§12. Liapunov Exponents and the Subadditive Ergodic Theorem.

Suppose that M is a smooth compact manifold of dimension n, and that f is a C^1-smooth map from M to itself. Then the *derivative* of f at x is a well defined linear map $v \mapsto Df_x(v)$ from the tangent vector space T_xM to $T_{f(x)}M$. Now choose some Riemannian metric on M, and let $\|v\|$ be the Riemannian norm of the tangent vector $v \in T_xM$. Then $\|Df_x\|$ is defined as the norm of Df_x considered as a linear map between normed vector spaces,

$$\|Df_x\| \;=\; \max_{v\neq 0} \|Df_x(v)\|/\|v\| ,$$

where v varies over $T_xM \smallsetminus \{0\}$. Similarly, the norm of the derivative of the k-fold iterate $f^{\circ k}$ at x is well defined. Using these concepts, we can make the following definition.

§12A. The Largest Liapunov Exponent. If the limit

$$\lambda_1 \;=\; \lambda_1(x) \;=\; \lim_{k\to\infty} \tfrac{1}{k} \log \|Df_x^{\circ k}\|$$

exists, then it is called the largest *Liapunov exponent* of f at x.

Remarks. Since M is compact, these numbers are independent of the particular choice of Riemannian metric. (Compare Problem 12-a.) It is easy to see that λ_1 is always bounded from above by the numbers $\max_x \log\|Df_x\|$. If every Df_x is non-singular, then it is also bounded from below. However, if f has *critical points*, where the transformation Df_x sends some non-zero vectors to zero, then $\lambda_1(x)$ will take the improper value $-\infty$ for some x.

Later, in §12C, we will define other Liapunov exponents $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ at x. The largest Liapunov exponent λ_1 is the most important one, since it provides an extremely precise measure of sensitive dependence at the point x. If $\lambda_1 > 0$, then $\|Df_x^{\circ k}\|$ grows exponentially as $k \to \infty$, roughly like $e^{k\lambda_1}$, and it follows that the orbits of suitably chosen nearby points get pushed away from the orbit of x at roughly this rate. On the other hand, if $\lambda_1 < 0$, then we should expect nearby orbits to be pulled in towards the orbit of x.

Now suppose that we are given some f-invariant probability measure μ on the σ-algebra $\mathcal{B}_M$ of Borel subsets of the smooth manifold M. Here M may be non-compact, but we assume that the support of μ is compact. Following is a basic result of [Oseledets], as it applies to λ_1.

> **Theorem 12.1.** *The Liapunov exponent $\lambda_1(x)$ exists for $[\mu]$-almost every point $x \in M$. Furthermore, $\lambda_1(x)$ is measurable as a function of x, and $\lambda_1(x) = \lambda_1(f(x))$ almost everywhere.*

If the measure μ is ergodic, evidently it follows that $\lambda_1(x)$ is independent of x for $[\mu]$-almost every x.

Remark. Evidently $\lambda_1(x)$ depends only on the point x and the smooth map f. It has nothing to do with the choice of measure. However, in the case of an ergodic measure, this common value $\lambda_1(x)$ is highly dependent on which particular ergodic measure is chosen. (Recall from 9.6 that different choices of ergodic invariant measure give rise to completely incompatible meanings for the term "almost every".) As a rather trivial example, suppose that μ is the Dirac measure δ_{x_0} concentrated at a fixed point $x_0 = f(x_0)$. Let $\eta_1, \ldots, \eta_n \in \mathbb{C}$ be the eigenvalues of the linear transformation Df_{x_0}. Then one can show that $\lambda_1(x_0)$ is equal to the largest of the numbers $\log|\eta_j|$. (Compare A.3 in the Appendix.)

There is no reason for this to be equal to the Liapunov exponent for any other point of M ; yet by definition $\lambda_1(x)$ is equal to $\lambda_1(x_0)$ for $[\mu]$-almost every x when $\mu = \delta_{x_0}$.

For the proof of 12.1, first consider the one-dimensional case. Let $\varphi(x) = \log\|Df_x\|$. Then for any orbit $x_0 \mapsto x_1 \mapsto \cdots$ it follows from the chain rule that

$$\log\|Df_x^{\circ k}\| \;=\; \varphi(x_0) + \varphi(x_1) + \cdots + \varphi(x_{k-1})$$

is just the k-fold Birkhoff sum for the function φ along the orbit of x_0 . Hence, dividing by k and taking the limit as $k \to \infty$, we see that $\lambda_1(x_0)$ is equal to the time average $A_\varphi(x_0)$, as studied in §9. If the function $|\varphi|$ is bounded, or more generally if $\int|\varphi|\,d\mu < \infty$, then it follows immediately from Birkhoff's Theorem then $\lambda_1(x)$ is defined μ-almost everywhere. Clearly $\lambda_1(x) = \lambda_1(f(x))$ (except in the special case where x is a critical point). In the ergodic case, it follows that $\lambda_1(x)$ is constant almost everywhere.

Examples. For the angle doubling map on the circle, using the obvious Riemann metric, we see that $\varphi(x) = \log\|Df_x\| \equiv \log 2$, and it follows easily that $\lambda_1(x) = \log 2$ for all x . Now consider the quadratic interval map $q(x) = 2x^2 - 1$ of §2A. Using the nearly smooth semi-conjugacy from the angle doubling map onto this quadratic map, as described in §2B, it follows that the Liapunov exponent of q is also equal to $\log 2$ almost everywhere. (See Problem 12-b.) On the other hand, there are certainly exceptional points. For example, at the fixed point $x = 1$ and its pre-image $x = -1$, the Liapunov exponent is $\log|q'(1)| = \log(4)$, while at the critical point 0 and its countably many iterated pre-images the Liapunov exponent is $-\infty$. For other examples with similar behavior, see Problem 12-c.

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§**12B. The Subadditive Ergodic Theorem.** This proof of 12.1 breaks down in higher dimensions. The chain rule in any dimension takes the form

$$D(f\circ g)_x \;=\; Df_{g(x)} \circ Dg_x \;,$$

which implies that

$$\|D(f\circ g)_x\| \;\le\; \|Df_{g(x)}\|\,\|Dg_x\| \;.$$

However, equality need not hold if the dimension is two or more. If we substitute $f^{\circ k}$ in place of f and $f^{\circ\ell}$ in place of g , and set

$$\varphi_k(x) \;=\; \log\|Df_x^{\circ k}\| \;, \qquad (12:1)$$

then it follows that $\varphi_{k+\ell}(x) \le \varphi_k(f^{\circ\ell}(x)) + \varphi_\ell(x)$, or briefly

$$\varphi_{k+\ell}(x_0) \;\le\; \varphi_k(x_\ell) \;+\; \varphi_\ell(x_0) \qquad (12:2)$$

for any orbit $x_0 \mapsto x_1 \mapsto \cdots$. Such a sequence of real valued functions φ_k satisfying (12:2) is sometimes called a *subadditive cocycle* for the dynamical system $(M\,,\,f)$.

The following generalization of Birkhoff's Theorem is due to [Kingman]. Let $(X, \mathcal{A}, \mu)$ be a probability space, and $f : X \to X$ a measurable function. To simplify the discussion, we consider only bounded real valued functions on X . In fact, we adopt the sharper hypothesis that $|\varphi_k(x)|$ grows at most linearly with k ,

$$|\varphi_k(x)| \;\le\; k\,B \qquad \text{for all} \quad x \in X \quad \text{and all} \quad k \ge 1 \;, \qquad (12:3)$$

where B is some fixed bound. Evidently this sharper hypothesis is always satisfied if φ_k is defined by $(12:1)$ on a compact manifold, provided that the smooth map f has no critical points, or in other words is locally a diffeomorphism.

Note that we can apply Birkhoff Theorem 9.1 to each of the measurable functions φ_k. Thus the time averages $A_{\varphi_k}(x)$ are defined almost everywhere, and are measurable, and f-invariant, with $|A_{\varphi_k}(x)/k| \le B$. Furthermore, it follows from (12 : 2) that

$$A_{\varphi_{k+\ell}} \le A_{\varphi_k} + A_{\varphi_\ell}$$

wherever these functions are defined. Proceeding as in §7.6, we see that the limit

$$\Phi(x) \;=\; \lim_{k\to\infty} A_{\varphi_k}(x)/k \;=\; \inf_{k\ge 1}\{A_{\varphi_k}(x)/k\} \tag{12 : 4}$$

exists whenever the $A_{\varphi_k}(x)$ are defined. Clearly $|\Phi(x)| \le B$ and $\Phi(x) = \Phi\circ f(x)$ wherever Φ is defined. Using 8.4, we see that this function Φ is measurable.

Theorem 12.2. Subadditive Ergodic Theorem. *If μ is an f-invariant probability measure on $(X, \mathcal{A})$, and if $\varphi_1, \varphi_2, \ldots$ are bounded measurable functions satisfying the inequalities* (12 : 2) *and* (12 : 3)*, then for $[\mu]$-almost every x the sequence of ratios $\varphi_k(x)/k$ converges, with*

$$\lim_{k\to\infty} \varphi_k(x)/k \;=\; \Phi(x)\ . \tag{12 : 5}$$

In particular, the function $x \mapsto \lim \varphi_k(x)/k$ is well defined, bounded, measurable, and f-invariant almost everywhere.

Evidently Theorem 12.1 will follow immediately from this statement, together with the discussion above.

Proof of 12.2. The proof will be based on [Katznelson and Weiss]. First note that the upper and lower limits

$$\overline{\varphi}(x) \;=\; \limsup_{k\to\infty}\, \varphi_k(x)/k \qquad \text{and} \qquad \underline{\varphi}(x) \;=\; \liminf_{k\to\infty}\, \varphi_k(x)/k$$

are well defined, measurable, and bounded between B and $-B$. To show that they are f-invariant almost everywhere we proceed as follows. Since

$$\varphi_{k+1}(x) \;\le\; \varphi_k(f(x)) \;+\; \varphi_1(x)$$

by (12 : 2), with φ_1 bounded, it follows that $\overline{\varphi}(x) \le \overline{\varphi}(f(x))$. Therefore,

$$\int \overline{\varphi} \;\le\; \int \overline{\varphi}\circ f \;=\; \int \overline{\varphi}\ .$$

Thus the functions $\overline{\varphi} \le \overline{\varphi}\circ f$ have the same integral over X, and hence must be equal almost everywhere. The argument for $\underline{\varphi}$ is similar.

Next note that

$$\overline{\varphi}(x) \;\le\; \Phi(x) \qquad \text{whenever} \quad \Phi(x) \quad \text{is defined}\,. \tag{12 : 6}$$

where the function $\Phi(x)$ is defined by (12 : 4). To prove this, fix some integer $n \ge 1$. For any fixed $1 \le i \le n$ and for an arbitrarily large k, we can set $k = r + nm + i$ with remainder $0 \le r < n$. Then

$$\varphi_k(x_0) \;\le\; \varphi_r(x_{mn+i}) \;+\; \left(\sum_{\ell=0}^{m-1} \varphi_n(x_{\ell n+i})\right) \;+\; \varphi_i(x_0)$$

by repeated application of (12 : 2), where $x_j = f^{\circ j}(x_0)$. Now sum from $i = 1$ to n and then divide by k . Taking the lim sup as $k \to \infty$ (and hence as $m \to \infty$), this yields

$$n\,\overline{\varphi}(x_0) \;\le\; 0 + A_{\varphi_n}(x_0) + 0\;.$$

Dividing by n and letting $n \to \infty$, this yields $\overline{\varphi}(x) \le \Phi(x)$, as asserted.

To complete the proof, we will show that

$$\int \Phi \;\le\; \int \underline{\varphi}\;. \tag{12 : 7}$$

Since $\underline{\varphi} \le \overline{\varphi} \le \Phi$, this will imply that $\underline{\varphi} = \overline{\varphi} = \Phi$ almost everywhere, as required. To prove this inequality (12 : 7), we first fix some small number $\epsilon > 0$. For any $x \in X$ there must be infinitely many integers $n \ge 1$ such that $\varphi_n(x)/n \le \underline{\varphi}(x) + \epsilon$. Choosing some positive integer N , let E_N be the set of all points x such that

$$\varphi_n(x)/n \;\le\; \underline{\varphi}(x) + \epsilon \tag{12 : 8}$$

for some integer n with $1 \le n \le N$. Now choose integers $n(x)$ and real numbers $\Psi(x)$ as follows:

For $x \in E_N$, let $n(x)$ be the smallest n satisfying (12 : 8), and set $\Psi(x) = \underline{\varphi}(x) + \epsilon$.

For $x \notin E_N$, choose $n(x) = 1$, and set $\Psi(x) = B + \epsilon$.

Since $\underline{\varphi}(x) \le B$ is f-invariant, it follows easily that

$$\varphi_{n(x)}(x) \;\le\; n(x)\,\Psi(x) \;\le\; \sum_{i=0}^{n(x)-1} \Psi(f^{\circ i}(x)) \tag{12 : 9}$$

in all cases. We now imitate the proof of 9.2. Starting with any point $x_0 \in X$, define an infinite sequence of integers

$$0 = i_0 \;<\; i_1 \;<\; i_2 \;<\; \cdots$$

by setting $i_{k+1} = i_k + n_k$ where $n_k = n(x_{i_k})$, with $1 \le n_k \le N$. Then (12 : 9) takes the form

$$\varphi_{n_k}(x_{i_k}) \;\le\; n_k\,\Psi(x_{i_k}) \;\le\; \sum_{i_k \le j < i_{k+1}} \Psi(x_j)\;.$$

It then follows easily from (12 : 2) that

$$\varphi_{i_k}(x_0) \;\le\; \varphi_{n_0}(x_{i_0}) + \varphi_{n_1}(x_{i_1}) + \cdots + \varphi_{n_{k-1}}(x_{i_{k-1}}) \;\le\; \sum_{0 \le j < i_k} \Psi(x_j)\;.$$

More generally, for any large integer L we can set $L = i_k + r$ with remainder $0 \le r < N$. Since the differences

$$\varphi_L(x_0) - \varphi_{i_k}(x_0) \;\le\; \varphi_r(x_{i_k}) \qquad \text{and} \qquad \Big|\sum_{i_k \le j < L} \Psi(x_j)\Big|$$

are both uniformly bounded, it follows that

$$\varphi_L(x_0) \;\le\; \sum_{0 \le j < L} \Psi(x_j) + B'\;,$$

where B' is a uniform bound. Integrating with respect to $d\mu(x_0)$, this yields

$$\int \varphi_L \;\le\; L \int \Psi + B'\;.$$

(The integral of $\Psi(x_j)$ is equal to the integral of $\Psi(x_0)$ since our measure μ is f-invariant.) Now dividing by L and passing to the limit as $L \to \infty$, since $B'/L \to 0$, this shows that the quantity

$$\lim_{L\to\infty} \int \varphi_L/L \;=\; \lim \int A_{\varphi_L}/L = \int \Phi$$

is less than or equal to the integral of Ψ,

$$\int_X \Phi \;\le\; \int_X \Psi\,.$$

Next note that

$$\int_X \Psi \;=\; \epsilon + \int_{E_N} \underline{\varphi} \;+\; \int_{X \smallsetminus E_N} B \;\;\le\;\; \epsilon + \int_X \underline{\varphi} \;+\; 2B\,\mu(X \smallsetminus E_N)\,.$$

But for fixed ϵ it is easy to check that we can make the measure $\mu(X \smallsetminus E_N)$ arbitrarily small by choosing N sufficiently large. This proves that

$$\int_X \Phi \;\le\; \int_X \Psi \;\le\; \epsilon + \int_X \underline{\varphi}\,.$$

Since ϵ can be arbitrarily small, it proves that $\int \Phi \le \int \underline{\varphi}$, as required. $\square$

12C. Some Linear Algebra. Before defining the full sequence of Liapunov exponents $\lambda_1(x) \ge \lambda_2(x) \ge \cdots \ge \lambda_n(x)$ for a smooth map from a compact Riemannian n-manifold to itself, we will need some elementary constructions from linear algebra.

Recall that the tangent space T_xM of a Riemannian manifold M at a point x is a Euclidean vector space. That is, it is a finite dimensional real vector space with a symmetric bilinear inner product $v\,,\, v' \mapsto v \cdot v' \in \mathbb{R}$, and with a positive definite norm $\|v\| = \sqrt{v \cdot v}$. If V and W are n-dimensional Euclidean vector spaces, then we can measure the "size" of a linear map $L : V \to W$ by n quantities

$$\mathcal{S}_1(L) \;\ge\; \cdots \;\ge\; \mathcal{S}_n(L) \;\ge\; 0$$

which are constructed as follows. First note that:

> *For any linear map $L : V \to W$, there exists an orthonormal basis $e_1, \ldots, e_n$ for V so that the images $L(e_i)$ are mutually orthogonal vectors in W.*

To find such a basis, start with any unit vector $e_1 \in V$ for which the length $\|L(e_1)\|$ attains its maximum. It follows by straightforward calculus that the orthogonal complement of e_1 in V maps to the orthogonal complement of $L(e_1)$ in W. The required basis is then constructed by induction. Note that the basis constructed in this way will automatically be ordered so that $\|L(e_1)\| \ge \|L(e_2)\| \ge \cdots \le \|L(e_n)\| \ge 0$. Given such a basis, we set

$$\mathcal{S}_i \;=\; \mathcal{S}_i(L) \;=\; \|L(e_i)\|\,, \qquad \text{with} \qquad \mathcal{S}_1 \;\ge\; \mathcal{S}_2 \;\ge\; \cdots \;\ge\; \mathcal{S}_n \;\ge\; 0\,.$$

Geometrically, this definition can be described as follows: *The linear map L carries the unit sphere in V onto a (possibly degenerate) ellipsoid in W with semi-axes $\mathcal{S}_1, \ldots, \mathcal{S}_n$.* Note that the largest semi-axis $\mathcal{S}_1(L)$ is just the norm $\|L\|$ of the linear map L, as used in §12A.

Another way of describing these numbers $\mathcal{S}_i$ is the following. Since V and W are Euclidean vector spaces, the map $L: V \to W$ has an *adjoint map* $L^*: W \to V$, where

$$L(v) \cdot w \;=\; v \cdot L^*(w)$$

for all $v \in V$ and $w \in W$. The composition $L^* \circ L$ is always a self-adjoint map from V to itself, with eigenvalues $\mathcal{S}_i^2$, in fact, if $L(e_i) = \mathcal{S}_i\, e'_i$, then $L^*(e'_i) = \mathcal{S}_i\, e_i$, hence $L^*(L(e_i)) = \mathcal{S}_i^2\, e_i$.

We can now state the main Theorem. Just as in §12.1, we assume that M is a compact n-dimensional Riemannian manifold and that $f: M \to M$ is a smooth map. Thus for each iterate $f^{\circ k}: M \to M$ the derivative $Df^{\circ k}_{x_0}$ is a linear map from the Euclidean vector space $T_{x_0}M$ to $T_{x_k}M$, where $x_k = f^{\circ k}(x_0)$. Again we assume that μ is an invariant Borel probability measure on M.

Theorem 12.3 (Oseledets). *For each $1 \le i \le n$, the limit*

$$\lambda_i(x) \;=\; \lim_{k\to\infty} \tfrac{1}{k}\, \log \mathcal{S}_i(Df^{\circ k}_x)$$

exists $[\mu]$-almost everywhere. Furthermore, the function $x \mapsto \lambda_i(x)$ is measurable, with $\lambda_i(f(x)) = \lambda_i(x)$ almost everywhere.

Again, if f has critical points, then $\lambda_i(x)$ may take the improper value $-\infty$. However, we only give a proof only in the case where f has no critical points. If μ is ergodic, it follows again that $\lambda_i(x)$ is independent of x almost everywhere.

The proof of 12.3 will be based on still another description of the quantities $\mathcal{S}_i(L)$. Recall that the m-th *exterior power* of an n-dimensional vector space W is a vector space $\bigwedge^m W$ of dimension $n(n-1)\cdots(n-m+1)/m!$ with the following properties. (Reference ??.) To any m vectors $w_1, \ldots, w_m \in W$ there is associated a m-linear skew-symmetric product

$$w_1 \wedge \cdots \wedge w_m \;\in\; \textstyle\bigwedge^m W\,.$$

Furthermore, if $\{e_1, \ldots, e_n\}$ is any basis for W, then the products $e_{i_1} \wedge \cdots \wedge e_{i_m}$ with $1 \le i_1 < \cdots < i_m \le n$ form a basis for $\bigwedge^m W$. Finally, if W is a Euclidean vector space, then $\bigwedge^m W$ is also Euclidean, with an inner product determined by the identity

$$(w_1 \wedge \cdots \wedge w_m) \cdot (w'_1 \wedge \cdots \wedge w'_m) \;=\; \det\,[w_i \cdot w'_j]\,.$$

Now consider a linear map $L: V \to W$ between Euclidean vector spaces. Then there is an induced linear map $\bigwedge^m L: \bigwedge^m V \to \bigwedge^m W$ which is defined by

$$v_1 \wedge \cdots \wedge v_m \;\mapsto L(v_1) \wedge \cdots \wedge L(v_m)\,.$$

If $\mathcal{S}_1 \ge \cdots \ge \mathcal{S}_n$ are the semi-axes associated with the map L, then it is easy to check that the various products $\mathcal{S}_{i_1} \cdots \mathcal{S}_{i_m}$ with $1 \le i_1 < \cdots < i_m \le n$ are the semi-axes associated with the map $\bigwedge^m L$. In particular, the largest semi-axis is given by the product

$$\|\textstyle\bigwedge^m L\| \;=\; \mathcal{S}_1(L) \cdots \mathcal{S}_m(L)\,. \qquad (12:11)$$

Proof of 12.3. As noted earlier, we only consider the case where $f: M \to M$ has no critical points. Consider the m-th exterior power of the derivative map

$$Df^{\circ k}_{x_0}: T_{x_0}M \;\to\; T_{x_k}M\,.$$

Arguing just as in the proof of 12.1, we see that the limit

$$\lim_{k\to\infty} \tfrac{1}{k} \log \|\bigwedge^m Df_x^{\circ k}\|$$

exists almost everywhere, and is f-invariant. Using $(12:11)$, this limit can be written as

$$\lim_{k\to\infty} \tfrac{1}{k} \sum_{i=1}^{m} \log \mathcal{S}_i(Df_x^{\circ k}) .$$

It then follows by induction on m that each limit

$$\lambda_m(x) \;=\; \lim_{k\to\infty} \tfrac{1}{k} \log \mathcal{S}_m(Df_x^{\circ k})$$

also exists. □

Remark 12.4. Smooth Volume Forms. On an n-dimensional manifold, the n-th exterior power $\bigwedge^n T_xM$ is a one-dimensional vector space, and the norm $\|\bigwedge^n Df_x\|$ is proportional to the absolute value of the Jacobian determinant. If the invariant measure μ comes from a smooth volume form (compare §12), then we can choose the Riemannian metric so that the function $x \mapsto \|\bigwedge^n Df_x\|$ is identically equal to $+1$. It then follows that the sum $\lambda_1 + \cdots + \lambda_n$ is identically zero.

Remark 12.5. Asymptotic Measures. One situation which may arise in applications is the following. Suppose that there is an asymptotic measure ν associated with some initial measure class $[\mu_0]$. (Compare §3B.) By definition this means that, for $[\mu_0]$-almost every point x_0, the orbit of x_0 is ν-evenly distributed. Now suppose that we forget about μ_0, but simply consider an arbitrary point x_0 whose orbit is ν-evenly distributed. By definition, this means that the time average $A_\varphi(x_0)$ is equal to $\int \varphi\, d\nu$ for every continuous φ. In particular, since the functions

$$\varphi_h(x) \;=\; \log \|Df_x^{\circ h}\|$$

are continuous, it follows that the time average $A_{\varphi_h}(x_0)$ is defined and equal to $\int \varphi_h\, d\nu$. Therefore, assuming as usual that φ_h/h is uniformly bounded so that we can apply the Lebesgue dominated convergence theorem, we see that the limit $\Phi(x_0) = \lim_{h\to\infty} A_{\varphi_h}(x_0)/h$ is given by

$$\Phi(x_0) \;=\; \lim_{h\to\infty} \left(\int \varphi_h\, d\nu/h\right) \;=\; \int \left(\lim_{h\to\infty} \varphi_h/h\right)\, d\nu \;=\; \int \lambda_1\, d\nu .$$

On the other hand, it follows from $(12:6)$ that the "upper Lyapunov exponent"

$$\overline{\lambda}_1(x_0) \;=\; \limsup_{k\to\infty} \varphi_k(x_0)/k$$

satisfies $\overline{\lambda}_1(x_0) \le \Phi(x_0)$. Thus we have proved that

$$\overline{\lambda}_1(x_0) \;\le\; \int \lambda_1\, d\nu .$$

If ν is ergodic, it follows that $\overline{\lambda}_1(x_0) \le \lambda_1(x)$ for $[\nu]$-almost every x.

Similarly, for each $1 \le m \le n$, using the m-th exterior power, we can define an upper limit

$$\overline{(\lambda_1 + \cdots + \lambda_m)}\,(x_0) \;=\; \limsup_{k\to\infty} \frac{1}{k} \sum_{i=1}^{m} \log \mathcal{S}_i(Df_{x_0}^{\circ k}) .$$

If the orbit of x_0 is ν-evenly distributed, an analogous argument shows

$$\overline{(\lambda_1+\cdots+\lambda_m)}\,(x_0) \;\le\; \int(\lambda_1+\cdots+\lambda_m)\,d\nu\ .$$

For $m=n$, since the n-th exterior power is one-dimensional, we can apply the Birkhoff Theorem (instead of the Subadditive Ergodic Theorem) and get the much sharper statement that $(\lambda_1+\cdots+\lambda_n)\,(x_0)$ is defined and equal to the average of $\lambda_1+\cdots+\lambda_n$. In particular, in the 1-dimensional case, $\lambda_1(x_0)=\int\lambda_1\,d\nu$. However, when $n>1$, I don't know how to prove that any $\overline{\lambda}_k(x_0)=\underline{\lambda}_k(x_0)$ so that $\lambda_k(x_0)$ is actually defined.

Some Problems.

Problem 12-a. For any smooth map on a compact Riemannian manifold, show that the Liapunov exponents do not depend on the particular choice of Riemannian metric.

Problem 12-b. For the quadratic map $q(x)=2x^2-1$ on the interval $|x|\le 1$, we can make the substitution $x=\cos(\pi t)$ for $0\le t\le 1$ to obtain an ergodic invariant probability measure

$$|dt| \;=\; \frac{dx}{\pi\sqrt{1-x^2}}\ .$$

(Compare §2B.) Show that the Liapunov exponent λ_1 takes the constant value

$$\lambda_1(x) \;=\; \int_{x=-1}^{1}\log|q'(x)|\,|dt(x)| \;=\; \int_{t=0}^{1}\log|4\cos(\pi t)|\,dt \qquad (12:12)$$

almost everywhere. Using the identity

$$2\sin(2\theta) \;=\; 2\sin(\theta)\,2\cos(\theta)\ ,$$

show that

$$\int_0^1\log|2\cos(\pi t)|dt \;=\; \int_0^1\log|2\sin(\pi t)|\,dt \;=\; 0\ ,$$

and conclude that the definite integral $(12:12)$ takes the value $\log 2$. Show, again using the coordinate t, that

$$\frac{dx_k}{dx_0} \;=\; \pm 2^k\sqrt{\frac{1-x_k^2}{1-x_0^2}}$$

for any orbit $x_0\mapsto x_1\mapsto\cdots$. Conclude that the upper Liapunov exponent takes the constant value $\overline{\lambda}_1(x_0)=\log 2$, except on the countable dense set consisting of points whose orbit $x_0\mapsto x_1\mapsto\cdots$ satisfies $x_k=1$ for large k. However, show that there is a much larger set of measure zero consisting of points whose orbit accumulates so closely on the fixed point $x=1$ that the lower Liapunov exponent $\underline{\lambda}_1(x)$ is strictly less than $\log 2$.

Problem 12-c. For any $0<a<1$, consider the modified tent map

$$f(x) \;=\; \begin{cases} x/a & \text{for}\quad 0\le x\le a\\ (1-x)/(1-a) & \text{for}\quad a\le x\le 1\,.\end{cases}$$

(This is not smooth at the single point $x=a$, but that doesn't matter for ergodic theory.) Check that the standard Lebesgue measure dx is invariant, so that the average Liapunov exponent $\int_0^1 \lambda_1(x)\,dx$ is defined and equal to

$$\int_0^1 \log|f'(x)|\,dx \;=\; a\,\log(1/a) + (1-a)\,\log(1/(1-a))\;. \qquad (12:13)$$

(Up to a set of measure zero, this dynamical system is measure theoretically isomorphic to the shift of two symbols, with a probability measure as described in §3.8. This system is ergodic, so the expression (12 : 13) is actually equal to $\lambda_1(x)$ for Lebesgue almost every x.) If we choose any parameter $a \ne 1/2$, this expression is definitely less than $\log 2$. In particular, the expression (12 : 13) definitely depends on the choice of a, although it can be shown that these maps are all topologically conjugate to each other. Show that there are many exceptional points with different Liapunov exponents. For example, if a is irrational, show that every periodic point has a Liapunov exponent which is different from the average $\int_0 \lambda_1(x)\,dx$.

Problem 12-d. For the map $F_1(z,w) = (z^2\,,\,z+\alpha\,w)$ on $S^1 \times \mathbb{C}$, with the dyadic solenoid as global attractor (compare §2E), show that the Liapunov exponents are defined everywhere, and take the constant values

$$\lambda_1 \;=\; \log 2\,, \quad \lambda_2 \;=\; \lambda_3 \;=\; \log|\alpha|\;.$$

Problem 12-e. For any linear map L from $\mathbb{R}^n$ to itself, show that the limits

$$\lim_{k\to\infty} \tfrac{1}{k}\,\log \mathcal{S}_k(L^{\circ k})$$

are defined and equal to the logarithms of the absolute values of the eigenvalues of L. (Compare Appendix B.) For a linear map of the torus $\mathbb{R}^n/\mathbb{Z}^n$, conclude that the Liapunov exponents are defined everywhere, and equal to the logarithms of the absolute values of the eigenvalues.

Problem 12-f. If f is a complex analytic mapping from a compact complex manifold to itself, show that the Liapunov exponents are equal in pairs,

$$\lambda_1 = \lambda_2 \;\ge\; \lambda_3 = \lambda_4 \;\ge\; \ldots \;\ge\; \lambda_{2n-1} = \lambda_{2n}\;.$$

(**Remark:** If f is a symplectic map from a compact symplectic $2n$-manifold to itself, it seems likely that the Liapunov exponents are opposite in pairs,

$$\lambda_1 + \lambda_{2n} = 0\,,\; \lambda_2 + \lambda_{2n-1} = 0\,,\; \ldots\,,\; \lambda_n + \lambda_{n+1} = 0\;.$$

In the 2-dimensional case this is true by 12.4, but I don't know whether it is true in higher dimensions.)