MAT 319, Quiz 3 (with solutions): Wednesday, 12 March 2003

1. Suppose $q(x)$ is a rational function such that $q(-1) = -1$ and $q(1) = 1$. Does the intermediate value theorem imply that q has a zero on the open interval $(-1, 1)$? Why?
SOLUTION: NO. The function may not be defined (hence not continuous) on the closed interval $[-1, 1]$. For example, the function $q(x) = \frac{(x-1)(x+1)}{x}$ does vanish at -1 and 1, but its derivative $q'(x) = \frac{x^2+1}{x^2}$ does not vanish anywhere.

2. Let f be a function which is defined and differentiable everywhere. Is $\frac{f(x)}{f(x)^2+1}$ differentiable everywhere? Why?
SOLUTION: YES because a ratio of two differentiable functions is differentiable at those points where the denominator does not vanish. The derivative is easily computed to be:
$$\frac{f'(x)(3f(x)^2 + 1)}{(f(x)^2 + 1)^2};$$
but is not needed to answer the question.

3. Give an example of a function that is continuous on the half-open interval $(0, 1]$, but is not uniformly continuous.
SOLUTION:$f(x) = \frac{1}{x}$ is such a function. It is not uniformly continuous because it is not even bounded on the interval $(0, 1]$.

4. Carefully state the Chain Rule.
SOLUTION: If g is differentiable at a and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a and
$$(f \circ g)'(a) = f'(g(a))g'(a).$$

5. Let $f(x) = \frac{1}{x}$. Prove from the definition of the derivative that $f'(x) = \frac{-1}{x^2}$.
SOLUTION:
$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{h}}{h} = \lim_{h \to 0} \frac{-h}{hx(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}.$$