Remarks on Piecewise Monotone Maps
Corrected Version

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Bremen, August, 2015
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When running this file in firefox the movies will display if you click the indicated button.

PM-maps $f : (\mathcal{I}, \partial\mathcal{I}) \to (\mathcal{I}, \partial\mathcal{I})$ where $\mathcal{I} = [0, 1]$.

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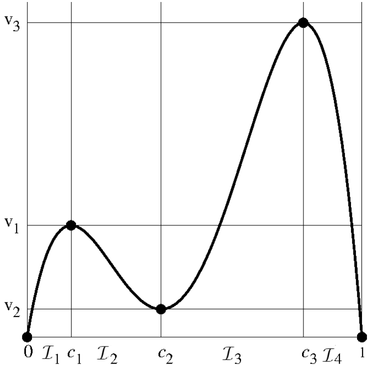

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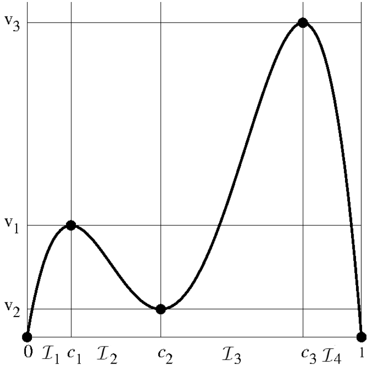

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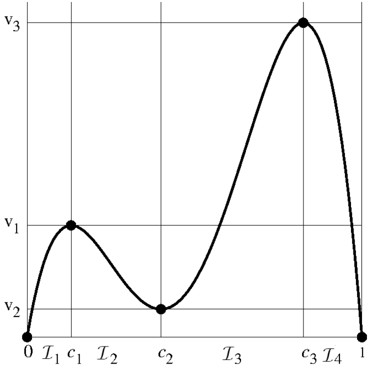

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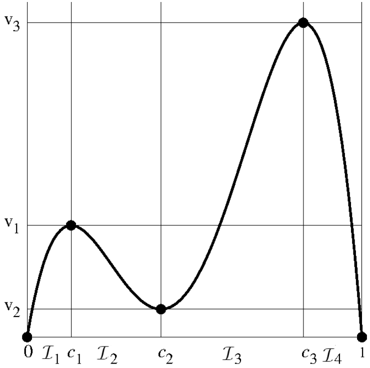

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Caution: In this talk the word “critical” will be used to mean local maximum or minimum point. Inflection points are not “critical”.

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Theorem. *Given a PM-map* $f(x)$ *with critical value vector* $(v_0, v_1, \ldots, v_d)$ *, there is one and only one polynomial PM-map* $g(x)$ *of degree* d *with the same critical value vector.*

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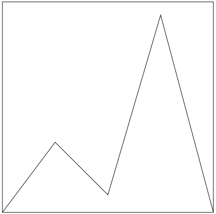

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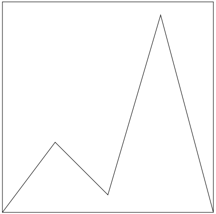

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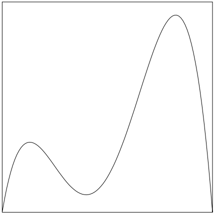

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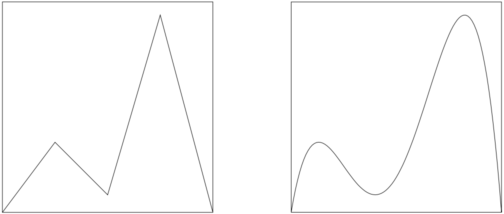

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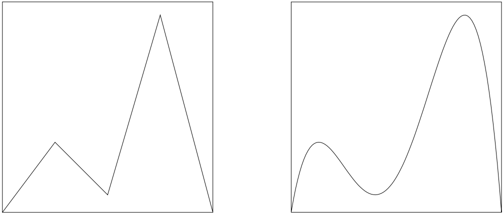

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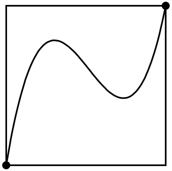
For the effective construction of $g(x)$, see [Bonifant-Milnor-Sutherland, 2021] in the list of references at the end.

Construction of polynomial map from critical values.



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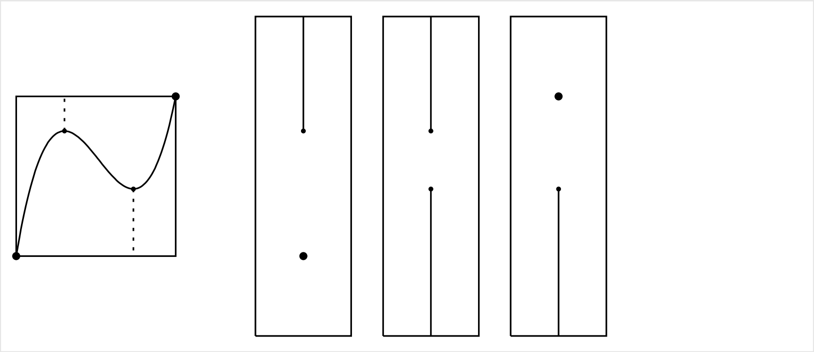

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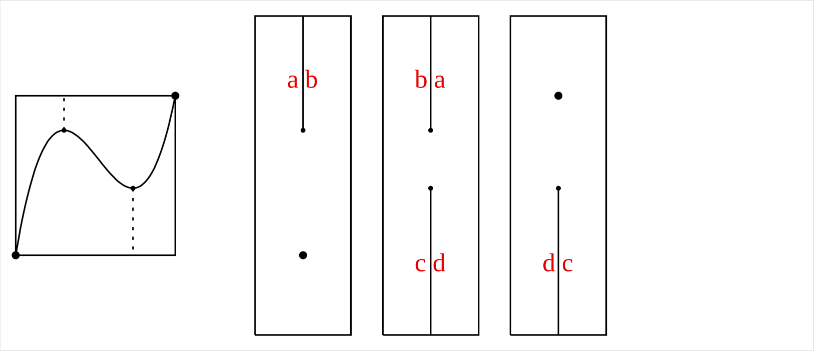

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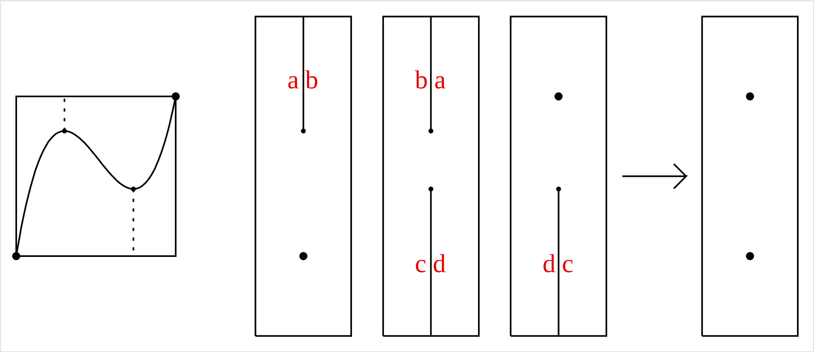

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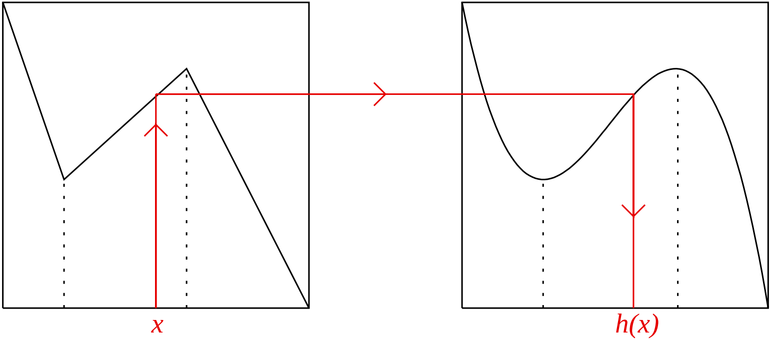

Graph of f , Graph of g .

The Tower Construction

$$\mathcal{I} \xrightarrow{\quad f_0 \quad} \mathcal{I}\ .$$

Suppose that we start with any PM-map f_0 .

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$$\begin{tikzcd}
\mathcal{I} \arrow[drr, "g_0"] & & \\
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Suppose that we start with any PM-map f_0.

Then there is a unique polynomial map g_0 of minimal degree with the same critical value vector.

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By the Lemma, there is a connecting homeomorphism

$$h_0 = h_{f_0, g_0} \qquad \text{with} \qquad g_0 \circ h_0 = f_0 \; .$$

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$$\begin{tikzcd}
{} \arrow[r, "f_2"] \arrow[dr, "g_1"] & {} \\
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{} \arrow[r, "f_3"] \arrow[dr, "g_2"] & {} \\
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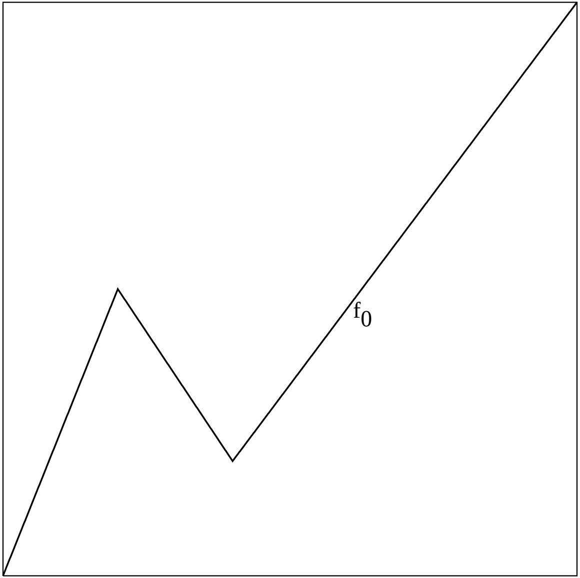
Caution: The tower algorithm bears a superficial resemblance to the Thurston algorithm; but they are not at all the same:

1. The Thurston algorithm is firmly documented and extremely stable. The tower algorithm may be easier to understand and to program; but it is speculative, and there are serious questions of stability.

2. The Thurston algorithm requires critical finiteness. The tower algorithm can be applied equally well to PM maps which are not critically finite; and also to other situations.

An Example with $d = 3$.



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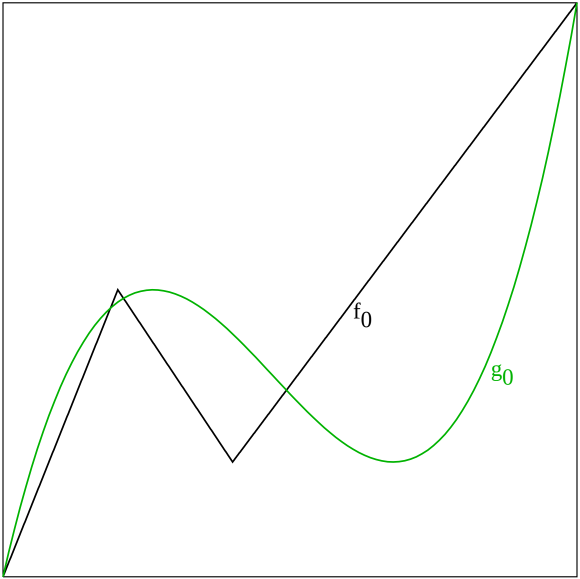

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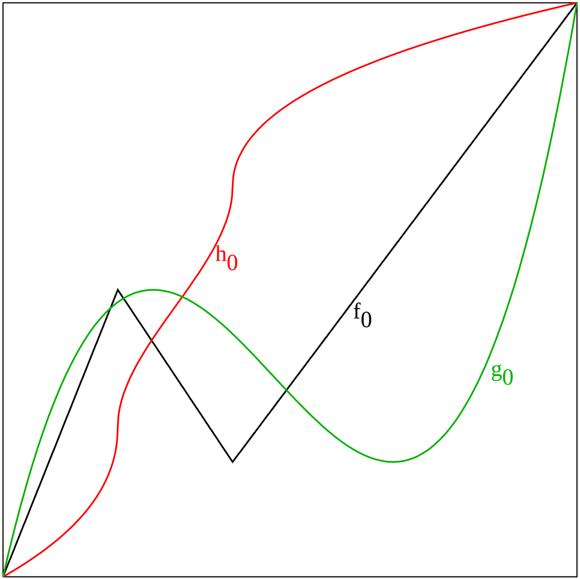

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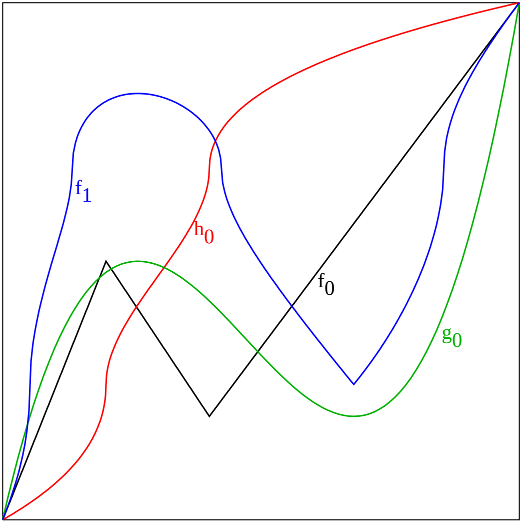

(movie 1)

A Critically Preperiodic Example

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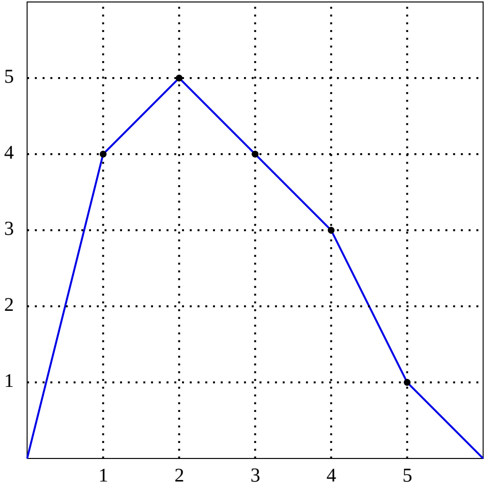

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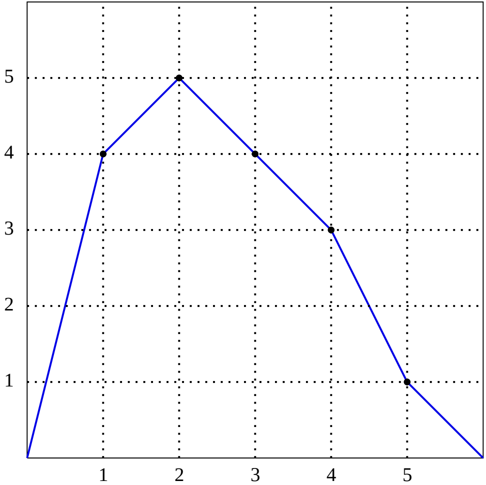
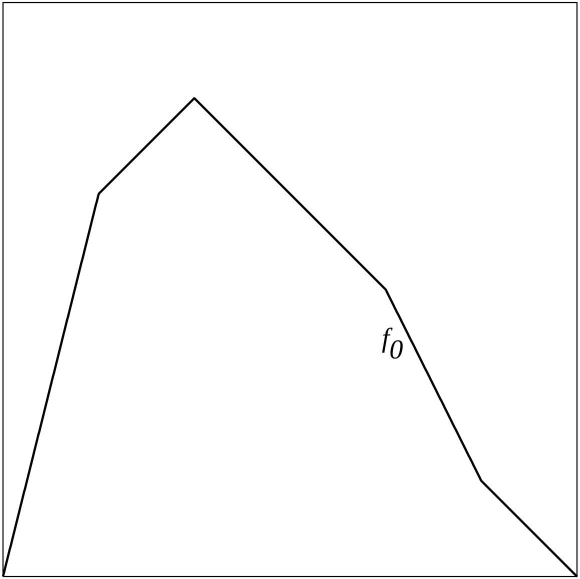


Here f_0 has critical orbit:

$$(2) \mapsto (5) \mapsto (1) \mapsto (4) \leftrightarrow (3)\ .$$

Critically preperiodic example (continued):



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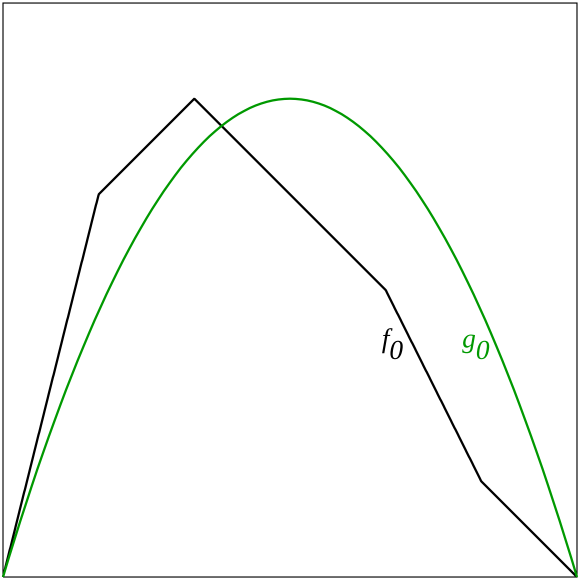

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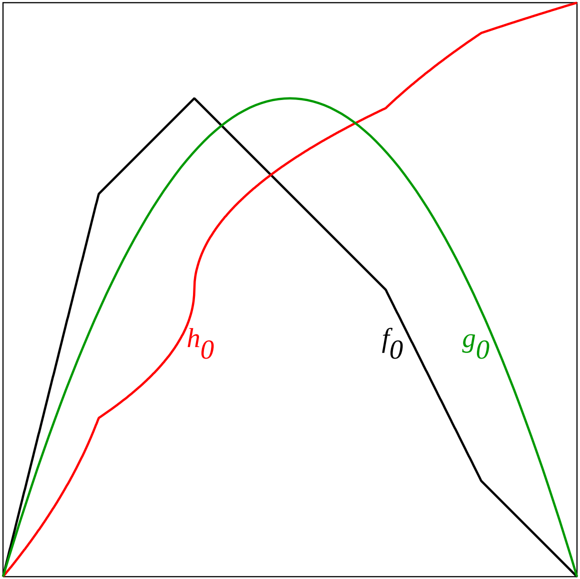

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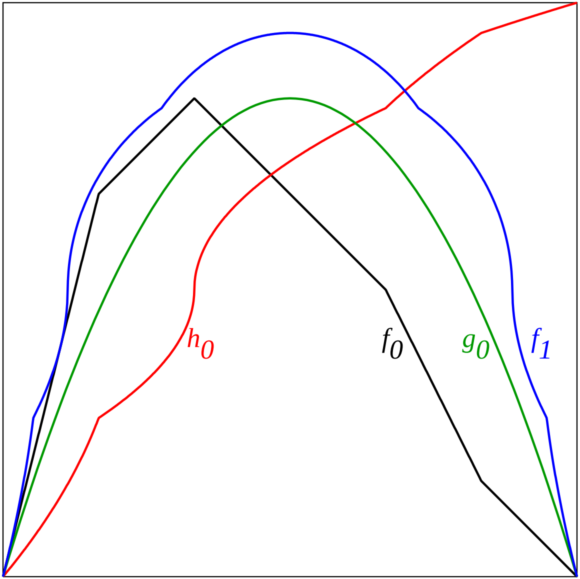

(movie 2)

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We can skip the intermediate steps and look at the topological conjugacy

$$\begin{tikzcd} \mathcal{I} \arrow[r, "f_n"] & \mathcal{I} \\ \mathcal{I} \arrow[r, "f_0"] \arrow[u, "H_n"] & \mathcal{I} \arrow[u, "H_n"'] \end{tikzcd}$$

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Thus $f_n = H_n \circ f_0 \circ H_n^{-1}$.

(movie 3)

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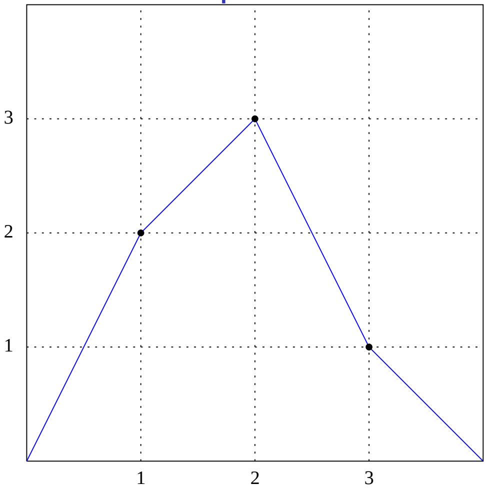

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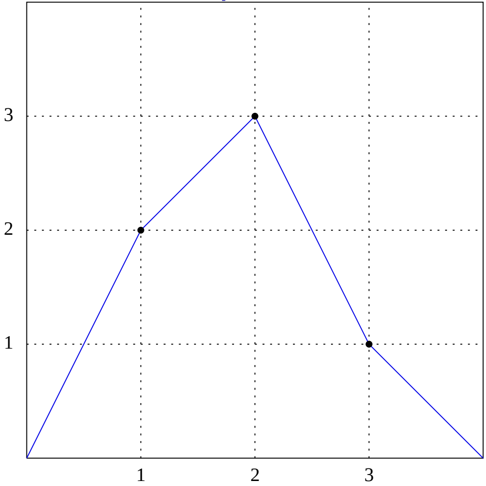

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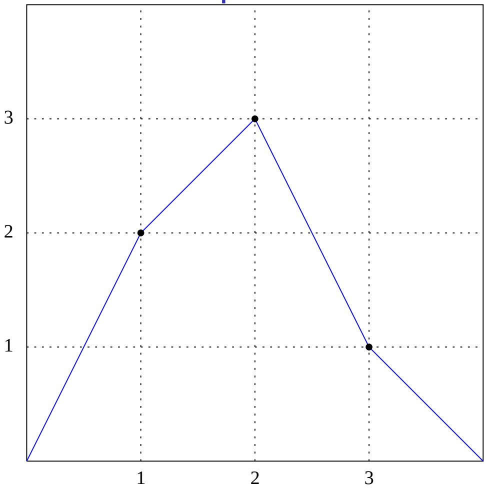

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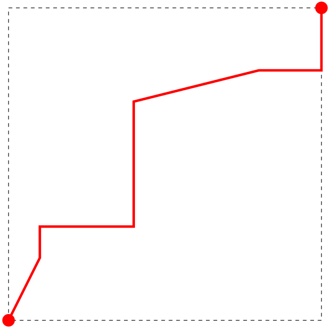

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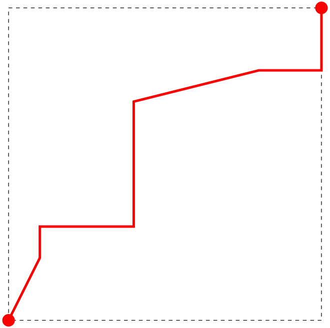

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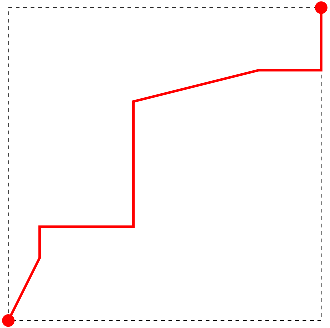

The **graph** of H_n does seem to converge to a limit in the **Hausdorff topology.**

The set of all limits of graphs of homeomorphisms forms a compact metric space.

Every such limit is a geodesic in the Manhattan metric.

$$|dx| + |dy| .$$

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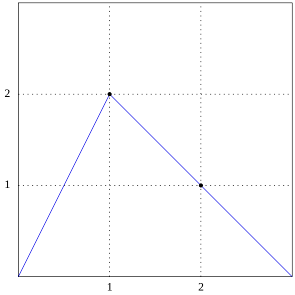

(movie 5)

A More General Construction.

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Given $d \geq 2$ and $v_0 \in \{0, 1\}$:

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For each such $\mathcal{G}$ there is an associated tower construction

$$\Theta_{\mathcal{G}} : f \mapsto h_{f,g} \circ g \qquad \text{where} \qquad g = g_f$$

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Now compute the critical points of g_f inductively $\cdots$. $\square$

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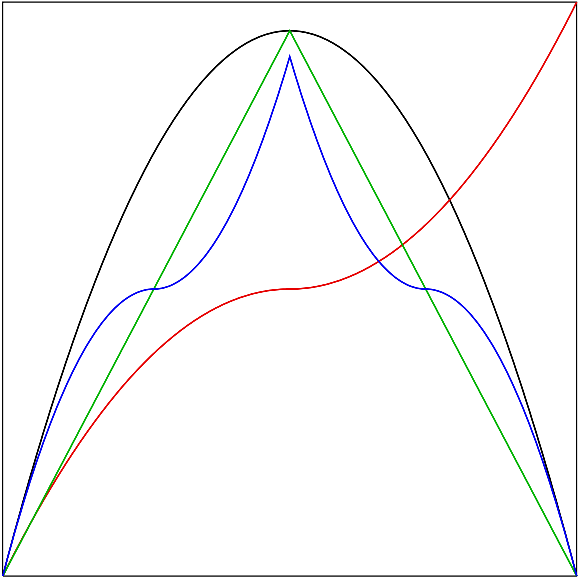

graphs of f_0 , f_1 , g_0 , h_0 , s = 1.900000

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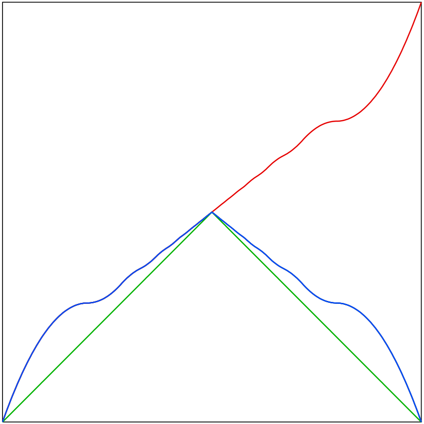

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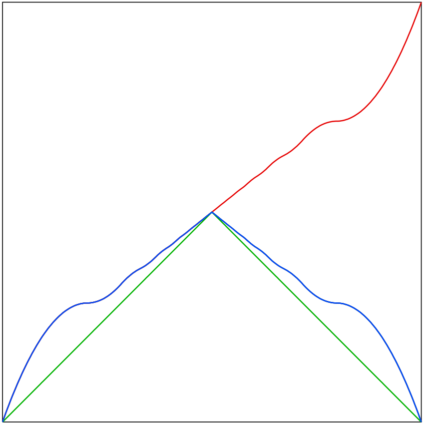

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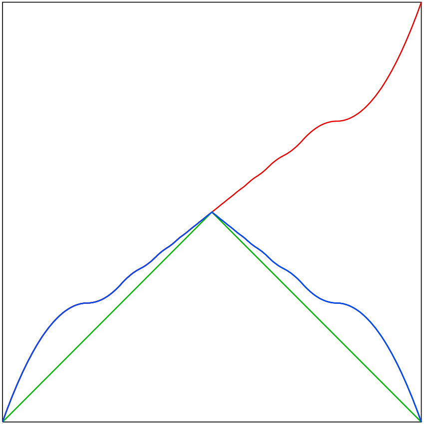

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Theorem (BST). *As $m \to \infty$, these upper and lower bounds both converge to $\exp(\mathbf{h}_{\rm top}(f))$.*

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