MAT 319 Foundations of Analysis – Midterm II with solutions

4/10/2003

Problem 1: (a) State carefully the intermediate value theorem.
SOLUTION: Let f be a continuous function on the closed interval $[a, b]$. The function assumes every value between $f(a)$ and $f(b)$.

(b) Let f be a continuous function whose domain and range are the closed interval $[0, 1]$. Show that there exists an $x \in [0, 1]$ such that $f(x) = x$.
HINT given during the examination. Draw graphs of possible functions f and the function $g(x) = x$ on $[0, 1]$.
SOLUTION: The function $h(x) = x - f(x)$ is continuous on $[0, 1]$, ≤ 0 at 0 and ≥ 0 at 1. Hence by the intermediate value theorem there is an $x \in [0, 1]$ such that $h(x) = 0$.

Problem 2: (a) Let f be a function defined in an open interval containing the point a. Define what it means for f to be differentiable at a.
SOLUTION: The function f is differentiable at a if and only if $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists.

Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$.
(b) Is f differentiable at 0. If yes, compute $f'(0)$; if not, explain why?
SOLUTION: The function is differentiable at 0 because $\lim_{h \to 0} \frac{f(h)-f(0)}{h} = h \sin \frac{1}{h} = 0$.

(c) Is f differentiable at $a \neq 0$. If yes, compute $f'(a)$; if not, explain why?
SOLUTION: The function is differentiable at $a \neq 0$ because it is obtained from functions that are differentiable through multiplication and composition. By the chain and product rule

$$f'(a) = a^2 \left(\cos \frac{1}{a}\right) \left(-\frac{1}{a^2}\right) + 2a \sin \frac{1}{a} = 2a \sin \frac{1}{a} - \cos \frac{1}{a}.$$

Problem 3: (a) State Rolle's theorem.
SOLUTION: If f is a continuous function on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = 0 = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

(b) State the second form of the fundamental theorem of calculus.
SOLUTION: Let f be a continuous function on the closed interval $[a, b]$ and define $F(x) = \int_a^x f$ for $x \in [a, b]$. Then F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.

(c) We have defined $\log(x) = \int_1^x \frac{dt}{t}$ for $x > 0$. Use partitions of $[1, 2]$ into intervals of equal length to show – you need not do the calculations – that you can estimate $\log(2)$ to three decimal places.
SOLUTION: We are integrating the function $g(x) = \frac{1}{x}$ and we need to compute $\int_1^2 g$. Let P be the partition of $[1, 2]$ into n intervals of equal length $\frac{1}{n}$: $\{1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, ..., 2\}$. Because g is a decreasing function, $U(P, g) = \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{1+\frac{j}{n}}$ and $L(P, g) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{1+\frac{j}{n}}$. We know that $L(P, g) < \log 2 < U(P, g)$ and that $U(P, g) - L(P, g) = \frac{1}{n}\left(1 - \frac{1}{2}\right) = \frac{1}{2n}$. For n sufficiently large (say $n = N$) this difference is less than .0005; producing 3 place accuracy for $\log 2$ by estimating $\log 2$ as *any* value between $L(P, g)$ and $U(P, g)$ for the partition P into N intervals.

Problem 4: (a) Let n be a positive integer. Compute the n-th derivative $f^{(n)}$ for the function $f(x) = \log(x)$ defined in the previous problem.
SOLUTION: We have

$$f'(x) = \frac{1}{x},$$

$$f''(x) = -\frac{1}{x^2},$$

$$f'''(x) = \frac{1 \cdot 2}{x^3},$$

and for arbitrary $n \in \mathbb{Z}_{>0}$,

$$f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{x^n}.$$

The first calculation is justified by the fundamental theorem of calculus, the remaining ones by the usual rules of differentiation. Formally an induction proof is required.

(b) What is the n-th Taylor polynomial $p_{n,3}(x)$ for f at 3.
SOLUTION:

$$p_{n,3}(x) = \log 3 + \frac{1}{3}(x-3) - \frac{1}{2}\frac{1}{3^2}(x-3)^2 + \frac{1}{3}\frac{1}{3^3}(x-3)^3 + ... + (-1)^{n-1}\frac{1}{n}\frac{1}{3^n}(x-3)^n.$$

(c) Give any formula for the remainder term $r_{n,3}(x) = f(x) - p_{n,3}(x)$.
SOLUTION: Probably the most useful of the 3 formulae is

$$r_{n,3}(x) = (-1)^n \frac{1}{n+1}\frac{1}{c^{n+1}}(x-3)^{n+1},$$

for some c with $3 \leq c \leq x$.

(d) Find an upper bound for $r_{4,3}(5)$.
SOLUTION:

$$|r_{4,3}(5)| \leq \frac{1}{5}\frac{2^5}{3^5}.$$

Problem 5: True or false: (Circle the correct answer and justify it by quoting a theorem or proving a True statement and giving a counter example to a False statement.)

T F (a) Every continuous function is differentiable.
T F (b) Every integrable function is continuous.
T F (c) Every bounded sequence converges.
T F (d) Every bounded set of reals has a maximum.
T F (e) Every 4-th degree polynomial has at least two roots.

SOLUTION: (a) False. A counterexample is provided by $f(x) = |x|$ which is not differentiable at 0.

(b) False. A counterexample is provided by any step function.

(c) False. A counterexample is provided by $\{(-1)^n\}_{n=1}^{\infty}$.

(d) **CLARIFICATION** provided during examination. The term “has” is the same as “contains” in this context.
False. A counterexample is provided by $[0, 1)$.

(e) False. A counterexample is provided by $p(x) = x^4 + 1$.