MAT 320 Mid term #2 with SOLUTIONS Monday 11/5/01

Name: **ID Number:**

The examination consists of 6 questions; the first 4 constitute the track B examination, the last two are the *additional* questions for track A students. Each question will be graded on the basis of 0 to 10; for a total maximum track B score of 40 with a possible additional score of 20 for those attempting track A. Answer using complete sentences and GIVE THE REASONS FOR YOUR CONCLUSIONS.

1. Compute each of the following limits.
(a) $\lim_{n\to\infty} \frac{\sin n}{n^2} = 0$, because the numerator is bounded by 1 in absolute value and the denominator goes to infinity.

(b) $\lim_{n\to\infty} \frac{3n^2}{2n^2+7n+9} = \lim_{n\to\infty} \frac{3}{2+\frac{7}{n}+\frac{9}{n^2}} = \frac{3}{2}$.

(c) $\lim_{n\to\infty} \frac{n^3-3n}{2n^2+7n-9} = \lim_{n\to\infty} \frac{n-\frac{3}{n}}{2+\frac{7}{n}-\frac{9}{n^2}} = \infty$.

(d) $\lim_{n\to\infty} \frac{3n+123456000}{2n^2+49n+91} = \lim_{n\to\infty} \frac{3+\frac{123456000}{n}}{2n+49+\frac{91}{n}} = 0$.

(e) $\lim_{n\to\infty}(3+\cos(4n))^{\frac{1}{n}} = 1$, because

$$2^{\frac{1}{n}} \le (3+\cos(4n))^{\frac{1}{n}} \le 4^{\frac{1}{n}},$$

the extreme terms approach 1, and thus so does the middle term by the Squeeze Principle.

2. (a) Define the radius of convergence R of the power series $\sum_{n=0}^{\infty} a_n x^n$.
SOLUTION: $R = \sup\ \{|x| : \sum_{n=0}^{\infty} a_n x^n$ converges absolutely$\}$.

(b) Describe any formula for computing R. What conditions on the sequence $\{a_n\}$ are needed to guarantee the validity of the formula?
SOLUTION: $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, provided the limit exists.

(c) What is the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{x}{3}\right)^n$?
SOLUTION: $\lim_{n\to\infty} \left|\frac{\frac{1}{n+2}\left(\frac{x}{3}\right)^{n+1}}{\frac{1}{n+1}\left(\frac{x}{3}\right)^n}\right| = \lim_{n\to\infty} \left|\frac{x}{3}\frac{n+1}{n+2}\right| = \frac{|x|}{3}$. Thus $R = 3$.

(d) For what x does the above series converge?
SOLUTION:From part **(c)** the series converges absolutely for $|x| < 3$ and diverges for $|x| > 3$.

The only remaining issue is what happens at the end points $x = \pm 3$. At $x = 3$ we have $\sum_{n=0}^{\infty} \frac{1}{n+1}$ which diverges, while at $x = -3$ we have $\sum_{n=0}^{\infty} \frac{1}{n+1}(-1)^n$ which is the alternating harmonic series and thus converges. The conclusion therefore is that $\sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{x}{3}\right)^n$ converges only on $[-3, 3)$.

3. Let f be a function with domain D_f and range R_f.
(a) Show that the inverse function f^{-1} exists if f is strictly increasing.
SOLUTION: Since f is strictly increasing, it is injective. An injective function has an inverse defined on its range.

(b) Does the existence of f^{-1} imply that f is strictly increasing?
SOLUTION: No, the function f could be strictly decreasing; for example, $f(x) = -x$.

(c) Does the function $f(x) = x^3 - 4$ have an inverse? If yes, what is the domain of the inverse and what is the value of (formula for) $f^{-1}(x)$?
SOLUTION: Yes. $D_{f^{-1}} = \mathbb{R}$. $f^{-1}(x) = (x+4)^{\frac{1}{3}}$.

4. Consider the function $f(x) = \frac{1}{1+x^2}$.
(a) What are its domain D and range R?
SOLUTION: $D = (-\infty, \infty)$ and $R = (0, 1]$.

(b) Does it have an inverse?
SOLUTION: No, f is not injective; for example $f(1) = f(-1)$.

(c) Which of the following four numbers exist. Evaluate those that exist.

$\sup_D f(x) = 1$

$\max_D f(x) = 1$

$\inf_D f(x) = 0$

$\min_D f(x)$ does not exist.

5. Consider the series $\sum_{n=1}^{\infty} a_n$ and the strictly increasing sequence of integers $\{n_i\}_{i=0}^{\infty}$ with $n_0 = 0$. Define a new series $\sum_{k=1}^{\infty} b_k$, where

$$b_k = \sum_{n=n_{k-1}+1}^{n_k} a_n.$$

(a) Assume that $\sum_{n=1}^{\infty} a_n$ converges and that $\sum_{n=1}^{\infty} a_n = S$. Does $\sum_{k=1}^{\infty} b_k$ converge and if so what is its sum? (A proof or counterexample is required.)
SOLUTION: Consider the sequences $S_n = \sum_{i=1}^{n} a_i$ and $T_k = \sum_{i=1}^{k} b_i$ of partial sums. We know that $\lim_{n\to\infty} S_n = S$ exists. Now $T_k = S_{n_k}$ and $\{S_{n_k}\}$ is a subsequence of $\{S_n\}$. Since a subsequence of a convergent sequence converges to the same limit as the sequence, $\lim_{k\to\infty} T_k = S$.

(b) Assume that $\sum_{n=1}^{\infty} a_n$ diverges. Does $\sum_{k=1}^{\infty} b_k$ diverge? (Again, a proof or counterexample is required.)
SOLUTION: No. Consider the divergent series

$$1 - 1 + 1 - 1 + 1 - ....$$

Let $n_i = 2i$. Then $b_k = 0$ for all k and thus $\sum_{k=1}^{\infty} b_k = 0$.

6. Let $f(x)$ be a locally constant function defined on $[0, 1)$.
(a) Define what it means for $f(x)$ to be locally constant.
SOLUTION: For each $x_o \in (0, 1)$, $f(x)$ is constant near x_o and $f(x)$ is constant near 0^+.

(b) Define $S = \{a \in (0, 1) : f(x) \text{ is constant on } [0, a)\}$. Can S be the empty set?
SOLUTION: No. Since $f(x)$ is constant near 0^+, there exists a positive δ_0 with $f(x)$ constant on $[0, \delta_0)$. Thus $\delta_0 \in S$.

(c) Is S bounded from above? If yes, what is an upper bound for S.
SOLUTION: Obviously, 1 is an upper bound for S.

(d) Use the results of parts (b) and (c) to prove that if $f(x)$ is locally constant, then it is constant.
SOLUTION: Since S is nonempty and bounded from above, it has a supremum m. Obviously $\delta_0 \le m \le 1$. We first show that $m = 1$. If $m < 1$ (assumed in order to arrive at a contradiction), then choose a small enough positive δ such that $(m - \delta, m + \delta)$ is a subset of $[0, 1)$ and $f(x)$ is constant on $(m - \delta, m + \delta)$. Choose an $a \in S$ such that $m - \delta < a < m$. This is possible since otherwise $m - \delta$ would be an upper bound for S. Now $f(x)$ is constant on $[0, m)$ (say c_1) and on $(m - \delta, m + \delta)$ (say c_2). Since a is in the intersection of the two intervals $c_1 = c_2$ and $f(x)$ is constant on $[0, m + \delta)$. Thus m could not be an upper bound for S, and we have reached a contradiction. Thus $f(x)$ is constant on $\cup_{0<a<1}[0, a) = [0, 1)$.