A PRELUDE TO SERIES

MAT 320, Fall 2001

Determine whether each of the following sequences $\{S_n\}$ converges or diverges. If it converges attempt to compute the limit $\lim_{n\to\infty} S_n$. Justify your calculation or explain, if possible, why you cannot evaluate the limit.

1. $S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$.
This sequence is increasing but not bounded from above. Hence

$$\lim_{n\to\infty} S_n = \infty.$$

This example was discussed in class several times. One can also reach the same conclusion by comparing S_n to $\int_1^n \frac{1}{x} dx$.

2. $S_n = 1 - \frac{1}{2} + \frac{1}{3} + ... + (-1)^{n-1}\frac{1}{n}$.

This example was discussed many times. It is a conditionally convergent series, and

$$\lim_{n\to\infty} S_n = \log 2.$$

3. $S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}$.
An absolutely convergent series with

$$\lim_{n\to\infty} S_n = e.$$

See Question 1.4.2 and/or recall your work in calculus.

4. $S_n = 1 - \left(\frac{\pi}{6}\right)^2 \frac{1}{2!} + \left(\frac{\pi}{6}\right)^4 \frac{1}{4!} + ... + (-1)^n \left(\frac{\pi}{6}\right)^{2n} \frac{1}{(2n)!}$.
The ratio test works to establish absolute convergence. From calculus,

$$\lim_{n\to\infty} S_n = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

5. $S_n = \left(\frac{\pi}{6}\right) \frac{1}{1!} - \left(\frac{\pi}{6}\right)^3 \frac{1}{3!} + ... + (-1)^{n-1} \left(\frac{\pi}{6}\right)^{2n+1} \frac{1}{(2n+1)!}$.
The ratio test establishes absolute convergence. From calculus,

$$\lim_{n\to\infty} S_n = \sin\frac{\pi}{6} = \frac{1}{2}.$$

6. $S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}$
Special case of 8.

7. $S_n = 1 - \frac{1}{2^2} + \frac{1}{3^2} + ... + (-1)^{n+1}\frac{1}{n^2}$.
Absolute convergence is a special case of 8.

8. $S_n = 1 + \frac{1}{2^s} + \frac{1}{3^s} + ... + \frac{1}{n^s}$.
See Example 7.5A.