MAT 319, Spring 2003 Pretest with solutions Thursday 1/30/03

1 (a). What are the converse and the contrapositive of the proposition, "The sum of any three consecutive positive integers is divisible by 6."
SOLUTION. CONVERSE "Every positive integer that is divisible by 6 is the sum of three consecutive positive integers."
CONTRAPOSITIVE "If a positive integer is not divisible by 6, then it is not the sum of three consecutive positive integers."

(b). Which of the three statements are true?
SOLUTION. The **PROPOSITION** is false. Let x be a positive integer. Then

$$x + (x + 1) + (x + 2) = 3(x + 1)$$

is odd (hence not dfivisible by 6) for even x.
The **CONVERSE** is true. Write

$$3(x + 1) = 6N$$

and conclude that $x = 2N - 1$.
The **CONTRAPOSITIVE** is false since a proposition and its contrapositive always have the same truth value.

2. Let S be the set consisting of all nonnegative integers that are divisible by 6 and not divisible by 3. Is S empty (that is, has no elements in it), finite or infinite? If the empty set, explain why. If S is not empty, give an alternate description (formula) for the elements in S and list its first three elements?
SOLUTION. Any integer divisible by 6 is also divisible by 3. Hence $S = \emptyset$.

3. Let $[0, 2]$, $[0, 4]$ and $[1, 3]$ be three closed intervals in $\mathbb{R}$. Express their union and intersection as closed intervals.
SOLUTION.

$$[0, 2] \cup [0, 4] \cup [1, 3] = [0, 4]$$

and

$$[0, 2] \cap [0, 4] \cap [1, 3] = [1, 2].$$

4. Consider a set $P(1), P(2), P(3), ..., P(n), ...$ of statements. You are told that for each positive integer n, $P(n + 1)$ is true whenever $P(n)$ is. Is $P(3)$ true?
SOLUTION. $P(3)$ need not be true. Let $P(n)$ be the statement "n is a negative integer."

5. Let a and b be nonnegative integers. If $(a + b)^2 = a^2 + b^2$, what, if anything, can you conclude about a and b?
SOLUTION. Since $(a+b)^2 = a^2 + 2ab + b^2$, the hypothesis tells us that $ab = 0$; thus either

$a = 0$ or $b = 0$.

6. True or False?
Let a, b and c be arbitrary real numbers.
(a). There always exists an $x \in \mathbb{R}$ such that $x^2 + ax + b = 0$.
SOLUTION. False. Take $a = 0$ and $b = 1$.

(b). There always exists an $x \in \mathbb{R}$ such that $x^3 + ax^2 + bx + c = 0$.
SOLUTION. True. The polynomial is negative for $x << 0$ and positive for $x >> 0$. It must hence be zero somewhere (in between).

7. Compute and if possible simplify in terms of sines and cosines of $2x$:
(a). $\frac{d}{dx}(\cos^2(x) + \sin^2(x))$.
SOLUTION. $\frac{d}{dx}(\cos^2(x) + \sin^2(x)) = 2\cos x(-\sin x) + 2\sin x \cos x = 0$.

(b). $\frac{d}{dx}(\cos^2(x) - \sin^2(x))$.
SOLUTION. $\frac{d}{dx}(\cos^2(x) - \sin^2(x)) = 2\cos x(-\sin x) - 2\sin x \cos x = -4\sin x \cos x = -2\sin(2x)$.

(c). Do you find either answer to be surprising? Why?
SOLUTION. Neither answer is surprising since $\cos^2(x)+\sin^2(x) = 1$ and $\cos^2(x)-\sin^2(x) = \cos(2x)$.

8. Compute $\int x \ln x dx$.
SOLUTION. $\int x \ln x dx = \frac{1}{2}\int \ln x d(x^2) = \frac{1}{2}\left(x^2 \ln x - \int x^2 \frac{1}{x} dx\right) = \frac{1}{2}\left(x^2 \ln x - \frac{x^2}{2}\right)$.

9. Use logarithms to estimate from below and from above $\sum_{i=1}^{n} \frac{1}{i}$.
SOLUTION. Comparing areas under the curve $y = \frac{1}{x}$ to areas of appropriate rectangles one sees that for all integers $n > 1$, that

$$\int_1^{n+1} \frac{1}{x} dx < \sum_{i=1}^{n} \frac{1}{i}$$

and

$$1 + \sum_{i=2}^{n} \frac{1}{i} < 1 + \int_1^{n} \frac{1}{x} dx$$

or

$$\ln(n+1) < \sum_{i=1}^{n} \frac{1}{i} < 1 + \ln n.$$

10. Which of the following series converge? Which diverge?
(a). $\sum_{n=1}^{\infty} \frac{1}{n}$.
SOLUTION. Diverges by the integral test.
(b). $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
SOLUTION. Converges by the integral test.
(c). $\sum_{n=1}^{\infty} e^{-n}$.
SOLUTION. Converges because it is a geometric series.
(d). $\sum_{n=1}^{\infty} e^{n}$.
SOLUTION. Diverges because it is a geometric series.