MAT312/AMS351

Fall 2002

Work sheet # 5, Cosets and conjugations.

This worksheet is an exploration of cosets and conjugation. Although none of the problems *require* a computer, MAPLE will simplify the experimentation that you need to do. Recall that if G is a group, then for all $g \in G$, the map $\theta(x) = gxg^{-1}$, $x \in G$, defines an isomorphism known as *conjugation* of G onto itself.

(1) Assume that G is an abelian group and θ is a conjugation of G onto itself. Can you compute what $\theta(x)$, $x \in G$ is?

(2) Let H be a subgroup of G and θ a conjugation of G onto itself. Show that $\theta(H)$ is a subgroup of G, called a *conjugate* of H in G.

(3) Let G be the group $\mathbb{Z}_{50}$ under addition. Let

$$H = 10G = \{[10a]_{50}; [a]_{50} \in G\}.$$

(a) Show that H is a subgroup of G. Identify a group $\mathbb{Z}_k$ that is isomorphic to H.
(b) List the 10 left cosets aH of H in G.
(c) List the 10 right cosets Ha of H in G.
(d) Is $aH = Ha$ for all $a \in G$? Why?
(e) What are the conjugates of H in G?

(4) Let G be the permutation group $S(4)$ and let H be the subgroup $A(4)$.
(a) List the 2 left cosets aH of H in G.
(b) List the 2 right cosets Ha of H in G.
(c) Is $aH = Ha$ for all $a \in G$? Why?
(d) What are the conjugates of H in G?

(5) Let G be the permutation group $S(4)$ and let H be the smallest subgroup of G containing the cycle $(1, 2, 3, 4)$.
(a) List the 6 left cosets aH of H in G.
(b) List the 6 right cosets Ha of H in G.
(c) Is $aH = Ha$ for all $a \in G$? Why?
(d) What are the conjugates of H in G?

(6) Let H be a subgroup of the finite group G and let $a \in G$. We define a *double coset* of H in G by

$$HaH = \{h_1 a h_2; h_1 \text{ and } h_2 \in H\}.$$

Show that

$$|H| \leq |HaH| \leq \min\{|H|^2, |G|\}.$$

Show that the above estimates are sharp; that is find examples for which
(a) $|H| = |HaH|$,
(b) $|HaH| = |H|^2$, and
(c) $|HaH| = |G|$.

(7) Construct an example of subgroup H of a finite group G and elements a and $b \in G - H$ such that $aH \neq Hb$.