Remarks on Piecewise Monotone Maps
Corrected Version

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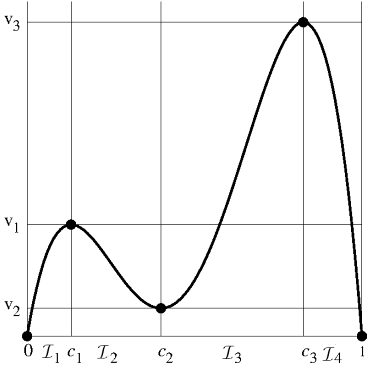
When running this file in firefox the movies will display if you click the indicated button.

PM-maps $f : (\mathcal{I},\ \partial\mathcal{I}) \to (\mathcal{I},\ \partial\mathcal{I})$ where $\mathcal{I} = [0, 1]$.



$(v_0 = v_4 = 0.)$

Maximal intervals of monotonicity: $\mathcal{I}_j = [c_{j-1},\ c_j]$ where

$$0 = c_0 < c_1 < \cdots < c_{d-1} < c_d = 1 .$$

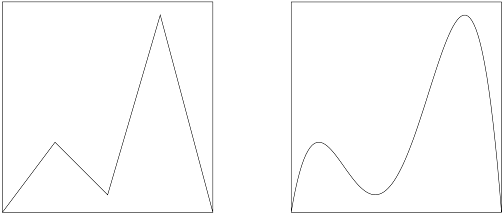
The vector $\mathbf{v} = (v_0,\ v_1,\ \ldots, v_d) \in \mathcal{I}^{d+1}$ where $v_j = f(c_j)$

will be called the **critical value vector.**

Caution: In this talk the word “critical” will be used to mean local maximum or minimum point. Inflection points are not “critical”.

The Polynomial Case.

Theorem. *Given a PM-map* $f(x)$ *with critical value vector* $(v_0, v_1, \ldots, v_d)$*, there is one and only one polynomial PM-map* $g(x)$ *of degree* d *with the same critical value vector.*

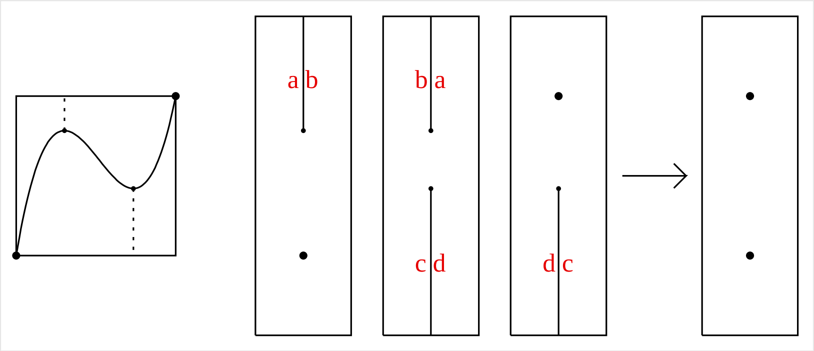


$f(x)$ $g(x)$

Proofs by deMelo and vanStrien, 1993; by Milnor and Tresser (and also by Douady and Sentenac in appendix), 2000.

For the effective construction of $g(x)$**, see [Bonifant-Milnor-Sutherland, 2021] in the list of references at the end.**

Construction of polynomial map from critical values.



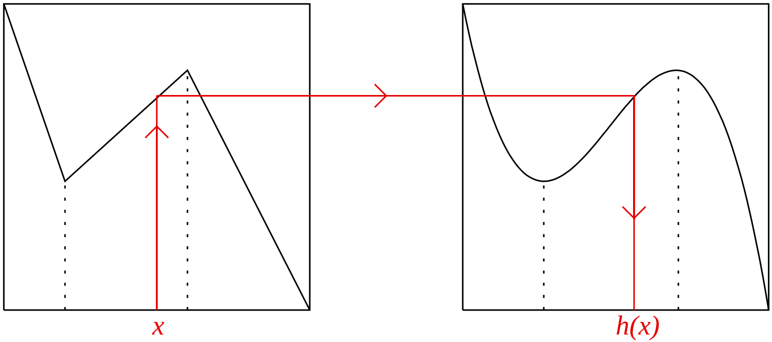
An Easy Lemma

Suppose that we are given two different PM-maps f and g with the same critical value vector.

LEMMA. *There is one and only one*

"connecting homeomorphism"

$h = h_{f,g}$ from $(\mathcal{I}, \partial\mathcal{I})$ to itself which maps each interval of monotonicity $\mathcal{I}_j(f)$ to the corresponding interval $\mathcal{I}_j(g)$ and which satisfies $g \circ h = f$.



Graph of f ,   Graph of g .

The Tower Construction

$$
\begin{tikzcd}
\mathcal{I} \arrow[r, "f_1"] \arrow[dr, "g_0"] & \mathcal{I} \\
\mathcal{I} \arrow[u, "h_0"] \arrow[r, "f_0"] & \mathcal{I} \arrow[u, "h_0"']
\end{tikzcd} .
$$

Suppose that we start with any PM-map f_0.

Then there is a unique polynomial map g_0 of minimal degree with the same critical value vector.

By the Lemma, there is a connecting homeomorphism

$$h_0 = h_{f_0, g_0} \quad \text{with} \quad g_0 \circ h_0 = f_0 .$$

Continue Inductively.

$$\begin{tikzcd}
 \arrow[r, dotted, dash] \arrow[dr, "g_3"] & \\
 \arrow[u, "h_3"] \arrow[r, "f_3"] \arrow[dr, "g_2"] & \arrow[u, "h_3"'] \\
 \arrow[u, "h_2"] \arrow[r, "f_2"] \arrow[dr, "g_1"] & \arrow[u, "h_2"'] \\
 \arrow[u, "h_1"] \arrow[r, "f_1"] \arrow[dr, "g_0"] & \arrow[u, "h_1"'] \\
 \arrow[u, "h_0"] \arrow[r, "f_0"] & \arrow[u, "h_0"']
\end{tikzcd}$$

Iteration.

*This construction defines a continuous correspondence $f_0 \mapsto f_1$
such that f_1 is topologically conjugate to f_0.*

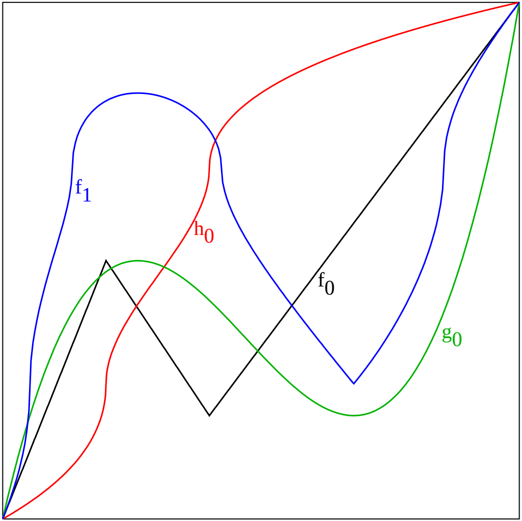
Iterating, we obtain an infinite sequence of topologically conjugate maps $f_0 \mapsto f_1 \mapsto f_2 \mapsto \cdots$.
Problem: For which f_0 does this sequence converge uniformly to a polynomial map?

Caution: The tower algorithm bears a superficial resemblance to the Thurston algorithm; but they are not at all the same:

1. The Thurston algorithm is firmly documented and extremely stable. The tower algorithm may be easier to understand and to program; but it is speculative, and there are serious questions of stability.

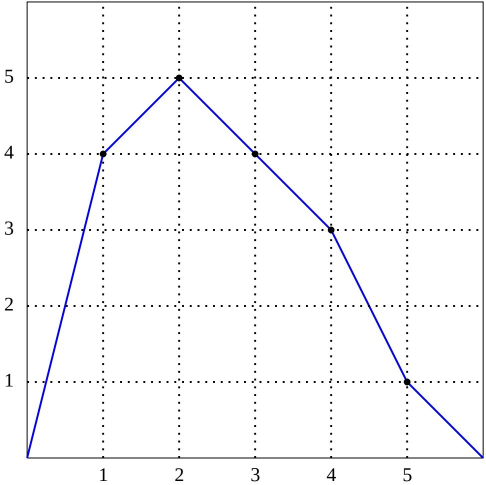
2. The Thurston algorithm requires critical finiteness. The tower algorithm can be applied equally well to PM maps which are not critically finite; and also to other situations.

An Example with $d = 3$.



(movie 1)

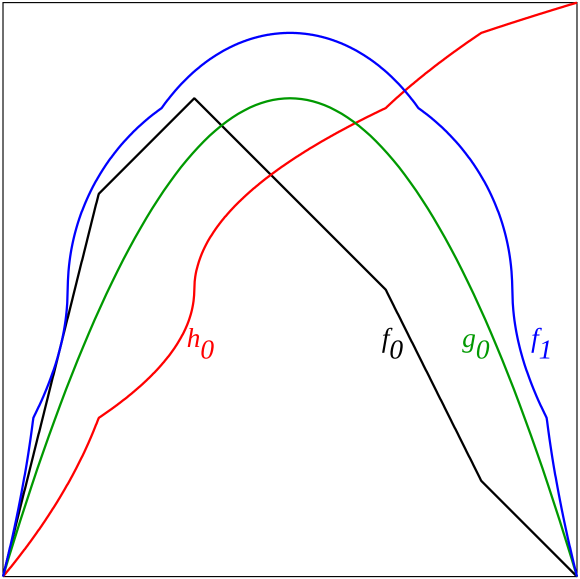
A Critically Preperiodic Example



Here f_0 has critical orbit:

$$(2) \mapsto (5) \mapsto (1) \mapsto (4) \leftrightarrow (3) \ .$$

Critically preperiodic example (continued):



(movie 2)

Jumping from f_0 to f_n.

We can skip the intermediate steps and look at the topological conjugacy

$$\begin{tikzcd} \mathcal{I} \arrow[r, "f_n"] & \mathcal{I} \\ \mathcal{I} \arrow[u, "H_n"] \arrow[r, "f_0"] & \mathcal{I} \arrow[u, "H_n"'] \end{tikzcd}$$

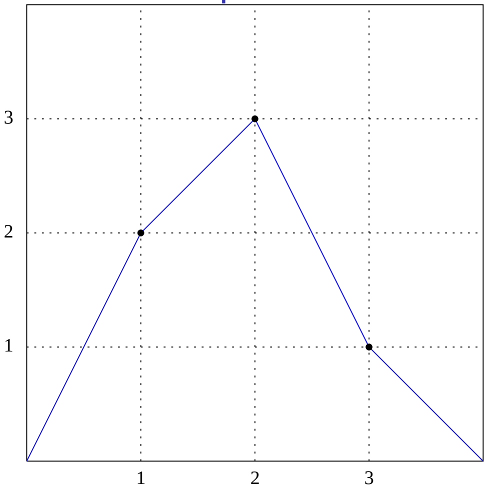
where

$$H_n = h_{n-1} \circ h_{n-2} \circ \cdots \circ h_1 \circ h_0 .$$

Thus $f_n = H_n \circ f_0 \circ H_n^{-1}$.

(movie 3)

A Critically Periodic Example



Critical orbit : $(2) \mapsto (3) \mapsto (1) \mapsto (2)$.

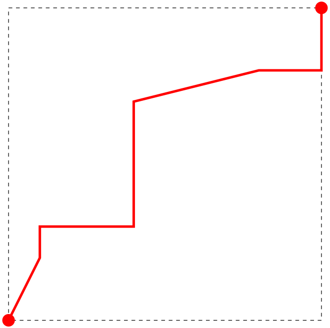
(movie 4)

Empirical conclusions.

"Good Convergence": For "many" choices of f_0 the sequences $\{f_n\}$ and $\{g_n\}$ seem to converge uniformly to a common polynomial limit f_∞, and the sequence $\{h_n\}$ seems to converge to the identity.

But this limit map f_∞ may not be topologically conjugate to f_0.

And the sequence of compositions $H_n = h_{n-1} \circ \cdots \circ h_1 \circ h_0$ may not converge to a homeomorphism.



The **graph** of H_n does seem to converge to a limit in the **Hausdorff topology.**

The set of all limits of graphs of homeomorphisms forms a compact metric space.

Every such limit is a geodesic in the Manhattan metric.

$$|dx| + |dy| .$$

Numerical Problem:

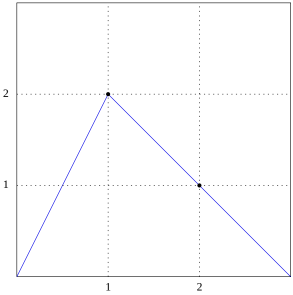
We have

$$f_n \;=\; H_n \circ f_0 \circ H_n^{-1}\,,$$

where the maps H_n and H_n^{-1} may have points with derivative tending to infinity as $n \to \infty$.

Therefore computation of f_n is likely to become ***very unstable*** *as $n \to \infty$.*

This seems to be particularly a problem for maps with topological entropy zero.



(movie 5)

A More General Construction.

Given $d \geq 2$ and $v_0 \in \{0, 1\}$:

Let $\mathcal{F} = \mathcal{F}(d, v_0)$ be the metric space consisting of all PM-maps f with d intervals of monotonicity and with $f(0) = v_0$, where

$$\mathrm{dist}(f, g) = \max_x \big(|f(x) - g(x)|\big) .$$

Definition. A subset $\mathcal{G} \subset \mathcal{F}$ is **parametrized by critical values** if, for any $f \in \mathcal{F}$ there is one and only one $g = g_f \in \mathcal{G}$ with the same critical value vector $\mathbf{v}$.

For each such $\mathcal{G}$ there is an associated tower construction

$$\Theta_{\mathcal{G}} : f \mapsto h_{f,g} \circ g \qquad \text{where} \qquad g = g_f$$

which maps each $f \in \mathcal{F}$ to a topologically conjugate map $\Theta_{\mathcal{G}}(f) \in \mathcal{F}$.

Examples of sets $\mathcal{G}$ parametrized by critical values.

1. Polynomials. The space $\mathcal{G}_{\rm poly}$ of polynomial maps of $(\mathcal{I},\,\partial\mathcal{I})$ with all critical points real and distinct, and in the interior of $\mathcal{I}$.

2. A trivial example. Take evenly spaced critical points $c_j = j/d$, and suppose that g is linear on each $\mathcal{I}_j = [c_{j-1}, c_j]$.

3. Constant Slope. By definition, a map f of the interval has **constant slope** $s \ge 0$ if f is piecewise linear with derivative satisfying $|f'(x)| = s$ almost everywhere.

> **Lemma.** *The set $\mathcal{G}_{\rm CS} \subset \mathcal{F}$ consisting of all PM-maps with constant slope is parametrized by critical values.*

Proof Outline: Suppose that $g_f \in \mathcal{G}_{\rm CS}$ has the same critical value vector as f. Then the slope s of g_f must be equal to the total variation of f (or of g_f):

$$s \;=\; \sum_{j=1}^{d} \,|v_j - v_{j-1}|\; > 0\;.$$

Now compute the critical points of g_f inductively $\cdots$. $\square$

Topological Entropy

Theorem of Misiurewicz and Slenk:

If $g : \mathcal{I} \to \mathcal{I}$ has constant slope $s \geq 0$, then its topological entropy is given by

$$\mathbf{h}_{\rm top}(g) = \log^+(s) \geq 0 .$$

Thus if iteration of $\Theta_{\mathcal{G}_{\rm CS}}$ converges to a map of constant slope, then we can easily compute the topological entropy of the limit map f_∞ .

Question. For which $f_0 \in \mathcal{F}$, does the sequence

$$\Theta_{\mathcal{G}_{\rm CS}} : f_0 \mapsto f_1 \mapsto f_2 \mapsto \cdots$$

converge uniformly to a map of constant slope, **with the same entropy?**

A Degree Four Example

(movie 6)

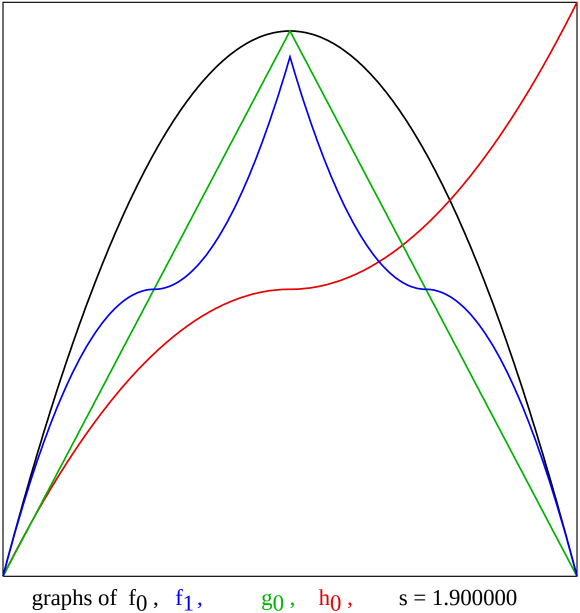
In this example, f_n converges to the standard **tent map**, and s converges to 2. Therefore

$$\mathbf{h}_{\mathrm{top}}(f_0) = \log(2) \quad ?$$

Conjecture. For any (reasonable ?) f_0, the associated sequence of constant slope maps g_n converges, and yields the correct topological entropy $\mathbf{h}_{\mathrm{top}}(f_0) = \log^+\big(s(g_\infty)\big)$.

(However, the sequence of topologically conjugate maps f_n does not always converge to a constant slope map; and the sequence of h_n does not always converge to the identity map.)

Example: $f_0(x) = 3.8\,x(1-x)$.



(movie 7)

Anomalous Convergence: $f_0(x) = 2.8\,x(1-x)$.

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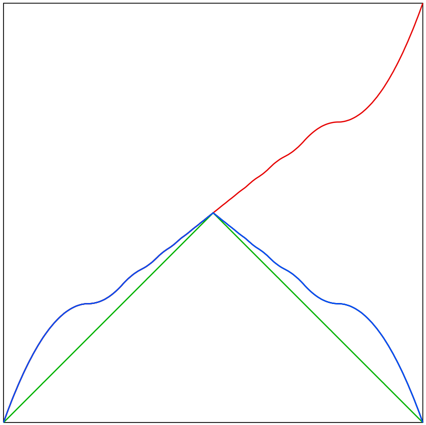

There seems to be uniform convergence:

$$f_n \to f_\infty\,, \quad g_n \to g_\infty\,, \quad h_n \to h_\infty \quad \text{as} \quad n \to \infty\,;$$

but $f_\infty \neq g_\infty$, and the homeomorphism h_∞ is not the identity map.

A Conditional Result.

Theorem. *If the sequence* $\{f_n\}$ *converges uniformly, then the sequences* $\{g_n\}$ *and* $\{h_n\}$ *also converge uniformly; and the limit maps* f_∞, g_∞, *and* h_∞ *commute with each other. Furthermore*

$$\mathbf{h}_{\rm top}(f_\infty) \;=\; \mathbf{h}_{\rm top}(g_\infty) \;=\; \log^+\big(s(g_\infty)\big)\;;$$

and if $\mathbf{h}_{\rm top} > 0$ *we have "good convergence":*
$f_\infty = g_\infty$, *and* h_∞ *is the identity map.*

Lemma. *If* $g = g_\infty$ *has constant slope* $s > 1$, *then no non-trivial orientation preserving homeomorphism* $h = h_\infty$ *can commute with* g.

Proof:

Step 1. Precritical points of g are everywhere dense,

Step 2. Any precritical point of g must be fixed by h. $\square$

But does $\mathbf{h}_{\rm top}(f_\infty) = \lim_{n\to\infty} \mathbf{h}_{\rm top}(f_n)$?

Appendix: The Balmforth-Spiegel-Tresser Algorithm

(Phys. Rev. Let. **72**, 1994; or arXiv, 1993)

Given a PM-map f with critical points c_j, let $P_m \subset \mathcal{I}$ be the finite set consisting of all $f^{\circ h}(c_j)$ with $0 \le h < m$.
This subdivides $\mathcal{I}$ into finitely many intervals $J_1, \cdots, J_N$.

Construct an $N \times N$ matrix $M = [a_{i\,k}]$ with

$$a_{i\,k} = \begin{cases} 1 & \text{if} \quad f(J_i) \supset J_k \,, \\ 0 & \text{if} \quad f(J_i) \text{ is disjoint from the interior of } J_k \,, \\ .5 & \text{if} \quad f(J_i) \text{ covers part of } J_k \,. \end{cases}$$

If we replace each .5 by a zero, we get a matrix M_0 whose leading eigenvalue is a lower bound for $s = \exp\big(\mathbf{h}_{\mathrm{top}}(f)\big)$.
Similarly, if we replace each .5 by a one, we get a matrix M_1 whose leading eigenvalue is an upper bound for s.

Theorem (BST). *As $m \to \infty$, these upper and lower bounds both converge to $\exp(\mathbf{h}_{\mathrm{top}}(f))$.*

Coda: For Hubbard's 70-th Birthday Fest.

It began with a classic Mechoui

And with Misha and Carsten and Cui

But time's running out

So let's get up and shout

Three cheers for John Hamal Hubbard,

and for Dynamical Holomorphie !

References

- The W. Thurston Algorithm Applied to Real Polynomial Maps, A. Bonifant, J. Milnor and S. Sutherland arXiv:2005.07800 [math.DS]; augmented version to appear in “Conformal Geometry and Dynamics”.
- Thurston’s Algorithm without Critical Finiteness, J. Milnor Linear and Complex. Analysis Problem Book 3, Part 2, Havin and Nikolskii editors, Springer Lecture Notes in Math. **1474**, p.434–436, 1994.
- Metrics on Trees I: The Tower Algorithm for Interval Maps. G. Tiozzo, (Work in Progress).