attractor":

Lemma 5.1. *A compact forward invariant set $A=f(A)$ is attracting if and only if it possesses a neighborhood U_0 whose forward images converge to A in the following strong sense: For any neighborhood V of A there should exist an integer $n_V\ge 0$ so that $f^{\circ n}(U_0)\ \subset\ V$ for all $n\ge n_V$.*

(Proofs later.) Ruelle calls such a U_0 a *fundamental neighborhood* of A . Here is another form of the definition which is apparently much weaker.

Lemma 5.2. *A compact forward invariant set $A\subset X$ is attracting if and only if there exists a neighborhood U so that A is equal to the intersection $\bigcap_{n\ge 0} f^{\circ n}(U)$ of the forward images.*

Here there is no requirement that $f(U)\subset U$. (Compare [Smale, 1967], [Conley].) Such a neighborhood U is called a *forward isolating* neighborhood for A .

Note that compactness is essential for this statement. As an example, if A is the x-axis in the (x,y)-plane and $f(x,y)=(x+1\,,\,y)$, then the neighborhood $U=\{(x,y)\ :\ |y|<e^x\}$ has the stronger property that

$$U\supset f(U)\supset f^{\circ 2}(U)\supset\cdots$$

with intersection A , yet A is certainly not an attracting set.

One standard method for locating attracting sets can be described as follows. A compact set $T\subset X$ will be called a *trapping region* if its image $f(T)$ is contained in the interior of T . It will be convenient to use the notation $A=f^{\circ\infty}(T)$ for the intersection of the nested sequence

$$T \supset f(T) \supset f^{\circ 2}(T) \supset \cdots .$$

We will also say that T is a *trapping neighborhood* of this intersection A. It is easy to check that A is compact, forward invariant, non-vacuous, and that T is a neighborhood of A.

Lemma 5.3. *A subset $A \subset X$ is compact and attracting if and only if it has a trapping neighborhood T. In fact, if A is compact and attracting, then every neighborhood of A contains such a trapping neighborhood.*

Remarks. If T is connected, then the associated attracting set A will also be connected. It may happen that nearly every point in T is attracted to some finite subset of T (as in Figure 23, right). However, the associated attracting set A will always be large enough to connect all of the points in this finite set.

Even if the map f and the trapping neighborhood T are quite smooth, the successive forward images $f^{\circ n}(T)$ may be folded in a more and more complicated manner, so that the associated $A = \bigcap_n f^{\circ n}(T)$ is an extremely complicated "fractal" set. Figure 24 illustrates how this can happen in the piecewise smooth case. For smoother and more natural examples, see Figure 12 in §2E, with a solid torus as trapping region, and with the Smale-Shub solenoid as associated attracting set, as well as Figure 4 of §1D, showing a trapezoid as trapping region for a Hénon map. (For other examples, see [Devaney, §2.5].) The solenoid example is very well understood, but Hénon attractors remain much more mysterious in spite of a great deal of study. (See especially [Benedicks-Carleson].) For any specific choice of parameter values, such as that in Figure 4, it is very difficult to be certain about the dynamics of the attracting set A. For example it is possible (although very unlikely) that this particular A actually contains an attracting periodic orbit, necessarily of very high period.

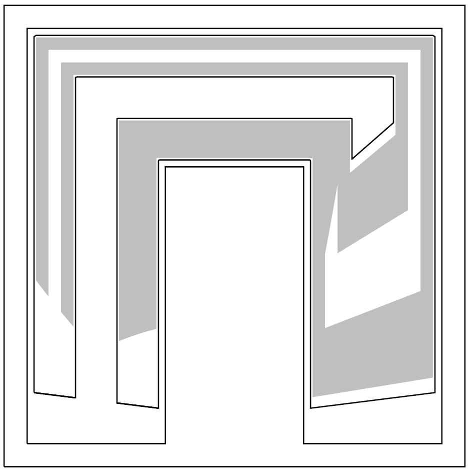

Figure 24. Showing the boundaries of the images of the unit square I^2 under the first two iterates of a piecewise smooth map f which sends I^2 homeomorphically onto a horseshoe shaped subset. The third image $f^{\circ 3}(I^2)$ has been shaded.

As an easy consequence of 5.3, we see that compact attracting sets are reasonably robust under perturbation of the map f. Suppose that the space X is locally compact metric, with distance function $\mathbf{d}(x, y) \geq 0$, and let $T \subset X$ be a trapping region, with associated attracting set $A = f^{\circ\infty}(T)$. We will say that a map $g : X \to X$ is *uniformly ϵ-close* to f throughout T if $\mathbf{d}(f(x)\,,\,g(x)) < \epsilon$ for all $x \in T$.

Lemma 5.4 (Upper semi-continuity). *For any neighborhood U of a compact attracting set $A = f^{\circ\infty}(T)$ there exists a number $\epsilon > 0$ with the following property. If $g : X \to X$ is uniformly ϵ-close to f throughout T , then g has a compact attracting set A' which is contained in U , and which has this same region T as trapping neighborhood: $A' = g^{\circ\infty}(T)$.*

Thus the attracting set cannot either disappear or become much larger under perturbation of f . However, it can become much smaller:

Example. Let

$$g_0(x) \;=\; 2\,x^2 - x^3$$

on the real line. Then g_0 has trapping region $T = [-\frac{1}{2}\,,\,2]$. The corresponding attracting set $A = g_0^{\circ\infty}(T)$ is the unit interval $[0, 1]$, which maps homeomorphically onto itself. (The image $g_0(T)$ is the interval $[0\,,\,\frac{32}{27}]$, which is contained in the interior of T as required. It is interesting to note however, that this image $g_0(T)$ is *not* a neighborhood of A .)

Now suppose that we perturb to the map $g_\epsilon(x) = (1 - \epsilon)\,(2\,x^2 - x^3)$. The interval $T = [-\frac{1}{2}\,,\,2]$ is a trapping region for g_ϵ also, but the corresponding attracting set $A' = \bigcap g_\epsilon^{\circ n}(T)$ consists of the single point 0 . Briefly we say that 0 is an *attracting fixed point* for g_ϵ .

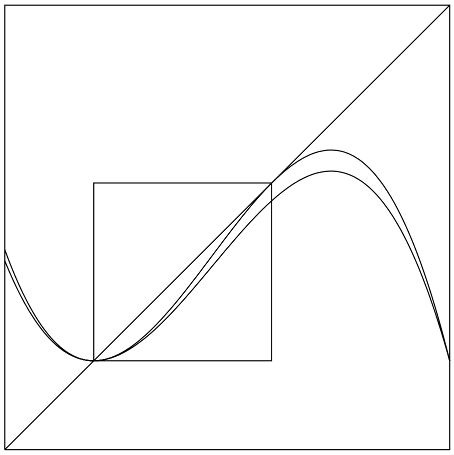

Figure 25. Graphs of $x \mapsto 2\,x^2 - x^3$ *and* $x \mapsto \frac{9}{10}\,(2\,x^2 - x^3)$ *on* $T = [-\frac{1}{2}\,,\,2]$.

Note. This example shows again that we may have one attracting set contained in another. For the origin is an attracting fixed point, not only for g_ϵ , but also for the original map g_0 . Thus g_0 has two attracting sets $\{0\} \subset [0, 1]$, one properly contained in the other.

The *minimal* attracting sets, that is, those for which no proper subset is attracting, are presumably of particular interest. However, these may not exist. As an example, for the map of Equations (5 : 1) of Figure 23 (left), each interval $[0\,,\,1/n]$ is an attracting set, but there is no minimal attracting set since the intersection $\{0\}$ of these intervals does not attract any neighborhood.

To conclude this section, we must prove the four lemmas.

Proof of 5.1, 5.2, and 5.3. (See [Hurley], as well as [Milnor, 1985 (correction)].) Let $A \subset X$ be a compact forward invariant set. First suppose that:

(i) *This set* A *is attracting.*

Choose a compact neighborhood N with the property that $\omega(x, f) \subset A$ for every $x \in N$. We will prove the following:

(ii) *Given any neighborhood* U *of* A, *there exists an integer* $n_U > 0$ *so that* $f^{\circ n}(N) \subset U$ *for* $n \ge n_U$.

In fact, by Liapunov stability, we can find a smaller open neighborhood V so that $f^{\circ n}(V) \subset U$ for all $n \ge 0$. For each $x \in N$ we can certainly find an integer $m_x \ge 0$ so that $f^{\circ m_x}(x) \in V$. It follows that $f^{\circ m_x}(y) \in V$ for all y in some open neighborhood W_x of x. Now cover N by finitely many of these open sets W_x and let n_U be the largest of the corresponding integers m_x. For $n \ge n_U \ge m_x$ it follows that $f^{\circ n}(W_x) \subset U$, and the required statement (ii) follows.

Clearly (ii) implies that this compact neighborhood N has the following property:

(iii) *The intersection of the forward images* $f^{\circ n}(N)$ *is equal to* A.

Next we will show that (iii) implies the following:

(iv) *Any neighborhood* $U \supset A$ *contains a compact neighborhood* $K \supset A$ *so that* $K \supset f(K) \supset f^{\circ 2}(K) \supset \cdots$, *with intersection* A.

To prove this statement, choose a compact neighborhood $N \subset U$ satisfying (iii), and define a nested sequence of compact neighborhoods

$$N = N_0 \supset N_1 \supset N_2 \supset \cdots$$

as follows. Let

$$N_k = N \cap f^{-1}(N) \cap \cdots \cap f^{-k}(N)$$

be the set of all $x \in N$ such that $f^{\circ m}(x) \in N$ for $0 \le m \le k$. This is a neighborhood of A, since it contains the open set

$$\mathring{N} \cap f^{-1}(\mathring{N}) \cap \cdots \cap f^{-k}(\mathring{N}) \supset A ,$$

where $\mathring{N}$ is the interior of N. If k is sufficiently large, we claim that $N_k = N_{k+1}$. For otherwise, if $N_k \ne N_{k+1}$ for infinitely many k, then choosing $x_k \in N_k \smallsetminus N_{k+1}$ the point $y_k = f^{\circ k}(x_k)$ would belong to N, with $f(y_k) \notin N$. Choose a subsequence of points y_{k_i} converging to a point $y' \in N$. For any fixed m, since y_{k_i} belongs to the compact set $f^{\circ m}(N)$ whenever $k_i \ge m$, it follows that the limit point y' also belongs to $f^{\circ m}(N)$. Hence y' belongs to $\bigcap f^{\circ m}(N) = A$. On the other hand, since $f(y_k) \notin N$ it follows that $f(y')$ cannot belong to the interior $\mathring{N}$. This is impossible, since $f(y') \in f(A) = A \subset \mathring{N}$.

Thus we can choose k so that $N_k = N_{k+1} = \bigcap_{i \ge 0} N_i$. Evidently $f(N_{k+1}) \subset N_k$, so it follows that the neighborhood $K = N_k$ satisfies $f(K) \subset K$. Thus we have constructed a compact neighborhood $K \subset N$ of A so that $K \supset f(K) \supset f^{\circ 2}(K) \supset \cdots$ with intersection A, proving (iv).

Next we show that (iv) implies that:

(v) *Any neighborhood* $U \supset A$ *contains a trapping neighborhood* T *for* A .

Choosing $K \subset U$ as in (iv), an easy compactness argument shows that for any neighborhood V of A there exists an integer n with $f^{\circ n}(K) \subset V$. In particular, we can take V equal to the interior $\overset{\circ}{K}$. Let n be the smallest integer with $n \geq 1$ and with $f^{\circ n}(K) \subset \overset{\circ}{K}$. If $n = 1$ then we can choose $T = K$, and are done. If $n > 1$, then setting $K = K_n$ we will construct neighborhoods $K_n \supset K_{n-1} \supset \cdots \supset K_1$ by induction so that $f^{\circ m}(K_m) \subset \overset{\circ}{K}_m$ for each m . Taking $T = K_1$, this will prove (v).

By hypothesis, $f^{\circ n-1}(K_n)$ intersects the boundary ∂K_n . Let $Y_n \subset K_n$ be the compact non-vacuous set consisting of all points $x \in K_n$ such that $f^{\circ n-1}(x) \in \partial K_n$. Since $f^{\circ n}(K_n)$ is contained in the interior $\overset{\circ}{K}_n$, we know that $f(K_n) \cap Y_n = \emptyset$. Let $\delta_n > 0$ be the minimum distance from $f(K_n)$ to Y_n , and let K_{n-1} be the closed $(\delta_n/2)$-neighborhood of $f(K_n)$ in K_n . Evidently K_{n-1} is a compact neighborhood of A with

$$f(K_{n-1}) \subset f(K_n) \subset K_{n-1} \subset K_n .$$

Note that every point in the boundary ∂K_{n-1} must either have distance $\delta_n/2 > 0$ from $f(K_n)$ or else must belong to ∂K_n . Now suppose that K_{n-1} contained a point x with $f^{\circ n-1}(x) \in \partial K_{n-1}$. Since $n > 1$, such a point $f^{\circ n-1}(x)$ must certainly belong to $f(K_n)$, and hence must belong to ∂K_n . Therefore $x \in Y_n$, which is impossible since $K_{n-1} \cap Y_n = \emptyset$. This completes the inductive step, showing that (iv) implies (v).

Since it is easy to see that (v) implies (i), this shows that the five conditions (i) through (v) are mutually equivalent. Evidently this proves 5.1, 5.2, and 5.3. □

Proof of 5.4. Choose a smaller trapping neighborhood $T' \subset U$ for f , and choose n so that $f^{\circ n}(T) \subset \text{interior}(T')$. If g is uniformly ϵ-close to f throughout T , with ϵ sufficiently small, then we have:

(1) $g(T) \subset \text{interior}(T)$,

(2) $g(T') \subset \text{interior}(T')$, and

(3) $g^{\circ n}(T) \subset \text{interior}(T')$.

Thus T and T' are trapping regions for g also, and the associated attracting set $A' = g^{\circ\infty}(T)$ is equal to $g^{\circ\infty}(T')$. Thus A' has both $T' \subset U$ and T as trapping neighborhoods under the perturbed map g . □

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§5B. Attractors.[1] The concept of "*attractor*" has been defined in many different ways in the literature. The various definitions represent different attempts to describe the

[1] In talking to a group of evolutionary biologists, I was startled to discover that they strongly objected to this word "attractor", since to them it implied some supernatural or teleological mechanism which controls the course of evolution (i.e., the behavior of orbits in a corresponding evolutionary dynamical system). Perhaps I must say explicitly that no such meaning is intended. In dynamics, this word simply describes where typical orbits actually go. It does not suggest that there is some agency, located in the attractor, which pulls orbits towards it.

asymptotic limits for "typical" orbits. Nearly all of these definitions have the following two properties in common:

(α) *Many orbits converge towards the attractor.*

(β) *Every part of the attractor is essential in some sense.*

Many of these definitions involve not only topology but also measure theory, and hence do not fit into a section on topological dynamics.[2] However, several topologically invariant definitions have been given. This section will present two of these, and describe rather trivial examples to distinguish between them. In order to avoid confusion, we must invent different names for these different concepts.

One commonly used concept of attractor is the following. We will say that $A \subset X$ is a *strong topological attractor* if A is a compact attracting set, as defined in §5A, and if the restriction $f|A$ mapping A onto itself is topologically transitive (§4C).

(See for example [Devaney].) Examples are provided by attracting fixed points, or attracting periodic orbits, or by the solenoid of §2E. A strong topological attractor is an extremely nice object when it exists. However, such strong attractors do not always exist; and even when there is a strong attractor, the existence proof may be very difficult.

As an example, there is no such attractor for the map (5 : 1) since the only topologically transitive subsets are fixed points, and none of these attracts an entire neighborhood. Similarly, for the differential equation illustrated in Figure 17 (§3B), there are no topologically transitive subsets other than the three fixed points. (In this case, the only compact attracting sets are the figure eight curve, and this curve with both lobes filled in.) A more interesting example the *Feigenbaum map* of the interval, as described in §6D. For this map, Lebesgue almost every orbit is attracted to a compact invariant Cantor set K, so that the associated basin B has full measure, although it has no interior.

Ruelle has given a very different definition, based on the "chain partial ordering" $x \succ y$ of a dynamical system, as described in §4B. Recall that a *chain component* Γ of a chain recurrent point x_0 is the set of all $y \in X$ such that both $x_0 \succ y$ and $y \succ x_0$.

Definiton. By a *chain-attractor* for the compact dynamical system $f : X \to X$ we will mean a chain component $\Gamma \subset X$ which is *minimal* in this partial ordering in the following sense. If $x \succ y$ with $x \in \gamma$, then y must also belong to Γ. More explicitly, for $x_0 \in \Gamma$ and $y \in X$, if there is an ϵ-chain from x_0 to y for every $\epsilon > 0$, then there must also be an ϵ-chain from y to x_0 for every $\epsilon > 0$, so that $y \in \Gamma$ also. (If X is compact, then a completely equivalent condition would be that there is no chain component $\Gamma' \neq \Gamma$ with $\Gamma \succ \Gamma'$.)

This is a simple and useful concept, but it must be emphasized that a chain-attractor may not be an attracting set as defined above. As an example, in Figure 23 (left) the single

[2] For example in [Milnor 1985] a (measure theoretic) attractor is defined as a closed forward invariant set whose attractive basin has positive measure, and such that no proper subset has these properties. [Viana] defines an attractor as a closed topologically transitive set whose basin has positive measure. One example which shows that such definitions may be useful is the Feigenbaum map, as discussed later on this page. A more startling example, due to [Kan], is a smooth self-map of a cylinder with two attractive basins which are thoroughly intermingled with each other. (See §8.6.)

point $\{0\}$ is the unique chain attractor, but attracts no other point.

Lemma 5.5. *If the space X is compact, then every map $f : X \to X$ has at least one chain-attractor. In fact, for every $x \in X$ there exists a chain-attractor Γ made up out of points y with $x \succ y$.*

Proof Outline. Let K_x be the set of all $y \in X$ with $x \succ y$. Then K_x is compact (Problem 4-a), and $K_x \supset K_y$ whenever $x \succ y$. The conclusion then follows easily from Zorn's Lemma.

(If X is compact metric, so that there exists a countable basis $\{U_i\}$ for the open sets, then we can avoid Zorn's Lemma as follows. Starting with $x = x_0$, inductively construct points

$$x_0 \succ x_1 \succ x_2 \succ \cdots$$

as follows. For each $i \geq 0$, if there is any point $y \in K_{x_i}$ with $K_y \cap U_i = \emptyset$, then choose such a point y as x_{i+1} . Let $\Gamma = \bigcap K_{x_i}$. Clearly Γ is not vacuous. Furthermore, any two points $x, y \in \Gamma$ satisfy $x \succ y$. For otherwise, choosing i so that U_i contains x and is disjoint from K_y , we would contradict the choice of x_{i+1} .) $\square$

As a first example, consider the map $x \mapsto 2x^2 - x^3$ on the interval $[-1/2\,,\,2]$, as shown in Figure 25. Both of the fixed points $\{0\}$ and $\{1\}$ are chain components which attract entire intervals of points; but only $\{0\}$ is minimal, so that $\{0\}$ is the unique chain-attractor. Similarly, for the map shown in Figure 23(left), each fixed point $1/n = g(1/n)$ with $n > 1$ is a chain component attracting an entire interval, but their limit point $\{0\}$ is the unique chain attractor. It would be easy to modify this example, constructing a strictly monotone map $g(x) \leq x$ with a sequence of fixed points tending to zero, so that $\{0\}$ would still be the unique chain attractor, but the orbit of every point $x > 0$ would remain strictly bounded away from zero. *Thus a chain-attractor does not necessarily attract many orbits.* Intuitively, perhaps we should think of a chain-attractor as a set which attracts an entire neighborhood under a slightly perturbed mapping.

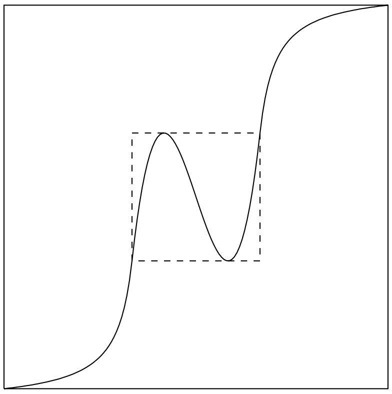

Figure 26. Showing a map of the unit interval such that each endpoint is an attractor. Here the middle interval $1/3\,,\,2/3]$ is a transitive chain-component which is contained in the "chain-basin" for each of these attractors.

If Γ is a chain-attractor, perhaps the closed set B_Γ consisting of all $x \in X$ with $x \succ \Gamma$ should be described as the associated *chain-basin*. Note however that two such chain-basins, corresponding to distinct Γ , may have a large intersection, as shown in Figure 26.

§5C. Repelling Sets and Repellors. The concept of "attraction", as expressed by Lemma 5.2, suggests a dual concept of "repulsion". In the special case of a homeomorphism, we can simply define a repelling set (or a repellor) for f to be an attracting set (or an attractor) for f^{-1}. Then every result proved above for attracting sets or attractors would have an immediate analogue for repelling sets or repellors. The problem is to make reasonable definitions when f is not a homeomorphism.

Definition. A compact subinvariant set $Y \supset f(Y)$ will be called a *repelling set* if there exists a neighborhood U which is "backward isolating", in the sense that Y is equal to the intersection $\bigcap_{n\geq 0} f^{-n}(U)$ of the iterated pre-images. It will be called a *repellor* if it is repelling and also topologically transitive.

Thus a set Y is repelling if and only if it satisfies $f(Y) \subset Y$, and has a neighborhood U such that every orbit

$$x_0 \mapsto x_1 \mapsto x_2 \mapsto \cdots$$

which starts at a point $x_0 \notin Y$ must contain at least one point x_n which lies outside of U.

Remarks. This orbit may well return to U later. For example, in the case of the doubling map on the circle of §2B, the zero point is clearly a repelling fixed point. Yet we have seen in 3.8 that most orbits return to an arbitrarily small neighborhood of zero infinitely often.

It might seem more natural to require that a "repelling set" Y must be forward invariant, $f(Y) = Y$. However, this would be somewhat restrictive. As an example, consider the map $f(x) = x\,(1-x)$ from the real line to itself. Then the unit interval $Y = [0,1]$ is a repelling set, as defined above, with $f(Y) = [0\,,\,1/4] \subset Y$. However, all bounded orbits converge to zero, so the only compact forward invariant set is $\{0\}$, which is not repelling.

Here is another example. If $B \subset X$ is the basin of an attracting set A in a compact space X, then the complement $Y = X \smallsetminus B$ is a backward invariant repelling set. In fact, if $N \subset B$ is a compact neighborhood of A, then any orbit which remains in $X \smallsetminus N$ forever must lie in Y. One important example is provided by polynomial maps of degree two or more from $\mathbf{C} \cap \infty$ to itself. In this case, the point at infinity is a strong topological attractor. The complement of its basin, consisting of all points with bounded orbit, is a repelling set called the *filled Julia set.* (See [Douady and Hubbard], [Milnor 1999], and compare Problem 5-d.)

In contrast to Lemma 5.4, a repelling set can vanish under a small perturbation of the map. As an example, the map $f_0(x) = 2x(1-x)$ on the unit interval clearly has 0 as a repelling fixed point. But for $f_\epsilon(x) = 2x(1-x) + \epsilon$, all orbits converge to an attracting fixed point, which forms the unique compact forward invariant set. (Figure 27.)

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§5D. Problems for the Reader.

Problem 5-a. The maximal attracting set. If $f : X \to X$ maps a compact space into itself, show that the intersection of the images $f^{\circ n}(X)$ is the unique maximal attracting set. (Of course if $f(X) = X$, then this attracting set is the entire space.) On the other hand, for the map $f(x) = x^2 + 1$ on the real line, show that all orbits diverge to infinity, so that there is no compact attracting set.

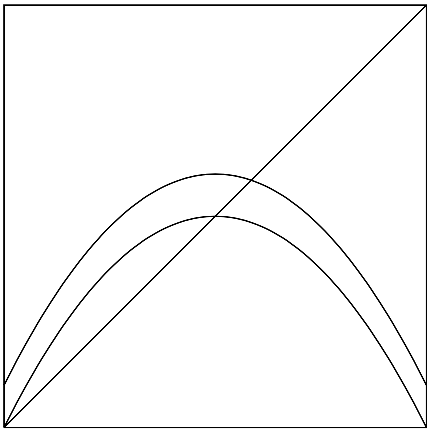

Figure 27. Graphs of $x \mapsto 2x(1-x)$ *and* $2x(1-x)+0.1$ *on the unit interval.*

Problem 5-b. Chain-repellors. Let us call a chain component $\Gamma \subset X$ a *chain repellor* if $x \succ y$ with $y \in \Gamma$ implies that $x \in \Gamma$. Such a chain component is necessarily "maximal" in the chain partial ordering in the sense that no chain-component $\Gamma' \neq \Gamma$ satisfies $\Gamma' \succ \Gamma$. If X is compact and f maps X *onto* itself, show that every maximal chain component is a chain-repellor, and show that there exists at least one chain-repellor.

Problem 5-c. "Expelling" regions. In analogy with the concept of "trapping region", as studied in 5.3, define a *expelling region* to be a compact set $P \subset X$ such that the intersection

$$Y = \bigcap_{n \geq 0} f^{-n}(P) \tag{5 : 3}$$

is non-vacuous, and such that $f(\partial P) \cap P = \emptyset$. If P is such a expelling region, show that Y is a repelling set contained in the interior of P. Such a P could be called a *expelling neighborhood* of Y. I don't know whether every repelling set has such a expelling neighborhood.

It might seem more natural to assume[3] that $f(P) \supset P$. This would certainly imply that the intersection Y is non-vacuous, and would also imply that $Y = f(Y)$. However, there are interesting examples where $f(P)$ does not contain in P. For example if $f(x) = x(1-x)$ with $P = [-1, 2]$ and $Y = [0, 1]$, then $f(P) = [-2\,,\,1/4]$ does not contain P.

Problem 5-d. Julia sets. (For readers with some knowledge of holomorphic dynamics.) Show that the Julia set of a rational map of the Riemann sphere is a repellor if and only if every orbit outside of the Julia set converges towards an attracting or super-attracting orbit.

[3] One important example is provided by the polynomial-like mappings of holomorphic dynamics, with the filled Julia set as associated repelling set. See [Douady and Hubbard, 1985].