MAT 320 SOLUTIONS XSMid term #1 Monday 10/1/01

Name: **ID Number:**

The examination consists of 6 questions; the first 4 constitute the track B examination, the last two are the *additional* questions for track A students. Each question will be graded on the basis of 0 to 10; for a total maximum track B score of 40 with a possible additional score of 20 for those attempting track A. Answer using complete sentences and GIVE THE REASONS FOR YOUR CONCLUSIONS. Throughout the examination $\{a_n\}$ is a sequence of real numbers indexed by the positive integers and $L \in \mathbb{R}$.

1. (a) Define what it means for the sequence $\{a_n\}$ to be bounded.
SOLUTION The sequence $\{a_n\}$ is *bounded* iff there exists an $M \in \mathbb{R}$ such that $|a_n| < M$ for all $n \in \mathbb{N}$.

(b) Define what it means for the sequence $\{a_n\}$ to be strictly increasing.
SOLUTION The sequence $\{a_n\}$ is *strictly increasing* iff $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

(c) Let $a_n = \frac{n^n}{n!}$. Show that this sequence is strictly increasing but not bounded.
SOLUTION $\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)n^n} = \left(1 + \frac{1}{n}\right)^n \geq 1 + 1 = 2$. This shows that $a_{n+1} \geq 2a_n$: Hence the sequence is strictly increasing and not bounded ($a_n \geq 2^{n-1}a_1 = 2^{n-1}$).

2. (a) Let $\epsilon > 0$. Define what the statement "$a_n \overset{\approx}{\epsilon} L$ for $n >> 1$" means.
SOLUTION $a_n \overset{\approx}{\epsilon} L$ for $n >> 1$ iff there exists an $n \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n > N$.

(b) Show that if for some fixed $L \in \mathbb{R}$ and some positive ϵ, $L \overset{\approx}{\epsilon} 1$ and $L \overset{\approx}{\epsilon} 2$, then $\epsilon \geq \frac{1}{2}$.
SOLUTION $L \overset{\approx}{\epsilon} 1$ means that $|L - 1| < \epsilon$ and $L \overset{\approx}{\epsilon} 2$ means that $|L - 2| < \epsilon$. If $\epsilon < \frac{1}{2}$, then $|2 - 1| = |(L - 1) - (L - 2)| \leq |L - 1| + |L - 2| \leq \epsilon + \epsilon < 1$; an obvious contradiction.

3. (a) Define: $\lim_{n\to\infty} a_n = L$.
SOLUTION $\lim_{n\to\infty} a_n = L$ iff for all $\epsilon > 0$, $a_n \overset{\approx}{\epsilon} L$ for $n >> 1$.

(b) Prove that for all $r \in \mathbb{R}$, $\lim_{n\to\infty} \frac{r^n}{n!} = 0$.
{**Hint:** Let $a_n = \frac{r^n}{n!}$ and compare a_{n+1} to a_n.}
SOLUTION $\frac{a_{n+1}}{a_n} = \frac{r}{n+1} \leq \frac{1}{2}$ for $n >> 1$. Thus $a_{N+m} \leq \frac{a_N}{2^m}$ for all m and some large but fixed N. Since $\lim_{n\to\infty} a_n = \lim_{m\to\infty} a_{N+m} = 0$, we are done.

4. (a) Define what it means for the sequence $\{a_n\}$ to be a Cauchy sequence.
SOLUTION The sequence $\{a_n\}$ is *Cauchy* iff for all $\epsilon > 0$, there exists an N such that $|a_n - a_m| < \epsilon$ for all $n > N$ and all $m > N$.

(b) Let $a_n = \sqrt{n}$. Is this a Cauchy sequence?
SOLUTION NO. It is not bounded.

5. (a) State the Squeeze Theorem for Convergent Sequences.
SOLUTION Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Then $\lim_{n\to\infty} b_n = L$.

(b) Evaluate $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}\right)$.
{**Hint:** Recall that for all $n \in \mathbb{N}$, $\int_n^{2n} \frac{1}{x} dx = \ln 2$.}
SOLUTION From an appropriate picture and area comparisons, for all $n \in \mathbb{N}$,

$$\ln\left(\frac{2n+1}{n+1}\right) = \ln(2n+1) - \ln(n+1) = \int_n^{2n} \frac{1}{x+1} dx < \sum_{k=1}^{n} \frac{1}{n+k} < \int_n^{2n} \frac{1}{x} dx = \ln 2.$$

As $n \to \infty$, the extremes approach $\ln 2$. By the Squeeze Theorem, so does the sum.

6. Use Newton's method (as outlined below) to find $\sqrt{3}$ without the use of a calculator by finding a positive solution to $x^2 - 3 = 0$.
(a) Start with a good guess a_0 for $\sqrt{3}$. (Specify a_0.)
SOLUTION $a_0 = 2$.

(b) Give the recursion formula that describes the relation between successive approximations a_{n+1} and a_n to $\sqrt{3}$.
SOLUTION The tangent line to the curve $y = x^2 - 3$ at the point $(a_n, a_n^2 - 3)$ is $y - a_n^2 + 3 = 2a_n(x - a_n)$. The approximation a_{n+1} is obtained by setting $y = 0$ in the tangent line and solving for $x = a_{n+1}$. Thus

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right).$$

(c) Show that the sequence of approximations $\{a_n\}$ converges.
SOLUTION Let $e_n = \sqrt{3} - a_n$. Then

$$|e_{n+1}| = \frac{|e_n|^2}{2|\sqrt{3} - e_n|} \leq |e_n|^2 \text{ if } |e_n| < .5.$$

Since the last condition is easily achieved, we conclude by induction that $|e_n| \leq |e_0|^{2^n}$ which approaches zero very rapidly.

(d) What is $\lim_{n\to\infty} a_n$?
SOLUTION $\lim_{n\to\infty} a_n = \sqrt{3}$.