# MAT 320 Quiz #3 solution Friday 11/16/01

**Name:** **ID Number:**

**1.** (a) Show that $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ is not continuous at $x = 0$. (Hint: Sequential Continuity)
(Ans.) Let $x_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ which converges to 0. Now suppose that $f$ is continuous at 0, then $\lim_{n\to\infty} f(x_n) = f(0)$ from Sequential Continuity Thm. However $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin\left(2n\pi + \frac{\pi}{2}\right) = 1 \neq 0 = f(0)$.

(b) Can you make the $f$ continuous at 0 by redefining $f(0)$ ? Explain why.
(Ans.) No. Define $f(0) = L \in \mathbb{R}$. Let $y_n = \frac{1}{2n\pi}$ which converges to 0. If $f$ is continuous at 0, by Sequential Continuity, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = f(0)$, which implies $1 = 0 = L$.

**2.** Show that $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ is continuous at $x = 0$.
(Ans.) Given $\epsilon > 0$, set $\delta = \epsilon$. $|f(x) - f(0)| = |x \sin \frac{1}{x}| \leq |x| < \delta = \epsilon$ for $0 < |x| < \delta$.

**3.** Give a counterexample to the following statement.
If $|f(x)|$ is continuous, then $f(x)$ is continuous.
(Counterexample) Let $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$. Then $|f(x)| \equiv 1$ which is continuous everywhere, but $f$ has a jump discontinuity at 0.

**4.** Give a counterexample to the following.
If $f$ is not continuous at $g(a)$ and $g$ is not continuous at $a$, then $f \circ g$ is not continuous at $a$.
(Counterexample) Let $f(x) = g(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. So $g$ is not continuous at 0 and $f$ is not continuous at $g(0) = 0$. However $f(g(x)) \equiv x$ on $\mathbb{R}$ who is continuous everywhere.