MAT 320, Fall 2001
Introduction to Analysis
I. Kra

Course syllabus and assignments

Course lecture meetings: M, W and F 11:35-12:30pm in SBU 237.
Recitation meeting: F 12:40-1:35pm in Grad Chem 128.
Lecturer: I. Kra, Math tower 4-111, 2-8273, irwin@math.sunysb.edu
Office hours (M, August 27– Th, December 13) in Math 4-111, W 3-4 and Th 3-4, in Undergraduate Mathematics Office, Tu 1-2, and by appointment.
Graduate Assistant: W. Kim, Math tower 3-118, wkim@math.sunysb.edu
Office hours: W 1-2 and Th 2-3.

Text book *Introduction to Analysis* by Arthur Mattuck, Prentice Hall, 1999.

Two tracks The course will be run along two tracks: B and A. A student who completes only the track B work can earn at most a B in the course (before taking into account the special projects that can increase the grade to a B^+). To qualify for a higher grade, a student must attempt each of, or be excused for good reasons from, the supplemental parts of the two midterms and the final examination. Students will not be penalized for attempting the A supplements since the grade earned in track B will not be decreased as a result of track A work. However, before every midterm and before the final students will receive a list of topics that will be excluded from the track B part of the examinations and *may appear only* on the track A supplements. A student pressed for time *may* choose to concentrate on the track B material only. It is also recommended that students do all the assigned Problems since the grade on these will enter into the calculation of the track A grade and more will be learned as a result of doing these problems.

How to read the book Reading this text book (all mathematics text books) is not a spectator sport. It should be read slowly with paper and pencil at hand. All details are to be filled in and difficulties resolved. The author has been kind and wise in supplying a set of *Questions* after each short sections. Students should answer these questions, *before* going ahead to the next section and *before* reading the supplied answers. The *Answers* should be viewed as a model for handing in solutions to the *Exercises* and *Problems.*

Examinations There will be a half hour pretest (on M September 10) that will cover very basic material from the courses that are prerequisites for this course (mainly logical foundations and First year calculus), two midterm examinations (on M October 1 and M November

5) and a final examination (on W December 19, 11:00- 1:30). There will also be 3 twenty minute to half hour quizzes (on W October 10, F October 19 and F November 16 – the last during recitation). There are no make-ups for missed quizzes and midterm examinations; but for justified absences, an adjustment to the grading plan will be made to minimize the effect on the final grade.

Projects and homework Homework is an integral part of the course. Of the 15 problem sets (consisting of exercises for all students and (harder) problems for track A students, at least 12 sets should be handed in (during the Friday recitation meeting of the week for which they are listed) for grading; the homework grade will be based on the best 10 of these. Students may hand, for extra credit, at *most* two of the three research/scholarship/computing projects described below. Each project will be assigned a grade of 0 to 10. A combined total of 12 or more points on the two projects will increase your grade by one step (thus, for example, from a C^+ to a B^-; a A grade cannot be increased). The problems to be handed in are only a *minimum*. The textbook has many good problems. You should do at least three problems every night and you should discuss the solutions with fellow students. Students may, and indeed should, work with fellow classmates on the projects and homework. They may also consult other people, but they alone should write up the solutions.

Grading Track B: The final examination (excluding the track A supplement) will constitute 25% of the grade; each of the midterms (again,excluding the track A supplement), 15%; the pretest 10%; the quizzes, 5% each; the homework exercises 20%. The tests (excluding the track A supplements) and quizzes will be constructed so that at least 50% of each will consist of statements of definitions, description of key results and routine calculations. A total grade of at least 60% will be required for a C; 80% will guarantee a B.
Track A: The supplemental part on each midterm and the problem sets will contribute a maximum extra of 10 points each and the final 20 points for a total Track A maximum grade of 150 points. A score of 120 or above will guarantee a A^-.

Recitation section Attendance at the recitation section is just as important as attending lectures. In the recitation section problems will be discussed, students can and should ask questions about homework exercises and problems, about examinations, about obscure points in either lectures or the book. Students will also work in groups and be expected to present solutions to problems.

Special needs If you have a physical, psychiatric, medical or learning disability that may impact on your ability to carry out assigned course work, you may contact the Disabled Student Services (DSS) office (Humanities 133, 632-6748/TDD). DSS will review your concerns and determine, with you, what accommodations may be necessary and appropriate. I will take their findings into account in deciding what alterations in course work you require.

All information on and documentation of a disability condition should be supplied to me in writing at the earliest possible time AND is strictly confidential. Please act early, since I will not be able to make any retroactive course changes.

READING & HOMEWORK ASSIGNMENTS

Wk of/Date	Topic (chapter)	B exercises	A (extra) problems
8/27	1. Real Numbers and Monotone Sequences	1.2.1, 1.3.1, 1.3.4 1.4.2, 1.5.1 and 2	1-1, 1-2, 1-3
9/3	2. Estimations and Approximations	2.1.1, 3 and 5, 2.2.1 2.3.1, 2.4.2, 3 and 6 2.5.1 and 5, 2.6.1 and 4	2-1, 2-4
9/10	3. The Limit of a Sequence	3.1.2, 3.2.2 and 3, 3.3.3 3.4.1 and 2, 3.6.1	3-1, 3-4
M Sept. 10	Half hour pretest		
9/17	4. Error Term Analysis	4.1.1, 4.2.1, 4.3.1, 4.4.2	4-1, 4-3
9/24	5. The Limit Theorems 6. The Completeness Property	5.1.1 and 4, 5.2.1 and 3 5.3.1 and 5,4.2, 5.5.1 6.1.1, 6.2.1 and 2, 6.3.1 6.4.1 and 3 , 6.5.1 and 4	5-1, 5-4 6-1, 6-5
M Oct. 1	First midterm	Chapters 1-6	
10/1	Catch up		
10/8	7. Infinite Series	7.1.1 and 3, 7.2.1 and 2 7.3.1 and 5, 7.4.1 and 2 7.6.1 and 2, 7.7.1	7-2 and 7-6
W Oct. 10	Quiz # 1		
10/15	8. Power Series	8.1.1 and 3, 8.2.1 8.3.1, 8.4.1	8-1
F Oct. 19	Quiz # 2		
10/22	9. Functions of One Variable	9.2.1, 3 and 4 9.3.1, 2 and 4, 9.4.1	9-1 and 9-3
10/29	10. Local and Global Behavior	10.1.1, 4 and 7, 10.2.2 10.3.1 and 4, 10.4.1	10-1 and 10-3
11/5	11. Continuity and Limits	11.1.1, 2 and 6, 11.2.1 and 3 11.3.1 and 2, 11.4.1 11.5.1, 2 and 5	11-2 and 11-3
M Nov. 5	Second midterm	Chapters 1-10	
11/12	12. The Intermediate Value Theorem	12.1.1 and 5, 12.2.2 and 4 12.3.1, 12.4.2	12-2
F Nov. 16	Quiz # 3	In recitation	

READING & HOMEWORK ASSIGNMENTS (continued)

Wk of/Date	Topic (chapter)	B exercises	A (extra) problems
11/26	13. Continuous Functions on Compact Intervals	13.1.1 and 2, 13.2.1, 13.3.2 13.4.1, 13.5.2	13-4
12/3	14. Differentiation: Local Properties	14.1.4, 5 and 6 14.2.4, 14.3.2 and 3	14-2, 14-3
12/10	15. Differentiation: Global Properties	15.1.2 and 4, 15.2.1 and 2 15.3.1 and 2, 15.4.2 and 4	15-1 and 15-2
W Dec. 19	Final examination	(11:00 to 1:30)	Comprehensive

PROJECTS

For these projects you may use and read any text book, you are free to consult anyone, and you may work in groups. The final write ups must be your work alone. Completed projects should be handed in to Mr. Kim on or before the last recitation meeting on Friday, December 7, 2001.

Project 1 Find up to five errors (not trivial typos, but typos that change the meaning are to be considered errors) in the book. For each error, describe in good English and using good mathematical formulation what is wrong and how to correct the text. Each correct corrections will receive 2 points towards the grade on the project.

Project 2 Write a program to determine whether a series $\sum_{n=1}^{\infty} a_n$ converges. The input consists of a sequence $a_1, a_2, ...,$. The terms a_n of the sequence are defined using a single formulae involving rational functions, trigonometric functions, exponentials and logarithms. Specify what restrictions, if any, you impose on the input. You may use any existing packages and the program should be written in MAPLE or any other language acceptable to the recitation instructor.

Hand in a description of your program (using good English sentences and appropriate mathematical formulae), an outline of the program, the code for the program, and a reasonable amount of program output. You will also be asked by the recitation instructor to execute the program in his presence. You will be given 3 "test" sequences as input and you will be expected to produce the output. Your grade will be based on 2 points for the description of the program, 2 for the outline and code and 2 points for each correct answer to the "test" cases.

Project 3 Do any 3 of these problems (some problems consist of several parts).

1. (a) Given any real number $t > 0$, use one of the equivalent completeness properties to establish the existence of the square root $\sqrt{t}$, i.e. the positive solution of the equation $x^2 - t = 0$.

(b) Recursively construct a sequence a_n that will converge to $\sqrt{t}$. Give a complete argument that $\lim_{n\to\infty} a_n = \sqrt{t}$.

(c) Without the use of a calculator, use (b) to compute $\sqrt{6}$ to 3 decimals. Show that your error is indeed less than 0.0005.

2. (a) Consider the power series

$$\sum_{0}^{\infty} \frac{x^n}{n!} .$$

Prove that it converges absolutely for all x.

(b) Think of the exponential function e^x as being defined by this series. Multiply two power series to prove the functional equation

$$e^{u+v} = e^u e^v .$$

Justify the method as completely as you can. Why does the above equation imply that $e^{-u} = 1/e^u$?

(c) Now again for any fixed x consider the sequence

$$a_n = \left(1 + \frac{x}{n}\right)^n .$$

Suppose first that $x > 0$. Use the Binomial Theorem (term by term) to show that a_n is strictly increasing. Furthermore, conclude that $a_n \leq s_n(x)$, the nth partial sum of the exponential series above. Why does this imply the convergence of a_n and what can be said about its limit so far? Try to prove the convergence of a_n similarly for $x < 0$. Is this possible?

(d) Prove that for any fixed x,

$$\lim_{n\to\infty} (s_n(x) - a_n) = 0 ,$$

and thus,

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x .$$

(e) Give an alternate proof of the functional equation for the exponential function in (b) based on the limit formula in (d).

3. (a) Consider the trigonometric functions sin and cos to be defined by the power series

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} , \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} .$$

What is the radius of convergence of these series?

(b) Prove the addition theorems

$$\sin(x+y) = \sin x \cos y + \cos x \sin y\ , \quad \cos(x+y) = \cos x \cos y - \sin x \sin y$$

by multiplying power series.

(c) Consider the function

$$\sec x = \frac{1}{\cos x}\ .$$

Assuming that there is a power series representation

$$\sec x = \sum_{n=0}^{\infty} c_n x^{2n}\ ,$$

near 0 (why are there no odd powers?), find the first 6 coefficients $c_1, ..., c_6$. Use only the above power series for cos and a suitable method of long division.

(d) Show that the division method in (c) actually yields a power series centered about 0 with radius of convergence $R = \pi/2$, which represents $\sec x$ on its interval of convergence. Why are the coefficients c_n uniquely determined by the last condition?

4. Axiom of Choice: Read pp. 18-19 of "Real Analysis" by H.L. Royden (this book is on reserve in the library), and then solve the problem on p. 19:
Let $f: X \to Y$ be a mapping *onto* Y. Then there is a mapping $g: Y \to X$ such that $f \circ g$, the composition of f with g, is the identity map on Y.

5. Axiomatic approach to the real number system: Read Chapter 2, Section 1, of Royden's "Real Analysis", cf. problem 4. Then do Problems 1 and 3 on p. 32.

6. Conditionally convergent series: Cf. Section 7.7 of our text; but you may also consult other sources. Consider the series

$$(*) \qquad \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}\ .$$

(a) Argue that this series is convergent, but it is not absolutely convergent.

(b) Use an error term analysis modelled after Example 4.2 on pp52-53 (building on a suitable geometric series) to show that we have the following power series representation.

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\ ,$$

valid for $-1 \le x \le 1$. Why does (a) now imply without any further computation that the last power series has radius of convergence $R = 1$? Give the value of the series (*).

(c) Give three explicit constructions of a rearrangements of (*) which yield divergent series that have the limit sum $+\infty$ or $-\infty$, respectively, or even no limit sum in any sense.

(d) Show that there is a rearrangement of (*) converging to 0. Use the Nested Interval Theorem to to actually establish the convergence.

(e) More generally, fix any real number S. Prove that (*) can be rearranged to converge with sum S. Describe a procedure as precisely as possible; you may want to use the Nested Interval Theorem again.

7. Basic concepts for functions:

(a) Examine the *boundedness, sup, inf, max*, and *min* of the following functions on their domains:

$$\textbf{(i)}\ \ f(x) = \frac{1}{1+x^2}\ , \quad \textbf{(ii)}\ \ f(x) = \left(\frac{x}{x-1}\right)^2 .$$

(b) Give an $f(x)$ such that $1/f(x)$ is defined for all x and is
(i) unbounded, **(ii)** bounded, but $f(x)$ is decreasing.

(c) Suppose $f(x)$ is defined for all x; under what hypotheses on $f(x)$ will $1/f(x)$ be bounded for all x?

(d) Suppose $f(x)$ and $g(x)$ are defined for all x. Prove if true; give a counterexample if false:
(i) $f(x)$ bounded $\Longrightarrow f(g(x))$ bounded
(ii) $f(x)$ bounded $\Longrightarrow g(f(x))$ bounded.

8. Cluster points:

(a) Consider the set of all ordered pairs

$$S = \{(p,q)\ :\ 1 < p < q,\ \text{p and q integers}\}\ .$$

Read pp. 329-332 of the text and use some form of Cantor's diagonal argument to construct an enumerative sequence a_n of S as explicitly as you can, i.e. give a bijection $f : \mathbf{N} \to S$, so that $f(n) = a_n$. Now by assigning to each pair (p,q) in the sequence a_n the fraction p/q we obtain a sequence r_n of rational numbers. Explain why the sequence r_n contains all rational numbers in the open interval $(0,1)$, with repetitions.

(b) Is r_n monotone? Show that all real numbers in the compact interval $[0,1]$ are cluster points of r_n.

9. Limits of functions and sequential convergence:

Do Exercises 11.5.5, 11.5.6, and Problem 11-4 on p. 169 of the text. Take the hint seriously and write careful arguments.

10. Uniform convergence:

(a) Study Chapter 22, pp. 305-322. This is very important and beautiful material. Then prove the Cauchy criterion of Exercise 22.2.5.

(b) Do Problems 22-2 and 22-3 on Bessel's equation of order 0 and derive the classical integral formula for the zero-th order Bessel function $J_0(x)$.

11. A C^∞-function which is not real analytic:

(a) Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} .$$

Show that f has derivatives of all orders at $x = 0$ and $f^{(n)}(0) = 0$ for all n.
Plot this function. Compute a few derivatives to see the pattern. In addition to the usual differentiation rules, you will have to use the definition of derivative, as well as l'Hospital's rule.

(b) Argue that there is no representation of $f(x)$ as a power series near 0, i.e. f is not *real analytic*. (You may want to read a few facts in Chapter 22, or recall Taylor's Theorem.)

12. Picard Iteration:

(a) Study Appendix C of our text, pp. 427-431. Work out Problem C-2 on p. 432.

(b) Study part of Appendix E, pp. 445-453. Then do Exercise E.1(a) on p. 453.

Good luck!

Last revision on: 10/12/01