§16. Dependence on Parameters.

Let F be a lift of a degree one circle map. How does the translation interval $\mathbf{TI}(F)$ change as we vary the map F ? As a first step, we should check continuity; but in order to make sense of this, we must choose an appropriate topology.

§16A. The Space of Circle Maps. For maps on any compact space, the following topology is often useful.

Definition. For any compact space X and any metric space Y , let $C^0(X,Y)$ be the space consisting of all continuous maps from X to Y , with the topology determined by the metric

$$\mathbf{d}(F\,,\,G) \;=\; \max_{x\in X}\,\mathbf{d}(F(x)\,,\,G(x))\;. \tag{16:1}$$

This is called the *topology of uniform convergence*, or the C^0*-topology*. We can use this same formula to define a metric, and hence a topology, on the space $C^0_1(\mathbb{R}\,,\,\mathbb{R})$ consisting of lifts of degree one circle maps, or in other words on the space of maps from $\mathbb{R}$ to itself satisfying $F(x+1) = F(x)+1$. (For any $F\,,\,G \in C^0_1(\mathbb{R}\,,\,\mathbb{R})$, note that the difference $F(x)-G(x)$ is a periodic function, and hence attains a finite maximum.)

Lemma 16.1. *If X , Y and Z are compact metric, then the correspondence $(F,G)\mapsto G\circ F$ defines a continuous mapping*

$$C^0(X,Y)\times C^0(Y,Z) \;\to\; C^0(X,Z)\;.$$

Similarly, if $C^0_1(\mathbb{R},\mathbb{R})$ is the space of lifts of degree one circle maps, then the composition mapping

$$C^0_1(\mathbb{R}\,,\,\mathbb{R})\times C^0_1(\mathbb{R}\,,\,\mathbb{R}) \;\to\; C^0_1(\mathbb{R}\,,\,\mathbb{R})\;.$$

is continuous.

Proof. This is easily checked, making use of the fact that every element of $C^0(Y,Z)$ (or of $C^0_1(\mathbb{R}\,,\,\mathbb{R})$) is actually uniformly continuous. $\square$

We can now prove the following.

Lemma 16.2. *Both endpoints $\mathbf{tn}^-(F)$ and $\mathbf{tn}^+(F)$ of the translation interval depend continuously on F , as F varies through the space of lifts of degree one circle maps with the C^0-topology.*

In particular, if we consider only *monotone* degree one circle maps f , so that the rotation number $\mathbf{rn}(f)\in\mathbb{R}/\mathbb{Z}$ is well defined, then it follows easily that $\mathbf{rn}(f)$ depends continuously on f .

Proof of 16.2. For any large integer q , choose the smallest p so that

$$\mathbf{tn}^+(F) \;<\; p/q\;.$$

According to Theorem 14.6, this is equivalent to the statement that

$$F^{\circ q}(x) \;<\; x+p$$

for all x . Now if G is sufficiently close to F in the C^0-topology, then $G^{\circ q}$ will be uniformly close to $F^{\circ q}$ by 16.1, so that $G^{\circ q}(x) < x+p$ for all x also, which implies that $\mathbf{tn}^+(G) < p/q$. Now let $p' = p-2$, so that $\mathbf{tn}^+(F) > p'/q$. Then the inequality

$F^{\circ q}(x_0) \;>\; x_0 + p'$ must hold for *some* x_0. Hence the same must be true for any G which is sufficiently close to F. It follows that $p'/q \le \mathbf{tn}^+(G) < p/q$, and hence that $|\mathbf{tn}^+(G) - \mathbf{tn}^+(F)| < 2/q$. Since q can be arbitrarily large, this proves the continuity of $\mathbf{tn}^+$. The argument for $\mathbf{tn}^-$ is completely analogous. $\square$

Here is an application of continuity.

Lemma 16.3. *Let F be a lift of a degree one circle map with non-degenerate translation interval $\mathbf{TI}(F)$. Then for every constant $c \in \mathbf{TI}(F)$ there exists an orbit $F : x_0 \mapsto x_1 \mapsto \cdots$ so that the translation number $\mathbf{tn}(F\,,\,x_0)$ is well defined and equal to c.*

Proof (following [Alseda et al.]). First consider the leftmost point $\mathbf{tn}^-(F)$ of the translation interval. Define a modified map $F_0(x)$ by the formula

$$F_0(x) \;=\; \min\{F(y)\;;\; y \ge x\} \;\le\; F(x)\;.$$

Then it is easy to check that F_0 is well defined and monotone, with $F_0(x+1) = F_0(x)+1$. Thus the translation number $\mathbf{tn}(F_0)$ is well defined. Note also that F_0 is locally constant throughout the open set where $F_0(x) < F(x)$. It follows that there is at least one point $x_0 \in \mathbb{R}$ whose orbit under F coincides with its orbit under F_0. Otherwise, for each $x \in \mathbb{R}$ there exists an $n > 0$ so that $F_0^{\circ n}(x)$ takes a constant value throughout some neighborhood of x. Covering $\mathbb{R}/\mathbb{Z}$ by finitely many such neighborhoods, we see that $F_0^{\circ n}$ must be constant for n sufficiently large. But this is impossible since $F_0(x+1) = F_0(x)+1$. It now follows easily that

$$\mathbf{tn}(F\,,\,x_0) \;=\; \mathbf{tn}(F_0)$$

is equal to the leftmost point $\mathbf{tn}^-(F)$ of the translation interval $\mathbf{TI}(F)$.

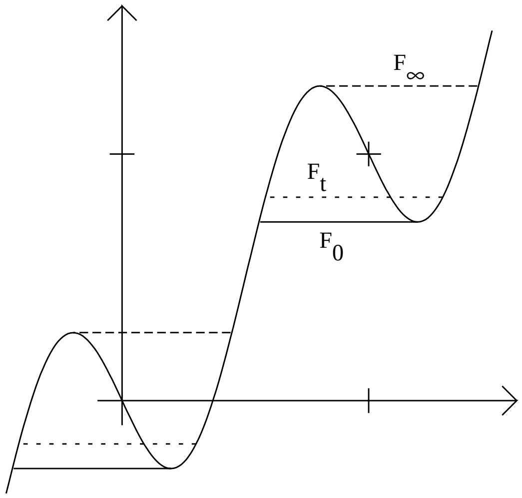

Figure 53. The monotone functions $F_0 \le F_t \le F_\infty$ associated with a non-monotone function F.

Similarly, define

$$F_\infty(x) \;=\; \max\{F(y)\;;\; y \le x\} \;\ge\; F(x)\;.$$

Then a completely analogous argument constructs a point x_0 so that

$$\mathbf{tn}(F\,,\,x_0) \;=\; \mathbf{tn}(F_\infty) \;=\; \mathbf{tn}^+(F)\ .$$

In order to construct points with translation number in the interior of the interval $\mathbf{TI}(F)$, we construct a one-parameter family of maps $F_t:\mathbb{R}\to\mathbb{R}$ for $t\ge 0$ by the formula

$$F_t \;=\; \left(\min(F\,,\,F_0+t)\right)_\infty\ .$$

The following facts are easy to check:

(i) In the special case $t=0$, this new definition of F_0 coincides with the old.

(ii) $F_t=F_\infty$ for t sufficiently large.

(iii) F_t is continuous and monotone, and depends continuously on the parameter t.

Furthermore, each F_t is locally constant throughout the open set where $F_t(x)\ne F(x)$. In fact, F_t is clearly locally constant in the region where $F_t(x)>\min\left(F(x)\,,\,F_0(x)+t\right)$. But if $F_t(x)=\min\left(F(x)\,,\,F_0(x)+t\right)\ne F(x)$, then $F_t(x)=F_0(x)+t<F(x)$, and again it follows easily that F_t is locally constant. Using these facts together with 16.2, the conclusion now follows easily. □

As one application of this argument, we obtain a monotonicity statement. Again let F and G be lifts of degree one circle maps.

Lemma 16.4. *If* $F(x)\le G(x)$ *for all* x, *then* $\mathbf{tn}^-(F)\le\mathbf{tn}^-(G)$ *and* $\mathbf{tn}^+(F)\le\mathbf{tn}^+(G)$.

Proof. If F and G are monotone, this is clear, since $F\le G$ implies that $F^{\circ q}\le G^{\circ q}$, hence $\mathbf{tn}(F)\le\mathbf{tn}(G)$. In the non-monotone case, we need an extra step, using the construction above:

$$F\le G \quad\Rightarrow\quad F_0\le G_0 \quad\Rightarrow\quad \mathbf{tn}(F_0)=\mathbf{tn}^-(F)\;\le\;\mathbf{tn}(G_0)=\mathbf{tn}^-(G)\ . \qquad \square$$

§**16B. The Standard Family.** To illustrate the way that translation numbers change as we vary the map, consider the "*standard family*" of circle maps

$$f(\xi) \;=\; f_{c,\kappa}(\xi) \;=\; \xi + c + \frac{\kappa}{2\pi}\sin(2\pi\xi)\ , \tag{16: 2}$$

depending on two parameters $c\in\mathbb{R}/\mathbb{Z}$ and $\kappa\in\mathbb{R}$. We can distinguish three cases: If $|\kappa|<1$, then it is easy to check that $f'(\xi)>0$ everywhere, so that f is a circle diffeomorphism. In the *critical case* $\kappa=\pm1$, the map f is still a circle *homeomorphism*, but the derivative $f'(\xi)$ has a zero, so that f^{-1} is not differentiable. Finally, for $|\kappa|>1$ the map f is no longer monotone.

Thus the rotation number $\mathbf{rn}(f_{c,\kappa})$ is well defined whenever $|\kappa|\le 1$. The graph of $\mathbf{rn}(f_{c,1})$ as a function of c is shown in Figure 55. Here we have chosen the critical case $\kappa=1$, since it is the most extreme. However, we would obtain a qualitatively similar graph for any fixed κ satisfying $0<|\kappa|\le 1$. The essential features of this function $c\mapsto\mathbf{rn}(f_{c,\kappa})$ are the following: It is monotone and continuous (see 16.2 and 16.4). Furthermore, for each rational value $\mathbf{rn}(f_{c,\kappa})=p/q$ there is an entire interval of corresponding c-values, so that the graph exhibits a "plateau" at height p/q. (Compare 16.6 and 16.7.) Of course, in the limiting case $\kappa=0$ the map f is just a rotation by c, so these plateaus shrink to points.

We can get a better picture by plotting the locus consisting of all pairs $(c\,,\,\kappa)$ for which

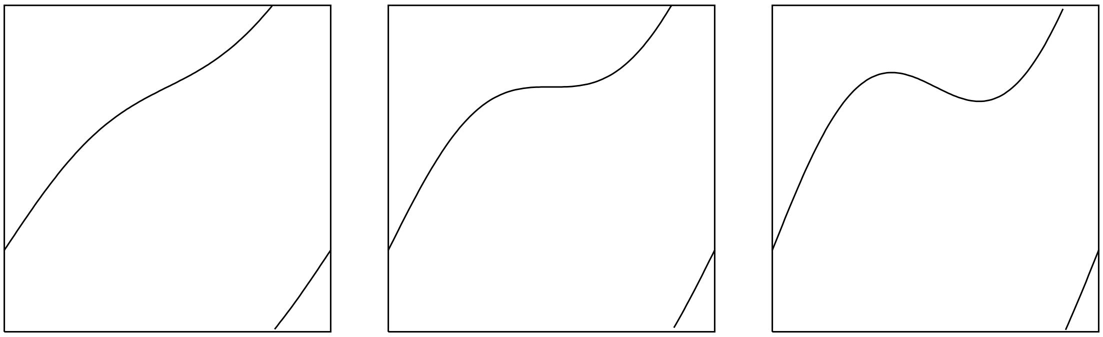

Figure 54. Graphs of the standard circle map f *for* $\kappa = .5$ *,* $\kappa = 1$ *, and* $\kappa = 1.5$ *respectively. Here* $c = .25$ *for all three graphs.*

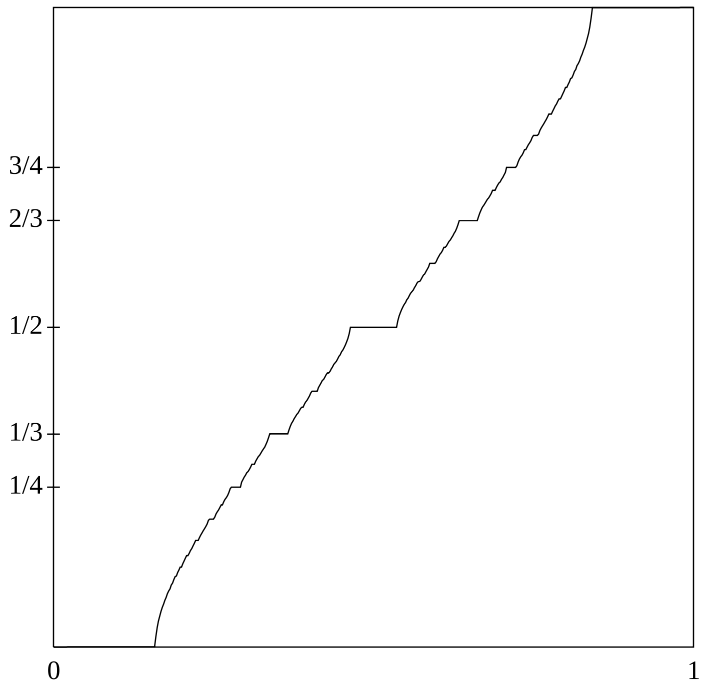

Figure 55. Graph of $\mathbf{rn}(f)$ *as a function of the parameter* c *in the critical case* $\kappa = 1$ *, with* $f(\xi) \;=\; \xi + c + \frac{1}{2\pi}\sin(2\pi\xi)$ *.*

$f_{c,\kappa}$ possesses an orbit of rotation number p/q . These wedge shaped regions, which taper down to points as we approach the line $\kappa = 0$, are sometimes known as *Arnold tongues*. (Compare Figure 56. More precisely, working in the universal covering space, the p/q-tongue can be defined as the set of all $(c, \kappa) \in \mathbb{R} \times \mathbb{R}$ such that p/q belongs to the translation interval $\mathbf{TI}(F_{c,\kappa})$, where $F_{c,\kappa}(x) = x + c + \kappa\sin(2\pi x)/2\pi$.)

It follows from Theorem 14.10 that these tongues are mutually disjoint as long as we restrict attention to the region $|\kappa| \le 1$ for which our maps are monotone. This is roughly the bottom 45% of Figure 56. However in the top part, where $|\kappa| > 1$, the maps are no longer monotone, and in fact the rotation number of f is often not uniquely defined, hence

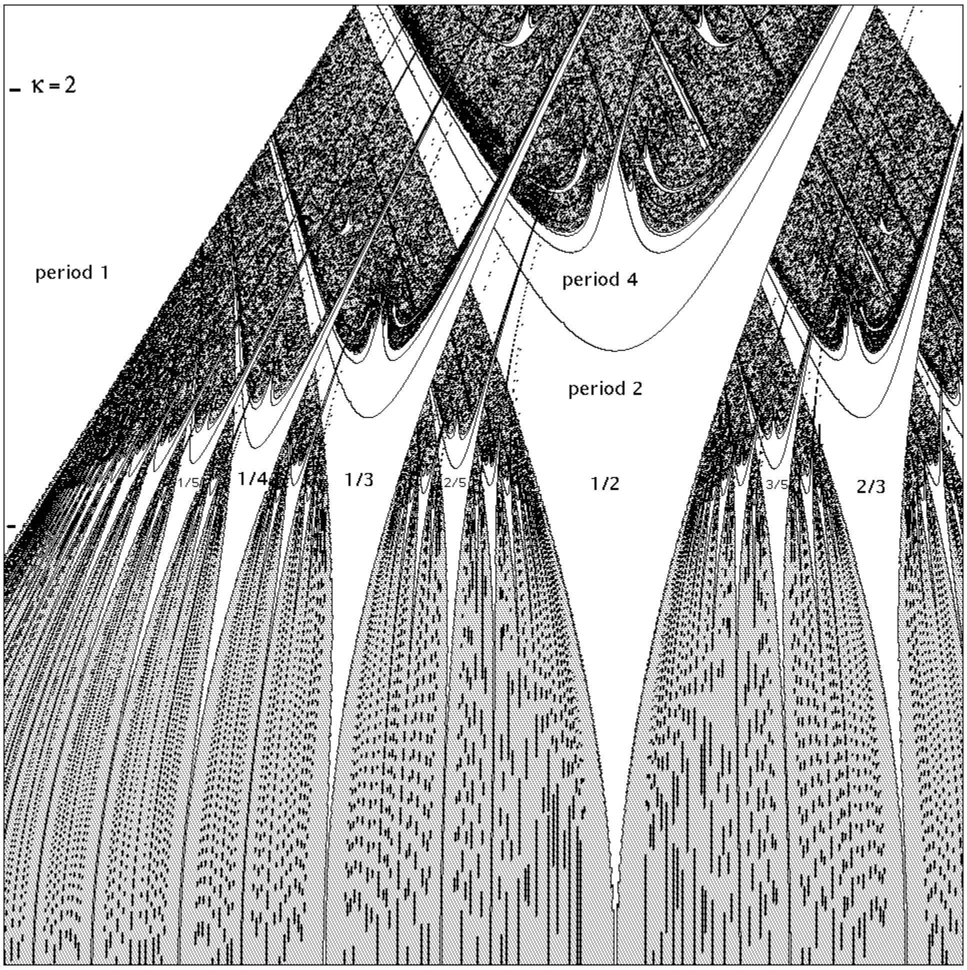

Figure 56. The rectangle $.15 \le c \le .7$ *and* $0 \le \kappa \le 2.2$ *in the* $(c\,,\,\kappa)$*-plane, showing regions with an attracting periodic orbit of rotation number* p/q *.*

these tongues may well cross each other. As an example, consider the map

$$F(x) \;=\; x + \tfrac{1}{2} + \tfrac{1}{2}\sin(2\pi x)\ .$$

This has $\{0\,,\,1/2\}$ as a period two orbit (modulo $\mathbb{Z}$), with translation number $1/2$. However, it also has $x = 3/4$ as a fixed point, with translation number zero, and $x = 1/4$ as fixed point (modulo $\mathbb{Z}$) with translation number $+1$. It follows from 14.6 that the corresponding circle map, with $c = 1/2$ and $\kappa = \pi$, belongs simultaneously to every Arnold tongue! (Compare the discussion of a linearly conjugate example following Figure 45.)

In fact the various tongues actually cross each other, for appropriate choice of c , whenever $\kappa > 1$.

Lemma 16.5. *For any* $\kappa > 1$ *, there exists a constant* c *so that the map* $F_{c,\kappa}$ *has non-degenerate translation interval, and hence belongs to infinitely many distinct Arnold tongues.*

Proof. Since the upper translation number $\mathbf{tn}^+(F_{c,\kappa})$ depends continuously on c , we can choose c so that it is irrational. It then follows from 15.4 that the translation interval $\mathbf{TI}(F_{c,\kappa})$ is non-degenerate, and hence contains infinitely many rational numbers. □

One way of understanding this statement intuitively is to consider the sum of the widths of all of these tongues,

$$w(\kappa) \;=\; \sum_{p/q\in\mathbb{Q}/\mathbb{Z}} \ell\{\, c \;\;;\;\; p/q \in \mathbf{TI}(F_{c,\kappa})\,\}\;, \tag{16:3}$$

as a function of the height κ. Here ℓ stands for *length* or one-dimensional *Lebesgue measure*. According to a Theorem of [Swiatek, 1988], as κ increases from 0 to 1, this total width increases from 0 to $w(1) = 1$. As κ increases past 1, the width of each Arnold tongue continues to increase, and there is no longer any room for the tongues to remain disjoint. In fact this sum (16 : 3) jumps discontinuously[2] from 1 to ∞ as κ increases past $+1$. For it follows from 14.6 that as soon as a map f belongs to two different Arnold tongues, it must automatically belong to infinitely many.

In the region $\kappa > 1$ the dynamics can be quite complicated. For example, as κ increases with $c \approx 1/2$ the period two orbit with rotation number $1/2$ "bifurcates" to a period 4 orbit, then to a period 8 orbit, and so on until we reach regions of "chaotic" behavior. (For further discussion see [Chavoya-Aceves et al.] as well as [Milnor, 1992].)

§**16C. Stability.** Let us return to the study of families of *monotone* circle maps, and ask whether we can change the rotation number by a small perturbation of f. In the rational case $\mathbf{tn}(F) = p/q$, we know by 14.6 that the function $F^{\circ q}(x) - x - p$ must have at least one zero. We can distinguish four cases according as this function is identically zero, takes only non-negative values, only non-positive values, or takes both positive and negative values.

Lemma 16.6. *Suppose that $\mathbf{tn}(F) = p/q$. If $F^{\circ q}(x) \geq x + p$ for all x, then we can increase the translation number by an arbitrarily small perturbation, while if $F^{\circ q}(x) \leq x + p$ for all x, then we can decrease the translation number by an arbitrarily small perturbation. However, if $F^{\circ q}(x) - x - p$ takes both positive and negative values, then $\mathbf{tn}(G) = \mathbf{tn}(F)$ for all G which are sufficiently close to F in the C^0-topology.*

This local constancy of the translation number in the last case provides an explanation for the "plateaus", or regions of constant rotation number, in Figure 55.

Proof of 16.6. First suppose that $F^{\circ q}(x) \geq x + p$ for all x. For any small real number η we can consider the perturbed map $G(x) = F(x) + \eta$. If $\eta > 0$, then since F is monotone an easy induction shows that $G^{\circ n}(x) \geq F^{\circ n}(x) + \eta$ for all $n \geq 0$. Hence $G^{\circ q}(x) > x + p$ for all x, and it follows that $\mathbf{tn}(G) > \mathbf{tn}(F) = p/q$. Further details of the proof are straightforward, and will be left to the reader. □

Remark 16.7. In the exceptional case when $F^{\circ q}(x)$ is identically equal to $x + p$, it follows that we can perturb so as to either increase or decrease the translation number. However, this case can never occur for a map $F(x) = x + c + \kappa \sin(2\pi x)/2\pi$ in the standard family, with $\kappa \neq 0$. To see this, note that such an F extends to a holomorphic function from $\mathbb{C}$ to $\mathbb{C}$ which is well defined but not one-to-one, since it is easy to check that F has

[2] Compare the infinite sum $\sum_1^\infty \kappa^n/n(n+1)$, which behaves similarly.

complex critical points. (Compare the proof of 16.9 below.) But if $F^{\circ q}(x)$ were identically equal to the linear function $x \mapsto x + p$, for x real, then it would follow by analytic continuation that $F^{\circ q}$ must be one-to-one throughout the complex numbers $\mathbb{C}$, which is impossible.

For irrational rotation number we will show that $\mathbf{rn}(f)$ can always be changed by an arbitrarily small perturbation of f. Again let f be monotone of degree one with lift F.

Lemma 16.8. *If* $\mathbf{tn}(F)$ *is irrational, then* $\mathbf{tn}(F + \eta) \neq \mathbf{tn}(F)$ *for* $\eta \neq 0$.

Proof. First consider an orbit $\xi'_0 \mapsto \xi'_1 \mapsto \cdots$ under the irrational rotation $\xi \mapsto \xi + \mathbf{rn}(f)$. Choose some arbitrarily large integer m. According to Lemma 14.14 we can choose $q > 0$ so that ξ'_q lies between ξ'_0 and $\xi'_0 + 1/m$. It follows that the $m+1$ points $\xi'_0, \xi'_q, \xi'_{2q}, \ldots, \xi'_{mq}$ lie in positive cyclic order. Now let $f : \xi_0 \mapsto \xi_1 \mapsto \cdots$ be some orbit under f. Using 14.11, it follows that the points $\xi_0, \xi_q, \xi_{2q}, \ldots, \xi_{mq}$ also lie in positive cyclic order. Since the distance from ξ_0 to ξ_{mq} in the positive direction around the circle is less than 1, it follows that we can choose $k < m$ so that the distance η between $\xi = \xi_{kq}$ and $f^{\circ q}(\xi) = \xi_{(k+1)q}$ in the positive direction satisfies $0 < \eta < 1/m$. Choose $x \in \mathbb{R}$ lying over this point ξ_{kq}, and define $p \in \mathbb{Z}$ by the equation $F^{\circ q}(x) = x + p + \eta$, Setting $G = F - \eta$, it follows that $G^{\circ q}(x) \leq F^{\circ q}(x) - \eta \leq x + p$, hence

$$\mathbf{tn}(G) \;=\; \mathbf{tn}(F - \eta) \;\leq\; p/q \;<\; \mathbf{tn}(F)\;.$$

The proof that $\mathbf{tn}(F + \eta) > \mathbf{tn}(F)$ is similar. $\square$

Recall from §5 that a periodic point $f^{\circ q}(\xi_0) = \xi_0$ is called ***attracting*** if it has a neighborhood whose successive images under $f^{\circ q}$ shrink down to ξ_0, and ***repelling*** if it has a neighborhood so that no orbit starting at a point $\xi \neq \xi_0$ can remain in the neighborhood forever. In the one-dimensional case, there is just one other possibility for an isolated fixed point of a monotone map: The point may be attracting from one side but repelling from the other. (A typical example of a one-sided attracting orbit for a circle map is provided by the fixed point $\xi = 0$ for the map $\xi \mapsto \xi + \sin^2(\pi\xi)/10$, which is attracting from the left but repelling on the right.)

If f is differentiable, it is useful to consider the derivative

$$\lambda \;=\; df^{\circ q}(\xi)/d\xi$$

at a periodic point $f^{\circ q}(\xi_0) = \xi_0$. By definition, this derivative is called the ***multiplier*** of the periodic orbit. Evidently an attracting orbit must have multiplier satisfying $|\lambda| \leq 1$, and a repelling orbit must have multiplier satisfying $|\lambda| \geq 1$, while a one-sided attracting orbit must have multiplier precisely equal to $+1$.

If f is monotone, then this multiplier must satisfy $\lambda \geq 0$. Note that the isolated period q point ξ_0 can then be classified as either ***attracting***, or ***repelling***, or ***one-sided attracting*** according as the the graph of $f^{\circ q}$ crosses from above the diagonal to below the diagonal at ξ_0, or from below to above, or fails to cross, remaining on one side of the diagonal.

We conclude this section with a more detailed study of the standard family of circle maps

$$f(\xi) \;=\; f_{c,\kappa}(\xi) \;=\; \xi + c + \kappa \,\sin(2\pi\xi)/2\pi\;.$$

If $\kappa \neq 0$, then since f is real analytic, it follows from 16.7 that any fixed point of $f^{\circ q}$

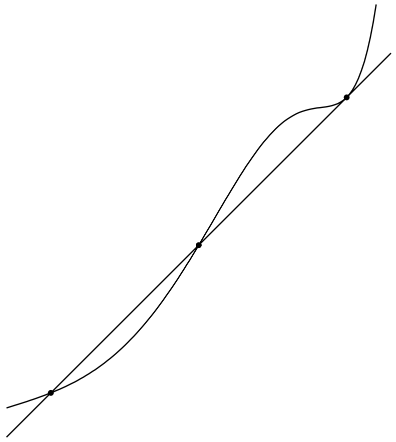

Figure 57. Graph of a map f with three fixed points which are respectively attracting, repelling, and one-sided attracting. The straight line is the graph of the identity map.

must be isolated. First consider the monotone case.

Theorem 16.9. *If $0 < |\kappa| \le 1$, and if this map f has rational rotation number p/q, then either f has exactly two periodic orbits, one attracting and one repelling, or else it has just one periodic orbit which is one-sided attracting.*

Proof. Note that the number of attracting period q orbits is equal to the number of times that the difference $F^{\circ q}(x) - x - p$ changes sign from $+$ to $-$ as x increases (modulo $\mathbb{Z}$). Evidently this is the same as the number of repelling period q orbits, where this difference changes sign from $-$ to $+$. Since there is at least one period q orbit by 14.6, it follows that there is at least one which is either attracting or one-sided attracting, and hence has multiplier $\lambda \in [0,1]$. Thus, to complete the proof, we need only show that there is at most one orbit with $\lambda \in [0,1]$.

The proof will make essential use of complex methods. First note that the circle map $f = f_{c,\kappa}$ extends naturally to an infinite-to-one holomorphic map

$$f(z) \;=\; z + c + \frac{\kappa}{2\pi}\sin(2\pi z)$$

from the cylinder $\mathbb{C}/\mathbb{Z}$ to itself. (Compare Remark 16.7 above.) We will only outline the theory of this map, which is described in more detail in [Keen] or [Kotus]. Using the substitution $w = e^{2\pi i z} \in \mathbb{C} \smallsetminus \{0\}$, we can write the derivative of f as

$$f'(z) \;=\; 1 + \kappa\cos(2\pi z) \;=\; 1 + \kappa\,(w + w^{-1})/2\ .$$

Thus the equation $f'(z) = 0$, reduces to a quadratic equation $w^2 + 2w/\kappa + 1 = 0$ provided that $\kappa \ne 0$, with just one solution if $\kappa = \pm 1$ and with exactly two solutions otherwise. Note also that $|f'(z)|$ tends to infinity as the imaginary part of z tends to $\pm\infty$, or in other words as $|w|$ or $1/|w|$ tends to infinity. In particular, $|f'|$ is bounded away from zero, outside a neighborhood of the two critical points. It follows easily that f has the following *path lifting property*: If $p : [0,1] \to \mathbb{C}/\mathbb{Z}$ is any smooth path avoiding the two critical values, then for any lift $f(z_0) = p(0)$ of its initial point there exists a unique path $\hat{p} : [0,1] \to \mathbb{C}/\mathbb{Z}$ which satisfies $f \circ \hat{p} = p$ with $\hat{p}(0) = z_0$. With just a little more work, one then sees that the map $f : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ is a branched covering, with exactly two branch points in general (but with only one branch point if $\kappa = \pm 1$, and no branch points if $\kappa = 0$).

By an immediate extension of the classical theory of Fatou and Julia, any attracting or parabolic periodic orbit in $\mathbb{C}/\mathbb{Z}$ can be located as the ω-limit set associated with one or more of these critical points. (By definition, a period q orbit of a holomorphic map is called *parabolic* if and only if the derivative λ of $f^{\circ q}$ at each orbit point is a root of unity. For a real periodic orbit, the only roots of unity which can occur are ± 1 .)

Still assuming that c and κ are real, we distinguish three cases as follows. If $|\kappa| > 1$ then the critical points in $\mathbb{C}/\mathbb{Z}$ are real and distinct, and there may well exist two distinct attracting periodic orbits. However, if $\kappa = \pm 1$ there is only one critical point, and if $0 < |\kappa| < 1$ then there are two complex conjugate critical points. In these last two cases, it follows that there can be at most one real periodic orbit which is attracting or parabolic, and hence has multiplier $\lambda \in [0, 1]$. □

Thus, we have also proved a result in the non-monotone case:

Lemma 16.10. *For $|\kappa| > 1$, at most two periodic orbits for the circle map $f = f_{c,\kappa}$ can have multiplier satisfying $|\lambda| \le 1$. Such an orbit with $|\lambda| \le 1$ can always be realized as the ω-limit set associated with one (or both) of the two critical points.*

By 16.5, we know that there may well be infinitely many periodic orbits, with periods q tending to infinity. According to 16.10, all but at most two of these orbits must be strictly repelling, with $|\lambda| > 1$.

Using the concept of *natural measure*, as developed in §3B, we can prove a rather different continuity result.

Theorem 16.11. *For $|\kappa| \le 1$, the map $f = f_{c,\kappa}$ possesses a unique natural measure μ_f . Furthermore, this measure depends continuously on the parameters c, κ in the following sense. For any continuous test function $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$, the correspondence*

$$(c\,,\,\kappa) \;\mapsto\; \int \phi \, d\mu_f \tag{16:4}$$

defines a continuous mapping from $(\mathbb{R}/\mathbb{Z}) \times [-1\,,\,1]$ to $\mathbb{R}$.

(In more technical language, the measure varies continuously provided that we give the space of all probability measures the "*weak* topology*", as in §10.) The proof will be divided into three cases, according as f has an attracting periodic orbit, a one-sided attracting periodic orbit, or has irrational rotation number.

Proof of existence and uniqueness. If f has one attracting orbit $\mathcal{O}$ and one repelling orbit $\mathcal{O}'$ (compare 16.9), then it is easy to see that every orbit $\xi_0 \mapsto \xi_1 \mapsto \cdots$ outside of $\mathcal{O}'$ must converge towards $\mathcal{O}$. Hence there is a unique natural measure μ_f , which is just the average of the Dirac measures concentrated at the various points of $\mathcal{O}$. If f has a one-sided attracting orbit $\mathcal{O}$, then *every* orbit must converge towards $\mathcal{O}$, so again there is a unique natural measure, concentrated at the points of $\mathcal{O}$. Finally, if the rotation number is irrational, then by 14.16 there exists a unique invariant probability measure μ_f , which is necessarily a natural measure.

Remark. In this last case, the measure μ_f is closely related to the Denjoy conjugating homeomorphism, $g = g_f$, as described in 14.13 and 15.1. For example, the push-forward $g_*(\mu_f)$ is equal to the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$. If g is C^1-smooth, then we can write this natural measure as

$$\mu_f(S) \;=\; \int_S dg(\xi) \;=\; \int_S g'(\xi)\, d\xi\;.$$

The proof that the map (16 : 4) is continuous at (c, κ) will again be divided into three cases.

Case A. If the map $f = f_{c,\kappa}$ has an attracting periodic orbit, then this orbit deforms continuously under perturbation of f, hence the associated natural measure also deforms continuously.

Case B. If f has only a one sided attracting orbit $\mathcal{O}$, of period $q \geq 1$ so that the graph of $f^{\circ q}$ is tangent to the diagonal along this orbit, then as shown in 16.6 above, we can perturb f so that this periodic orbit disappears. However, it is easy to see that it will still remain as a "*transient*". More precisely, we have the following statement: Given any neighborhood U of the orbit $\mathcal{O}$ and given any integer $N > 0$, there exists a neighborhood $\mathcal{F}$ of f with the following property. For any $g \in \mathcal{F}$, any g-orbit which *enters* the neighborhood U will remain within U for at least N iterations. On the other hand, there clearly exists an integer n_U which depends only on U, so that, for g close to f, any orbit which starts outside of U must enter U within at most n_U iterations. Now as g converges to f, we have $N \to \infty$, so the proportion of time that any orbit spends within U will be at least $N/(N + n_U)$, which converges to $+1$. Therefore, the associated natural measure can accumulate only on measures supported within the finite set $\mathcal{O}$. In fact it is not difficult to check that the q points of $\mathcal{O}$ must be weighted equally in the limit, as required. (Here $\mathbf{rn}(g)$ may well be either rational or irrational.)

Case C. If $c = \mathbf{rn}(f)$ is irrational, consider the corresponding rotation $\eta \mapsto \eta + c$. Given $\epsilon > 0$, choose q so that the distance around the circle from η to $\eta + qc$ is less than ϵ. Setting $c' \equiv q\,c$, choose $m > 1/\epsilon$ so that the points $0\,,\, c'\,,\, 2c'\,,\, \ldots\,,\, mc'$ lie in either positive or negative cyclic order, and cut the circle into m equal intervals plus one shorter interval. Then corresponding points $\xi_0\,,\, \xi_q\,,\, \xi_{2q}\,,\, \ldots\,,\, \xi_{mq}$ for an orbit $f : \xi_0 \mapsto \xi_1 \mapsto \cdots$ will also lie in positive or negative cyclic order, and will cut the circle into m intervals $I_0\,,\, I_1\,,\, \ldots\,,\, I_{q-1}$ with disjoint interiors, where $I_n = f^{\circ n}(I_0)$, together with one further interval which is properly contained in $f^{\circ q}(I_0)$. It follows that

$$1/(m+1) \;<\; \mu_f(I_0) \;=\; \cdots \;=\; \mu_f(I_{q-1}) \;<\; 1/m\;,$$

where μ_f denotes the natural measure for f. Now choose g so close to f that the first $(m+1)\,q$ points of the orbit $g : \xi_0 = \hat{\xi}_0 \mapsto \hat{\xi}_1 \mapsto \cdots$ will have the same cyclic order as the ξ_i. It follows that the circle can be cut up into m intervals

$$g \;:\; \widehat{I}_0 \;\mapsto \cdots \mapsto \widehat{I}_{q-1}$$

together with one complementary interval, where

$$1/(m+1) \;<\; \mu_g(\widehat{I}_0) \;=\; \cdots \;=\; \mu_g(\widehat{I}_{q-1}) \;<\; 1/m\;.$$

(Again $\mathbf{rn}(g)$ may well be either rational or irrational.) Furthermore, as g converges to f, each $\widehat{I}_j$ will converge to I_j. For any fixed test function $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$, it now follows easily that $\int \phi \, d\mu_g$ converges to $\int \phi \, d\mu_f$ as $g \to f$, as required. This completes the proof of 16.11. $\square$

§16D. Analytic Conjugacy and Herman Rings. To conclude this section, let us suppose that the rotation number is irrational, and discuss smoothness of the conjugating homeomorphism g, or equivalently of the natural measure μ_f. By definition, a measure on $\mathbb{R}/\mathbb{Z}$ is *singular* with respect to Lebesgue measure if it assigns full measure to some set having Lebesgue measure zero. (Compare 9.4.) For example, if $\mathbf{rn}(f)$ is rational, then the measure μ_f is concentrated in the finite set $\mathcal{O}$, and hence is certainly singular.

> **Lemma 16.12.** *For each $\kappa \neq 0$ in $[-1, 1]$, there exist uncountably many values of c so that the map $f = f_{c,\kappa}$ has irrational rotation number, and yet so that the measure μ_f on $\mathbb{R}/\mathbb{Z}$ is singular with respect to Lebesgue measure.*

(It follows that the conjugating homeomorphism g has derivative $g'(\xi)$ which is well defined and equal to zero for Lebesgue almost every ξ. All of the jumps in the value of g occur on a subset of Lebesgue measure zero.)

Proof of 16.12. In fact we will show that this behavior occurs for a *generic* choice of the rotation number $\mathbf{rn}(f)$. (Compare §4C.) To simplify the discussion, we will work with some fixed value of κ. First note that, for each $p/q \in \mathbb{Q}/\mathbb{Z}$ and for each $n \geq 1$ there exists a neighborhood $N_n(p/q)$ of p/q in $\mathbb{R}/\mathbb{Z}$ with the following property. *For any $f = f_{c,\kappa}$, if $\mathbf{rn}(f) \in N_n(p/q)$ then the natural measure μ_f assigns measure at least $1 - 1/n$ to a union of q intervals having total length $1/n$.* In fact this statement is certainly true when the rotation number is equal to p/q. It follows from 16.11 that it is also true for a map which is C^0-close to a map with rotation number p/q. But clearly any map with rotation number sufficiently close to p/q must be C^0-close to a map with rotation number p/q.

Now let U_n be the union over all p/q of the neighborhoods $N_n(p/q)$. Then a generic rotation number will belong to the intersection $\bigcap_n U_n$. If $\mathbf{rn}(f)$ belongs to this intersection, then for every n there exists a union $V(n)$ of intervals with total Lebesgue length $\ell(V(n)) = 1/n$ but with $\mu_f(V(n)) \geq 1 - 1/n$. Then $V(n) \cup V(n^2) \cup V(n^3) \cup \cdots$ has Lebesgue measure $\leq 1/(n-1)$, but has full measure under μ_f. Taking the intersection as $n \to \infty$, we obtain a set of Lebesgue measure zero having full measure under μ_f. $\square$

However, for *most* choices of rotation number, the behavior is rather different:

> **Theorem of Herman (1979).** *For Lebesgue almost every $\eta \in \mathbb{R}/\mathbb{Z}$ any real analytic circle diffeomorphism with rotation number η must actually be real analytically conjugate to a rotation.*

I will not attempt to give a proof.

Many variants or sharper versions of this statement are available. (Compare [Yoccoz, 1984, 1995], [Perez-Marco], and see the discussion in [de Melo, van Strien].) The basic principle is that rotation numbers which are very difficult to approximate by rational numbers correspond to smooth conjugating homeomorphisms. For example Roth in 1955, sharpening a much earlier result by Liouville, showed that for every irrational algebraic number η and for every real number $k > 2$ the products

$$q^k \left| \eta - \frac{p}{q} \right|$$

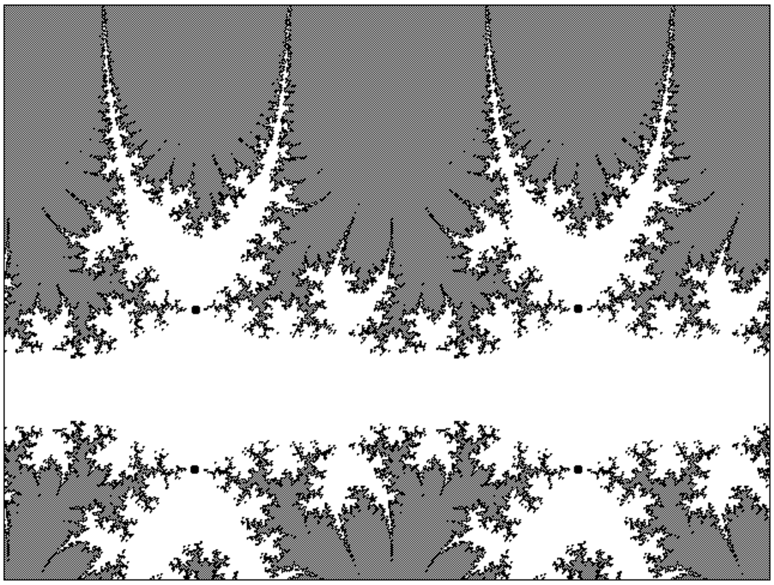

Figure 58. The Julia set (locus of sensitive dependence, shown in black) for the standard map

$$f(z) \;=\; z + c + 0.5\sin(2\pi z)/2\pi$$

in the cylinder $\mathbb{C}/\mathbb{Z}$, where the constant $c = 0.69985\cdots$ has been chosen so that the rotation number is $\sqrt{2}/2$. The region depicted is $[0\,,\,2] \times [-1/2\,,\,1]$; thus the left and right halves of this figure are to be identified by a unit translation. The critical points $n + 0.5 \pm (0.2096\cdots)\,i$ have been marked with black dots. Here the circle $\mathbb{R}/\mathbb{Z}$ forms an f-invariant subset of a Herman ring along the lower middle level of this figure. The complement of the Julia set (region of Liapunov stability, shown in white) consists of this Herman ring $\mathcal{H}$, together with countably many iterated preimages, each biholomorphic to the universal covering of $\mathcal{H}$ and extending infinitely far up or down in the cylinder.

are bounded away from zero. The conclusion of Herman's Theorem holds for any such η. By way of contrast, the rotation numbers constructed in 16.12 can be expressed as limits of very rapidly converging sequences of rational numbers.

As in the proof of 16.9, it is of interest to extend the standard circle map $f = f_{c,\kappa}$ to a map from the complex cylinder $\mathbb{C}/\mathbb{Z}$ to itself. The statement that the conjugating homeomorphism g is real analytic means that this homeomorphism extends as a biholomorphic map over some neighborhood of the circle $\mathbb{R}/\mathbb{Z}$ in $\mathbb{C}/\mathbb{Z}$. Thus we can find an entire annulus neighborhood where f is conjugate to a rotation. By definition, such a neighborhood is called a *Herman ring*. (Compare Figure 58.) In particular, it follows that f has no periodic points in an entire neighborhood of $\mathbb{R}/\mathbb{Z}$. On the other hand, in cases such as that described in 16.12 where there is no Herman ring, it follows from standard results in holomorphic dynamics that there are periodic points arbitrarily close to every point of $\mathbb{R}/\mathbb{Z}$. (See for example [Milnor 1999, §16 and §14].)