Estimation Under Heteroscedasticity

Eco321: Econometrics

Donghwan Kim
Department of Economics
SUNY at Stony Brook

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Example

Data

y	x
9.25	1
5.50	2
2.88	3
2.80	4
0.04	5
-2.75	6
-4.08	7
-8.00	8
-6.91	9
1.00	10

The Model

Model

$$y_i = \beta_0 + x_i\beta_1 + \epsilon_i \quad \text{or} \quad y = X\beta + \epsilon$$

$$\text{with } E(\epsilon|X) = 0, \; E(\epsilon\epsilon^{'}|X) = \sigma^2 I$$

Regress y on x

out = regstats(y, x, 'linear');

The OLS estimation and its standard errors

The other way to find OLS estimator

```
n = length(x); X = [ones(n,1) x] ;
beta_hat = inv(X'*X)*X'*y ;
e = y - X*beta_hat ; k = length(beta_hat) ;
s2 = e'*e/(n-k) ;
var = s2*inv(X'*X) ;
se = sqrt(diag(var)) ;
tstat = beta_hat./se ;
```

When the error term is heteroscedastic

- The white test

The variance-covariance matrix

$$
\begin{aligned}
\operatorname{var}(\hat{\beta}) &= E(\hat{\beta}-\beta)(\hat{\beta}-\beta)' \\
&= E(X'X)^{-1}X'\epsilon\epsilon' X(X'X)^{-1} \\
&= (X'X)^{-1}X'\Sigma X(X'X)^{-1} \\
&= (X'X)^{-1}X' \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ 0 & & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n^2 \end{bmatrix} X(X'X)^{-1}
\end{aligned}
$$

σ_i^2 is estimated by e_i^2

The white-corrected standard errors

$$\mathrm{var}(\hat{\beta}) = (X'X)^{-1}X' \begin{bmatrix} e_1^2 & 0 & \dots & 0 \\ 0 & e_2^2 & \dots & 0 \\ 0 & & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_n^2 \end{bmatrix} X(X'X)^{-1}$$

Matlab Commands

```
e2 = e.^ 2 ;
V = diag(e2) ;
var_w = inv(X'*X)*X'*V*X*inv(X'*X);
se_w = sqrt(diag(var_w)) ;
tstat_w = beta_hat./se_w ;
```

When Heteroscedasticity is of a known form

- Assume that $\sigma_i^2 = \sigma^2 \cdot x_i^2$
- The Breusch-Pagan-Godfrey (BPG) test

The variance-covariance matrix

$$\Sigma = \sigma^2 \begin{bmatrix} x_1^2 & 0 & \ldots & 0 \\ 0 & x_2^2 & \ldots & 0 \\ 0 & & \ldots & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_n^2 \end{bmatrix} = \sigma^2 V$$

The Weighted Least Squares (WLS) Estimation

The idea of WLS: Making the error term homoscedastic

$$\begin{aligned} \mathrm{var}(V^{-\frac{1}{2}}\epsilon) &= V^{-1}\mathrm{var}(\epsilon) \\ &= V^{-1}\sigma^2 V \\ &= \sigma^2 I \end{aligned}$$

The Weighted Least Squares (WLS) Estimation

Step 1: Multiply by $V^{-\frac{1}{2}}$ in both sides

$$V^{-\frac{1}{2}}y = V^{-\frac{1}{2}}x\beta + V^{-\frac{1}{2}}\epsilon$$
$$y^* = x^*\beta + \epsilon^*, \quad \epsilon \sim (0, \sigma^2 I)$$

Matlab Commands

```
x2 = x.^ 2 ;
v = diag(x2) ;
yy = v^ (-1/2)*y ;
XX = v^ (-1/2)*X ;
```

The Feasible WLS Estimation

Step 2: Regress y^* on x^* by OLS

```
beta_wls = inv(XX'*XX)*XX'*yy ;
residual = yy - XX*beta_wls ;
ssquare = residual'*residual/(n-k) ;
var_wls = ssquare*inv(XX'*XX) ;
se_wls = sqrt(diag(var_wls));
tstat_wls = beta_wls./se_wls ;
```

Table: Results of Estimations

Variables	OLS	Corrected OLS	Weighted LS
Constant	7.95	7.95	10.94
	(2.36)	(1.93)	(0.64)
x	-1.44	-1.44	-2.10
	(0.38)	(0.46)	(0.25)

Note: Standard errors are in parentheses