SPECTRA, UNORIENTED BORDISMS, AND STEENROD PROBLEM

References:

(1) J.F.Adams, *Stable homotopy theory* (Lectures delivered at the University of California at Berkeley, 1961);
(2) J.F.Adams, *Stable homotopy and generalised homology*;
(3) Haynes Miller, *Notes on Cobordism*
(4) J.P.May, *A Concise Course in Algebraic Topology*;
(5) May-Ponto, *More Concise Algebraic Topology: Localization, Completion, and Model Categories*
(6) Milnor-Stasheff, *Characteristic classes*
(7) Robert Stong, *Notes on cobordism theory.*

1. Steenrod's problem

Convention: Throughout, a space means a finite simplicial complex, and a map means a continuous map.

1.1. Steenrod's problem on "smoothability" of cycles.

Our motivating question is:

Question 1.1. *What is the picture of a cycle?*

The following exercise can be solved by drawing pictures of simplicial chains:

Exercise 1.2. Let X be a space.

(1) Let $\alpha \in H_1(X;\mathbb{Z})$. Then there exists a map
$$f\colon S^1 \sqcup \cdots \sqcup S^1 \to X$$
such that $\alpha = f_*[S^1 \sqcup \cdots \sqcup S^1]$.

(2) Let $\alpha \in H_2(X;\mathbb{Z})$. Then there exists a map
$$f\colon \Sigma_1 \sqcup \cdots \sqcup \Sigma_k \to X$$
where Σ_i are oriented surfaces ($i = 1, \ldots, k$), such that $\alpha = f_*[\Sigma_1 \sqcup \cdots \sqcup \Sigma_k]$.

In fact, Exercise 1.2 can be continued for α with higher and higher dimensions, although the proofs get significantly harder – as the dimension of the cycle increases, it becomes increasingly harder to "resolve the singularities" on the cycle.

Remark 1.3. Doing Exercise 1.2 might also convince you that all cycles are smooth outside of a codimension-2 subset. This is basically the reason why pseudo-cycles compute ordinary homology. See McDuff-Salamon (Chapter 6), and Zinger[1], for more details.

One can ask the following general question:

Question 1.4 (Steenrod's problem). *Given a cycle $\alpha \in H_k(X;\mathbb{Z})$, does there exist a closed oriented manifold M^k of dimension k and a continuous map $M \xrightarrow{f} X$ such that $f_*[M] = \alpha$?*

[1] Zinger (2008), Pseudocycles and integral homology

Famously Thom (1954) answers the question in the negative. However, in contrast,

Theorem 1.5 (Thom, 1954)**.** *Given a cycle $\alpha \in H_k(X;\mathbb{F}_2)$, there exists a closed k-manifold M^k together with a continuous $M \xrightarrow{f} X$ such that $f_*[M] = \alpha$.*

Combining these, Thom shows that for any integral homology class, some odd multiple of it can be represented by a manifold.

1.2. **A reformulation: bordism theories.** Before going into the proof of Theorem 1.5, we first give a (tautological) reformulation of the Steenrod problem in terms of a comparison between two homology theories.

Let X be a space. We define the *unoriented bordism group* of X, denoted by $\Omega_n^O(X)$, to be the set of all continuous maps $M \xrightarrow{f} X$ where M ranges over (homeomorphism classes of) all closed n-manifolds, with the group operation given by

$$(M \xrightarrow{f} X) + (N \xrightarrow{g} X) := (M \sqcup N \xrightarrow{(f,g)} X),$$

modulo the bordism relation:

- A closed manifold M is *null-bordant* if there exists a manifold W whose boundary is M;
- An element $M \xrightarrow{f} X$ of $\Omega_n^O(X)$ is *null-bordant* if there exists a W with $\partial W = M$ and a map $W \xrightarrow{F} X$ extending f;
- Two elements $M \xrightarrow{f} X$ and $N \xrightarrow{g} X$ are *bordant* if $M \sqcup (-N) \xrightarrow{(f,g)} X$ is null-bordant.

There is no set-theoretic issue since all closed manifolds can be embedded in $\mathbb{R}^N$.

Remark 1.6. In our un-oriented setting, notice that any element in $\Omega_n^O(X)$ has order 2, so $\Omega_n^O(X)$ is a vector space over $\mathbb{F}_2$.

Moreover, Cartesian products between manifolds give us the product map

$$\Omega_n^O(X) \otimes \Omega_m^O(Y) \to \Omega_{n+m}^O(X \times Y).$$

In particular, this makes $\Omega_*^O := \Omega_*^O(\mathrm{pt})$ into a graded-commutative ring, called the *unoriented bordism ring*.

Back to the Steenrod problem, notice that our assignment to each $M^n \xrightarrow{f} X$ a homology class $f_*[M] \in H_n(X)$ in fact defines a well-defined homomorphism $\Omega_n^O(X) \to H_n(X)$, natural with respect to X. The only thing that we need to verify for this claim is that

Exercise 1.7. If $M \xrightarrow{f} X$ is null-bordant, then $f_*[M] = 0$ in homology.

Thus we have reformulated the Steenrod problem into a problem of studying this "morphism" between these two generalized homology theories Ω_*^O and H_*. In typical topologist fashion, we study this problem by studying the "universal" objects for these two theories.

2. Homology theories and spectra

We begin by a general discussion on generalized (co)homology theories and spectra, before focusing on the homology theories relevant to the $\mathbb{F}_2$-Steenrod problem: the mod-2 ordinary homology theory and the unoriented bordism theory.

2.1. **Eilenberg-Steenrod axioms and generalized (co)homology theories.** We temporarily switch to cohomology for familiarity. First recall the Eilenberg-Steenrod axioms for ordinary cohomology. Fix an abelian group π.

Theorem 2.1 (Eilenberg-Steenrod axioms)**.** *For each $q \in \mathbb{Z}$, there exist a contravariant functor $H^q(-;\pi)$ from the homotopy category of pairs of spaces to the category of abelian groups, together with a natural transformation*

$$\delta\colon H^q(Y;\pi) \to H^{q+1}(X,Y;\pi)$$

where $H^q(X;\pi)$ is defined to be $H^q((X,\emptyset);\pi)$. These satisfy and are characterized by the following axioms:

(1) ***(Exact sequence)*** *For each pair (X,A),*

$$\cdots \to H^q(X,Y;\pi) \to H^q(X;\pi) \to H^q(Y;\pi) \xrightarrow{\delta} H^{q+1}(X,Y;\pi) \to \cdots$$

is exact;

(2) ***(Excision axiom)*** *If $(X;A,B)$ is an excisive triad, so that X is the union of the interiors of A and B, then the inclusion $(A, A\cap B) \hookrightarrow (X,B)$ induces an isomorphism*

$$H^*(X,B;\pi) \xrightarrow{\cong} H^*(A, A\cap B;\pi);$$

(3) ***(Additivity axiom)*** *If (X,Y) is the disjoint union of a set of pairs (X_i,Y_i), then the inclusions $(X_i,Y_i) \hookrightarrow (X,Y)$ induce an isomorphism*

$$H^*(X,Y;\pi) \xrightarrow{\cong} \prod_i H^*(X_i,Y_i;\pi);$$

(4) ***(Weak equivalence)*** *If $f\colon (X,A) \to (Y,B)$ is a weak equivalence, then*

$$f^*\colon H^*(Y,B;\pi) \to H^*(X,A;\pi)$$

is a weak equivalence;

(5) ***(Dimension axiom)*** *For $X = \mathrm{pt}$,*

$$H^*(\mathrm{pt};\pi) = \begin{cases} \pi, & * = 0 \\ 0, & * \neq 0 \end{cases}.$$

Definition 2.2. A *generalized cohomology theory* (a.k.a. *extraordinary cohomology theory*) is a functor $E^*(-)$ from the homotopy category of pairs of spaces to the category of abelian groups, together with a natural transformation

$$\delta\colon E^*(Y) \to E^{*+1}(X,Y)$$

for each pair (X,Y), where $E^*(X)$ denotes $E^*(X,\emptyset)$, satisfying all of the Eilenberg-Steenrod axiom but the dimension axiom.

One can check that Ω_*^O is a generalized cohomology theory.

2.2. **Representing (co)homology theories by spectra.** Recall that ordinary cohomology with coefficient in an abelian group π is "represented" by the *Eilenberg-MacLane spaces* $K(\pi,i)$, characterized by

$$\pi_k(K(\pi,i)) = \begin{cases} \pi, & k = i \\ 0, & k \neq i \end{cases},$$

in the sense that for any space X,

$$H^i(X;\pi) = [X, K(\pi,i)] \tag{2.1}$$

as sets (where for two spaces X and Y, $[X,Y]$ denotes the set of homotopy classes of maps $X \to Y$).

Exercise 2.3. The left-hand side of (2.1) is an abelian group whereas the right-hand side is a priori only a set – is there a natural abelian group structure on the right hand side, and how to describe it?

This seemingly miraculous fact is actually a consequence of the much more general *Brown representability theorem*. Without recalling the full statement, we now state its consequence for generalized cohomology theories.

Suppose that we are given a generalized cohomology theory $E^*(-)$, defined on (say) CW pairs. Since

$$E^n(X) = E^n(X, \mathrm{pt}) \oplus E^n(\mathrm{pt}),$$

we define the *reduced cohomology*

$$\widetilde{E}^n(X) := E^n(X, \mathrm{pt}).$$

The additivity axiom (in the reduced context, the Milnor-Brown wedge axiom) gives that the map

$$\theta\colon \widetilde{E}^n\left(\bigvee_{i\in I} X_i\right) \to \prod_{i\in I} \widetilde{E}^n(X_i)$$

given by the inclusions $X_i \hookrightarrow \bigvee_i X_i$ is an isomorphism (takes coproducts to products). This, together with the Mayer-Vietoris property (takes weak pushouts to weak pullbacks), allows us to invoke the Brown representability theorem, which says there are connected CW complexes E_n with basepoints, and natural equivalences

$$\widetilde{E}^n(X) \cong [X, E_n]$$

as sets, where X ranges over connected CW complexes.

The generalized cohomology E^* has one more data, the coboundary map δ, which reflects in the reduced theory as the suspension isomorphisms

$$\widetilde{E}^n(X) \xrightarrow{\sigma} \widetilde{E}^{n+1}(\Sigma X).$$

[In more detail, σ is given by

$$\begin{tikzcd}
E^n(X, \mathrm{pt}) \arrow[r, "\delta"] \arrow[ddr, "\sigma"'] & E^{n+1}(CX, X) \\
& E^{n+1}(\Sigma X, C'X) \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
& E^{n+1}(\Sigma X, \mathrm{pt})
\end{tikzcd}$$

where each of the map in the diagram is an isomorphism, and we view the (reduced) suspension ΣX as the union along X of two (reduced) cones $CX, C'X$ of X.] Now in terms of the representing spaces E_n, the suspension isomorphism σ becomes gives

$$[X, E_n] \xrightarrow{\sigma} [\Sigma X, E_{n+1}] \cong [X, \Omega_0 E_{n+1}]$$

(where $\Omega_0 X$ of a space is the component of basepoint in the based loop space ΩX). Thus we get an equivalence $E_n \to \Omega_0 E_{n+1}$ for all $n \in \mathbb{Z}$.

This naturally leads to the following definition:

Definition 2.4 (G.W. Whitehead, ...)**.** A *spectrum* E is a sequence of spaces E_n with basepoint, together with choices of $\epsilon_n \colon \Sigma E_n \to E_{n+1}$, or equivalently $\epsilon'_n \colon E_n \to \Omega E_{n+1}$. If we are working with connected spaces, then automatically $\epsilon'_n(E_n) \subset \Omega_0 E_{n+1}$, so a variant of the definition specifies maps $\epsilon'_n \colon E_n \to \Omega_0 E_{n+1}$.

A spectrum is an Ω-*spectrum* (resp. Ω_0-spectrum) if $\epsilon'_n \colon E_n \to \Omega E_{n+1}$ (resp. $\epsilon'_n \colon E_n \to \Omega_0 E_{n+1}$) is a weak equivalence for each $n \in \mathbb{Z}$.

To summarize, for each generalized cohomology theory, we have constructed an associated Ω_0-spectrum. (There is a version of construction where we allow our spaces X to be disconnected and construct an Ω-spectrum out of it. We will be vague about such details.)

Example 2.5. For an abelian group π, denote by $H\pi$ the spectrum associated to the ordinary cohomology theory $H^*(-;\pi)$. Thus $(H\pi)(n) = K(\pi, n)$.

2.3. **Homotopy theory for spectra.** Other than representing spaces of generalized cohomology theories, one other natural example is:

Example 2.6. Let X be a space. Define its *suspension spectrum* $\Sigma^\infty X$ by

$$\Sigma^\infty X(n) := \begin{cases} \Sigma^n X, & n \geq 0 \\ \mathrm{pt}, & n < 0 \end{cases}$$

with the obvious maps. Since a "jump" happens at $n = 0$, this is not an Ω-spectrum!

The suspension spectrum $\Sigma^\infty S^0$ of the 0-sphere plays a special role in the theory. We call it the *sphere spectrum* and denote it by $\mathbb{S}$, with $\mathbb{S}(n) = S^n$ for $n \geq 0$.

Recall the usual definition of stable homotopy group of a space X

$$\pi_r^{st}(X) := \varinjlim_{n\to\infty} \pi_{r+n}(\Sigma^n X).$$

Notice that this definition adapts nicely for spectra.

Definition 2.7. Let E be a spectrum. Its r-th *homotopy group* is defined by

$$\pi_r(E) = \varinjlim_{n\to\infty} \pi_{n+r}(E_n)$$

where the direct system is formed by maps

$$\pi_{n+r}(E_n) \xrightarrow{\Sigma} \pi_{n+r+1}(\Sigma E_n) \xrightarrow{\epsilon_n} \pi_{n+r+1}(E_{n+1}).$$

The direct limit is attained for Ω_0-spectra. For suspension spectra $\Sigma^\infty X$, the direct limit is also attained, thanks to Freudenthal suspension theorem.

From this viewpoint, spectra are "stable" topological spaces. We would therefore like to study their homotopy theory. Just like in the case of spaces, we need to first build and decorate our category of spectra by (1) fixing a nicely-behaving class of spectra, the CW spectra; (2) introduce various homotopical notions (what are the morphisms? what is a homotopy?) (3) introduce various homotopical operations and constructions (e.g. wedge and smash products, (co)fibrations) and undertand their properties. We only briefly sketch these for the sake of time. Throughout, one should notice various similarities of this category of spectra with the category of chain complexes of abelian groups.

The following follows Part III of Adams's book on *Stable Homotopy and Generalised Homology* very closely.

2.3.1. *CW spectra and subspectra.*

Definition 2.8. A spectrum E is a *CW spectrum* if (1) E_n are CW complexes; (2) $\epsilon_n \colon \Sigma E_n \to E_{n+1}$ is an isomorphism onto a subcomplex of E_{n+1}.

Remark 2.9. One can eventually prove a CW approximation theorem so that this is no essential loss of information.

From now on a spectrum means a CW spectrum.

Definition 2.10. A *subspectrum* of a spectrum X is a collection of subcomplexes $Y_n \subset X_n$ such that under the structure maps $\Sigma X_n \to X_{n+1}$, the subcomplex ΣY_n is mapped to Y_{n+1}.

Then we can define *relative homotopy groups* of spectra by

$$\pi_r(X, Y) := \varinjlim_{n\to\infty} \pi_{n+r}(X_n, A_n),$$

and one can show the sequence

$$\cdots \to \pi_* Y \to \pi_* X \to \pi_*(X, Y) \to \pi_{*-1} Y \to \cdots$$

is exact.

2.3.2. *Maps between spectra.* The definition of a morphism is trickier than one might have thought. Let us first define the notion one might naively guess:

Definition 2.11. A *function* f from one spectrum E to another F of degree r is a sequence of maps $f_n \colon E_n \to F_{n-r}$ such that the diagrams of the structure maps are strictly commutative for each n:

$$\begin{tikzcd} \Sigma E_n \arrow[r] \arrow[d, "\Sigma f_n"'] & E_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma F_{n-r} \arrow[r] & F_{n-r+1} \end{tikzcd}.$$

Example 2.12. We would like the Hopf map $S^3 \to S^2$ to induce a function $\mathbb{S} \to \mathbb{S}$ of degree -1. But Hopf map does not come from a suspension of a map $S^2 \to S^1$.

To remedy this, we should allow our map $E \to F$ to be ill-defined on certain places, but these ill-defined regions are "eventually" well-defined after enough suspensions.

Definition 2.13. Let E be a CW spectrum. A subspectrum $E' \subset E$ is said to be *cofinal* (or *dense*) if for all n and for all finite subcomplex $K \subset E_n$, there exists an m, depending on n and K, such that $\Sigma^m K$ is mapped into E'_{m+n} under the obvious map.

Definition 2.14. Let E be a CW spectrum and F a spectrum. A *map* from E to F is a function $f' \colon E' \to F$ where $E' \subset E$ is a cofinal subspectrum. Two such $E' \xrightarrow{f'} F, E'' \xrightarrow{f''} F$ are equivalent if there is a cofinal subspectrum E''' contained in E' and E'' such that the restriction of f' and f'' to E''' coincide. Compositions of maps are defined in an obvious way.

2.3.3. *Homotopy of maps between spectra.* To define the notion of homotopy, we need the notion of a cylinder.

Definition 2.15. Let $I^+ := [0,1] \sqcup \{*\}$. If E is a spectrum, define the *cylinder spectrum* $\mathrm{Cyl}(E)$ to have terms

$$(\mathrm{Cyl}(E))_n := I^+ \wedge E_n$$

and structure maps

$$(I^+ \wedge E_n) \wedge S^1 \xrightarrow{1\wedge\epsilon_n} I^+ \wedge E_{n+1}.$$

Denote the two inclusions of E into $\mathrm{Cyl}(E)$ by

$$i_0, i_1 \colon E \to \mathrm{Cyl}(E).$$

Definition 2.16. Two maps $f_0, f_1 \colon E \to F$ are *homotopic* if there is a map

$$h \colon \mathrm{Cyl}(E) \to F$$

such that $f_0 = hi_0, f_1 = hi_1$.

Denote by $[E, F]_r$ the set of homotopy classes of maps of degree r from E to F.

As a sanity check:

Proposition 2.17. *For a finite CW complex K and a spectrum F,*

$$[\Sigma^\infty K, F]_r \cong \varinjlim_{n\to\infty} [\Sigma^{n+r} K, F_n].$$

In particular the homotopy groups of a spectrum is given by

$$\pi_r(F) \cong [\mathbb{S}, F].$$

Theorem 2.18 (Stable Freudenthal suspension)**.** $\mathrm{Susp}_* \colon [X, Y] \to [\mathrm{Susp}(X), \mathrm{Susp}(Y)]$ *is a bijection.*

Ingredients of the proof. Consider

$$\begin{tikzcd}
{[X,Y]} \arrow[r, "\mathrm{Cone}"] \arrow[d, "\mathrm{Susp}"'] & {[\mathrm{Cone}(X), X; \mathrm{Cone}(Y), Y]} \arrow[d, "j_*"] \\
{[\mathrm{Susp}(X), \mathrm{Susp}(Y)]} \arrow[r, "j^*"'] & {[\mathrm{Cone}(X), X; \mathrm{Susp}(Y), \mathrm{pt}]}
\end{tikzcd}$$

- The map Cone is a bijection because we can define a restriction map

  $$\mathrm{res} \colon [\mathrm{Cone}(X), X; \mathrm{Cone}(Y), Y] \to [X, Y].$$

  To show that $\mathrm{Cone} \circ \mathrm{res}$ is the identity, we need to use a version of homotopy extension lemma for spectra and the fact that $\pi_* \mathrm{Cone}(Y) = 0$.
- j^* is clearly a bijection;
- j_* is a bijection by a relative version of Whitehead's theorem, using the fact that

  $$j_* \colon \pi_*(\mathrm{Cone}(Y), Y) \to \pi_*(\mathrm{Susp}(Y), \mathrm{pt})$$

  is a bijection (this requires the usual Frendenthal suspension theorem).

□

Let us explain how this makes $[X, Y]$ into an abelian group. The set $[\mathrm{Susp}(X), Y]$ is a group, since we can use the spare suspension coordinate S^1 to concatenate domains of the maps. Similarly $[\mathrm{Susp}^2(X), Y]$ is a group since we can use the S^2 to concatenate; but it is also abelian since we have space to homotope the two ways of concatenating. This is essentially the same reason why for an ordinary space K, $\pi_0(K)$ is a set, $\pi_1(K)$ is a group, and $\pi_2(K)$ is an abelian group.

In fact, if we don't stop at $\mathrm{Susp}^2(X)$ or π_2, we notice in $\mathrm{Susp}^3(X)$ or π_3, we have not just an abelian group, but also the various ways of homotoping the two concatenations are themselves homotopic to each other, and so on and so forth. Therefore in some sense, $[X, Y]$ is better than an abelian group – it is "infinitely abelian".

2.3.4. *Products and coproducts of spectra.*

Definition 2.19. For a collection $\{X_\alpha\}_{\alpha\in A}$ of spectra, define the *wedge product* $\bigvee_\alpha X_\alpha$ by

$$\left(\bigvee_\alpha X_\alpha\right)_n := \bigvee_\alpha (X_\alpha)_n$$

and the structure maps

$$\left(\bigvee_\alpha (X_\alpha)_n\right) \wedge S^1 = \bigvee_\alpha ((X_\alpha)_n \wedge S^1) \to \bigvee_\alpha (X_\alpha)_{n+1}.$$

Lemma 2.20 (Wedge product is a coproduct)**.**

$$\left[\bigvee_\alpha X_\alpha, Y\right] \xrightarrow{\cong} \prod_\alpha [X_\alpha, Y].$$

Next we would like to define products in the category of spectra. Unfortunately the construction is very complicated (it took over 30 pages in Adams's book!). Fortunately all that we will use are properties of the construction, which we shall state:

Theorem 2.21. *For X, Y CW spectra, there is a CW spectrum $X \wedge Y$ called the* smash product *of X and Y, such that*

(1) *$X \wedge Y$ is functorial in both X and Y;*
(2) *$\wedge$ is commutative, associative, and has the sphere spectrum $\mathbb{S}$ as a unit, up to coherent equivalences;*
(3) *The smash product is distributive over the wedge sum: let $i_\alpha\colon X_\alpha \to \bigvee_\alpha X_\alpha$, then the morphism*

$$\bigvee_\alpha (X_\alpha \wedge Y) \xrightarrow{\{i_\alpha\wedge 1\}} \left(\bigvee_\alpha X_\alpha\right) \wedge Y$$

is an equivalence;
(4) *If $X \xrightarrow{f} Y \xrightarrow{i} Z$ is a cofibering, then*

$$W \wedge X \xrightarrow{1\wedge f} W \wedge Y \xrightarrow{1\wedge i} W \wedge Z$$

is a cofibering.

For the precise meaning of cofibering in the context of spectra, see the upcoming section 2.3.5.

2.3.5. *Fiber and cofiber sequences.* As with spaces, we need to introduce the notions of cones and suspensions.

Definition 2.22. Let X be a spectrum. Let $\mathrm{Cone}(X)$ be the spectrum with

$$\mathrm{Cone}(X)_n := I \wedge X_n, \quad (I \wedge X_n) \wedge S^1 \xrightarrow{1\wedge\epsilon_n} I \wedge X_{n+1}.$$

Definition 2.23. Let X be a spectrum. Let $\mathrm{Susp}(X)$ be the spectrum with

$$\mathrm{Susp}(X)_n := S^1 \wedge X_n, \quad (S^1 \wedge X_n) \wedge S^1 \xrightarrow{1\wedge\epsilon_n} S^1 \wedge X_{n+1}.$$

Definition 2.24. Given a map $f\colon X \to Y$ between CW spectra, represented by a function $f'\colon X' \to Y$ where X' is a cofinal subspectrum, define $Y \cup_{f'} CX$ by

$$(Y \cup_{f'} CX)_n := Y_n \cup_{f'_n} (I \wedge X'_n)$$

with the obvious structure maps. Choosing different X' and f' gives equivalent result, so we denote it by $Y \cup_f CX$ and call it the *mapping cone.*

Then we have the following sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX.$$

Anything equivalent to this sequence is called a *cofiber sequence* or a *Puppe sequence.*

Example 2.25. Let $A \subset X$ be a subspectrum. We say A is *closed* if for every finite subcomplex $K \subset X_n$, $\Sigma^m K \subset A_{m+n}$ implies $K \subset A_n$. Denote by $i\colon A \to X$ the inclusion. Then we can form the quotient spectrum X/A where $(X/A)_n := X_n/A_n$, and there is a map

$$r\colon X \cup_i CA \to X/A$$

which is an equivalence (by a version of Whitehead theorem for spectra). The quotient sequence

$$A \xrightarrow{i} X \to X/A$$

is a cofibering.

As with spaces, continuing the sequence to the right gives

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \xrightarrow{j} \mathrm{Susp}(X) \xrightarrow{-\mathrm{Susp}(f)} \mathrm{Susp}(Y) \to \cdots$$

Proposition 2.26. *For each Z, the sequence*

$$[X,Z] \xleftarrow{f^*} [Y,Z] \xleftarrow{i^*} [Y \cup_f CX, Z] \xleftarrow{j^*} [\mathrm{Susp}(X), Z] \xleftarrow{-\mathrm{Susp}(f)^*} [\mathrm{Susp}(Y), Z] \leftarrow \cdots$$

is exact.

Proof. Exercise. □

Proposition 2.27 (Cofiberings are the same as fiberings)**.** *The sequence*

$$[W,X] \xrightarrow{f_*} [W,Y] \xrightarrow{i_*} [W, Y \cup_f CX]$$

is exact.

Proof. Chase the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "i"] & Y \cup_f CX \arrow[r, "j"] & \mathrm{Susp}(X) \arrow[r, "-\mathrm{Susp}(f)"] & \mathrm{Susp}(Y) \\
W \arrow[r, "1"] & W \arrow[u, "g"] \arrow[r, "i"] & \mathrm{Cone}(W) \arrow[u, "h"] \arrow[r, "j"] & \mathrm{Susp}(W) \arrow[u, "k"] \arrow[r, "-1"] & \mathrm{Susp}(W) \arrow[u, "\mathrm{Susp}(g)"]
\end{tikzcd}.$$

(If for $g \in [W,Y]$ we have $ig \sim 0$, then we get h and thus k, and that it maps to g under f_* follows from the rightmost square.) □

2.3.6. *Homology and cohomology.* Now that we have seen that a spectrum plays both the role of a "stable space" and a generalized cohomology theory, let us define

Definition 2.28 (G.W.Whitehead)**.** Given a spectrum E, we define the E-homology and E-cohomology of another spectrum X as follows:

(1) $E_n(X) = [\mathbb{S}, E \wedge X]_n$;
(2) $E^n(X) = [X, E]_{-n}$.

One can easily check, using properties we have previously stated, that these are generalized (co)homology theories satisfying (the spectral analogues of) the Eilenberg-Steenrod axioms. Moreover, the dimension axiom becomes

$$E_n(\mathbb{S}) = E^{-n}(\mathbb{S}) = \pi_n(E).$$

Before going into further details, let us mention a curious fact:

Theorem 2.29 (G.W.Whitehead)**.** *$E_n(X) \cong X_n(E)$.*

Proof. Since $E \wedge X \to X \wedge E$ is an equivalence,

$$[\mathbb{S}, E \wedge X]_n \cong [\mathbb{S}, X \wedge E]_n.$$

□

The proof is one-line, but note the following corollary, whose proof is more nontrivial if we don't have this language:

Corollary 2.30. *$(H\pi)_n(HG) \cong (HG)_n(H\pi)$.*

2.3.7. *Ring spectra.* The representing spectrum of a multiplicative cohomology theory has further structures:

Definition 2.31. A *ring spectrum* is a spectrum E together with maps $\mu\colon E \wedge E \to E$, $\eta\colon \mathbb{S} \to E$ of degree 0 such that the "obvious" diagrams commute.

One can similarly define commutative ring spectra and module spectra.

Given a ring spectrum, the cohomology theory it represents is multiplicative:

$$\widetilde{E}(X) \otimes \widetilde{E}(Y) \cong [X, E] \otimes [Y, E] \cong [X \wedge Y, E \wedge E] \xrightarrow{\mu} [X \wedge Y, E] \cong \widetilde{E}(X \wedge Y).$$

The multiplication $E(X) \otimes E(Y) \to E(X \times Y)$ is then induced from this by the plus construction in a standard way.

3. Thom's reformulation

Thom's brilliant insight is to "dualize" the Steenrod problem – in the words of Dennis[2], the Pontryagin-Thom construction illustrates "*basic duality between geometric covariant objects and algebraic contravariant objects*". Along the way, we will discover the representing spectrum for bordism theories.

Notation: Given a bundle η, we distinguish the bundle with its total space by calling the total space $E(\eta)$. The trivial k-plane bundle over a space X is denoted by $\epsilon^k = \epsilon^k_X$. The Thom space of a bundle η is denoted $\mathrm{Th}(\eta)$, and is defined by the quotient of the associated disc bundle $D(\eta)$ over the unit sphere bundle $S(\eta)$.

[2] Sullivan, Dennis. "René Thoms work on geometric homology and bordism." *Bulletin of the American Mathematical* Society 41.3 (2004): 341-350.

3.1. **Pontryagin-Thom construction.** Given $M \xrightarrow{f} X$, where M is a closed n-dimensional manifold, we will associate to it an algebraic contravariant object as follows.

- First, choose an embedding $i\colon M \hookrightarrow S^{n+k}$. Denote by ν the normal bundle of the embedding.
- Then the Gauss map gives us a classifying map which we still denote by $\nu\colon M \to BO(k)$, which pulls back the universal k-plane bundle ξ_k over $BO(k)$ to the normal bundle ν.
- Taking the product of $BO(k)$ with X, we obtain a bundle map

$$\begin{tikzcd} E(\nu) \arrow[r] \arrow[d] & E(\xi_k) \times X \cong E(\xi_k \times \epsilon^0_X) \arrow[d] \\ M \arrow[r, "\nu \times f"] & BO(k) \times X \end{tikzcd}.$$

  This induces a map on the Thom spaces: $\mathrm{Th}(\nu) \to \mathrm{Th}(\xi_k \times \epsilon^0_X) \cong \mathrm{Th}(\xi_k) \wedge X_+$ (where X_+ is the one-point compactification of X)[3];
- Composing the collapsing map $S^{n+k} \to \mathrm{Th}(\nu)$ by quotienting out everything outside a tubular neighborhood of ν with the map $\mathrm{Th}(\nu) \to \mathrm{Th}(\xi_k) \wedge X_+$, we obtain a map $S^{n+k} \to \mathrm{Th}(\xi_k) \wedge X_+$.

Definition 3.1. Denote by $MO(k) := \mathrm{Th}(\xi_k)$, the Thom space of the universal bundle.

Summarizing, starting from $M \xrightarrow{f} X$, we have associated with it an element $\pi_{n+k}(MO(k) \wedge X_+)$. We now need to study its dependence on various data chosen during the process:

- (Dependence on stabilization) Suppose that we chose a different embedding $i'\colon M \hookrightarrow S^{n+k'}$. Recall the following theorem in differential topology:

  Theorem 3.2. *Suppose $i\colon M \hookrightarrow \mathbb{R}^{n+k}$ and $i'\colon M \hookrightarrow \mathbb{R}^{n+k'}$, with extensions to embeddings of $E(\nu_i)$ and $E(\nu_{i'})$. Then for m large enough, the composite embeddings of $E(\nu_i)$ and $E(\nu_{i'})$ into $\mathbb{R}^{n+m}$ are isotopic.*

  Thus we just need to check what happens when we stabilize the embedding $M \hookrightarrow S^{n+k} \hookrightarrow S^{n+k+1}$.

  Exercise 3.3. The following diagram commutes:

$$\begin{tikzcd} S^{n+k+1} \arrow[r, Rightarrow] & \mathrm{Th}(\nu \oplus \epsilon^1) \arrow[r] & MO(k+1) \wedge X_+ \\ \Sigma S^{n+k} \arrow[r] \arrow[u, "\cong"] & \Sigma(\mathrm{Th}(\nu)) \arrow[r] \arrow[u, "\cong"] & \Sigma MO(k) \wedge X_+ \arrow[u, "\cong"] \end{tikzcd}$$

  Thus from $M \xrightarrow{f} X$ we obtain an element in

$$MO_n(X) := \varinjlim_k \pi_{n+k}(MO(k) \wedge X_+).$$

- (Dependence on bordism class) Given a null-bordism $X = \partial W$, we can embed W into D^{n+k+1} so that the boundary $\partial W \hookrightarrow \partial D^{n+k+1} = S^{n+k}$. Checking the commutativity

[3] *Exercise*: (1) $\mathrm{Th}(\eta \times \xi) \cong \mathrm{Th}(\eta) \wedge \mathrm{Th}(\xi)$; (2) Over a compact base, $\mathrm{Th}(\eta) \cong E(\eta)_+$.

of

$$\begin{tikzcd} S^{n+k} \arrow[r] \arrow[d, hook] & \mathrm{Th}(\nu) \arrow[r] \arrow[d, hook] & MO(k) \wedge X_+ \\ D^{n+k+1} \arrow[r] & \mathrm{Th}(\widetilde{\nu}) \arrow[ur] & \end{tikzcd},$$

gives a well-defined map

$$\Omega_n^O(X) \to \varinjlim_k \pi_{n+k}(MO(k) \wedge X_+).$$

Conversely, given a map $f\colon S^{n+k} \to MO(k) \wedge X_+$, we would like to obtain an element in $\Omega_n^O(X)$. Recall that the base space $BO(k)$ is the direct limit of Grassmannians $Gr_k(\mathbb{R}^{n+k})$, and the universal k-plane bundle $\xi_k \to BO(k)$ is the direct limit of bundles $\xi_{k,n} \to Gr_k(\mathbb{R}^{n+k})$. By compactness, the image of f lands in $\mathrm{Th}(\xi_{k,n})$ for some finite n. Transversality (near the zero section $Gr_k(\mathbb{R}^{n+k}) \hookrightarrow \mathrm{Th}(\xi_{n,k})$) gives us that $M := f^{-1}(Gr_k(\mathbb{R}^{n+k}))$ is a well-defined bordism class of manifold, and the composition $M \xrightarrow{f} MO(k) \wedge X_+ \xrightarrow{\mathrm{proj}_2} X_+$ gives us an element in $\Omega_n^O(X)$. One can check that this construction gives us a well-defined map

$$\varinjlim_k \pi_{n+k}(MO(k) \wedge X_+) \to \Omega_n^O(X),$$

which is clearly a two-sided inverse to our previous map.

The spaces $MO(k)$ together with the suspension map forms a ring spectrum MO.

Theorem 3.4 (Thom, 1954)**.** *The Thom spectrum MO represents the unoriented bordism theory $\Omega_n^O(-)$.*

We have thus further reduced the Steenrod problem from a comparison between two homology theories $H_*(-;\mathbb{F}_2)$ and $\Omega_*^O(-)$, to a comparison between their representing spectra $(K(\mathbb{F}_2, n))_n$ and $MO(n)$. Namely, we have constructed (by "pushing forward fundamenta classes") a morphism $MO \to H\mathbb{F}_2$, and we would like to construct a splitting $H\mathbb{F}_2 \to MO$.

For this we need to compute the topologies of these spaces.

4. Cohomology of $MO(n)$: Thom isomorphisms and characteristic classes

In this section we compute the cohomology of MO. Most of the content from this section can be found in Milnor-Stasheff.

4.1. Thom isomorphism.

Recall that Thom isomorphism says:

Theorem 4.1 (Thom, 1951; see Milnor-Stasheff Theorem 8.1)**.** *Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a rank-n, unoriented, real vector bundle. Denote by E_0 the complement of the zero-section in E, and $F_0 := F \setminus \{0\}$. Then there exists a unique class $U \in H^n(E, E_0; \mathbb{F}_2)$, called the* Thom class, *such that its restriction to each fiber*

$$U|_{(F,F_0)} \in H^n(F, F_0; \mathbb{F}_2) \cong \mathbb{F}_2$$

is the non-zero element, and that the correspondence

$$H^k(B; \mathbb{F}_2) \to H^{k+n}(E, E_0; \mathbb{F}_2); \quad x \mapsto \pi^* x \cup U$$

defines an isomorphism for every k.

Therefore, for each bundle η over B, the cohomology of the Thom space is given by

$$\widetilde{H}^*(\mathrm{Th}(\eta);\mathbb{F}_2) \cong H^*(E(\eta), E(\eta)_0;\mathbb{F}_2) \cong H^{*-\mathrm{rank}(\eta)}(B;\mathbb{F}_2).$$

Moreover, this isomorphism is an isomorphism of $H^*(B)$-modules, where the action on the relative group $H^*(E(\eta), E(\eta)_0;\mathbb{F}_2)$ is given by pulling back and taking cup product. Thus $\widetilde{H}^*(\mathrm{Th}(\eta);\mathbb{F}_2)$ is a free $H^*(B)$-module generated by the Thom class $U \in \widetilde{H}^{\mathrm{rank}(\eta)}(\mathrm{Th}(\eta);\mathbb{F}_2)$.

4.2. **Stiefel-Whitney classes.** For the case of $MO(n)$, the cohomology ring of the base space $H^*(BO(n);\mathbb{F}_2)$ is well-understood, since $BO(n)$ can be constructed by taking a direct limit of real Grassmannians:

Theorem 4.2 (See Milnor-Stasheff Theorem 7.1)**.** *The cohomology ring of $BO(n)$ is given by*

$$H^*(BO(n);\mathbb{F}_2) \cong \mathbb{F}_2[w_1,\dots,w_n], \quad w_i \in H^i(BO(n);\mathbb{F}_2),$$

i.e. a polynomial algebra over $\mathbb{F}_2$ freely generated by the Stiefel-Whitney classes $w_1(\xi_n),\dots,w_n(\xi_n)$ of the universal n-plane bundle.

5. Cohomology of $H\mathbb{F}_2$: the Steenrod algebra

Before going into Steenrod algebra, we should say one word about why they feature prominently in this story. We would like to study the cohomology $H^i(K(\mathbb{F}_2,n);\mathbb{F}_2)$ of the Eilenberg-MacLane spaces. By the fact that these spaces themselves represent the $H\mathbb{F}_2$ theory,

$$H^i(K(\mathbb{F}_2,n);\mathbb{F}_2) \cong [K(\mathbb{F}_2,n), K(\mathbb{F}_2,i)].$$

On the other hand, by "Yoneda", this set can be identified as the set of natural transformations from $H^n(-;\mathbb{F}_2)$ to $H^i(-;\mathbb{F}_2)$. These are, by definition, the (primary) cohomology operations. To be more honest, we only need the stable cohomologies of the Eilenberg-MacLane spaces, and the corresponding stable operations are governed by the Steenrod algebra. We now make these precise.

We very closely follow a combination of Adams Berkeley notes and Miller's notes.

5.1. **A brief overview of the mod-2 Steenrod algebra.** Recall that the mod-2 Steenrod operations are homomorphisms

$$\mathrm{Sq}^i\colon H^n(X,Y;\mathbb{F}_2) \to H^{n+i}(X,Y;\mathbb{F}_2)$$

satisfying (and are characterized by):

(1) Naturality: given $(X,Y) \xrightarrow{f} (X',Y')$,

$$\begin{array}{ccc} H^n(X',Y';\mathbb{F}_2) & \xrightarrow{f^*} & H^n(X,Y;\mathbb{F}_2) \\ \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} \\ H^{n+i}(X',Y';\mathbb{F}_2) & \xrightarrow[f^*]{} & H^{n+i}(X,Y;\mathbb{F}_2) \end{array} ;$$

(2) Stability: if $\delta\colon H^n(Y;\mathbb{F}_2) \to H^{n+1}(X,Y;\mathbb{F}_2)$ is the coboundary map, then $\mathrm{Sq}^i(\delta u) = \delta(\mathrm{Sq}^i u)$;
(3) Properties for small values of i:
    (a) $\mathrm{Sq}^0 u = u$;
    (b) $\mathrm{Sq}^1 u = \beta u$ where β is the mod-2 Bockstein;
(4) Properties for small values of n:

  (a) If $n = i$, $\mathrm{Sq}^i u = u^2$;
  (b) If $n < i$, $\mathrm{Sq}^i u = 0$;
(5) Cartan formula:
$$\mathrm{Sq}^i(u \cup v) = \sum_{j+k=i} (\mathrm{Sq}^j u) \cup (\mathrm{Sq}^k u);$$
(6) Adem relations: if $i < 2j$ then
$$\mathrm{Sq}^i \mathrm{Sq}^j = \sum_{\substack{k+\ell=i+j \\ k \geq 2\ell}} \lambda_{k,l} \mathrm{Sq}^k \mathrm{Sq}^\ell$$
  where $\lambda_{k,\ell}$ are certain binomial coeffcients.

We also write
$$\mathrm{Sq} := \sum_{i=0}^{\infty} \mathrm{Sq}^i$$
to be the total Steenrod operation, so that, for example, the Cartan formula becomes
$$\mathrm{Sq}(u \cup v) = \mathrm{Sq}(u) \cup \mathrm{Sq}(v).$$

We give a very brief sketch of the construction. Let X be a finite simplicial complex. In e.g. defining the cup product on $H^*(X)$, we must define a diagonal embedding $X \hookrightarrow X \times X$. However the "genuine" diagonal map $X \hookrightarrow X \times X$ is not a simplicial embedding (e.g. if $X = [0, 1]$, the diagonal embedding $[0, 1] \to [0, 1] \times [0, 1]$ by $t \mapsto (t, t)$ does not map cells to cells), and we must choose an *approximation to diagonal* $\widetilde{\Delta}$. Such choice is far from unique: let $\tau \colon X \times X \to X \times X$ be the map that switches the two factors, then $\tau \circ \widetilde{\Delta}$ is also an approximation to diagonal. In other words, we obtain an $\mathbb{F}_2$-invariant map
$$S^0 \times X \to X \times X,$$
where the $\mathbb{F}_2$-action on the domain is on S^0 and the $\mathbb{F}_2$-action on the target is by τ. But by acyclic carrier theorem, all such diagonal approximations are homotopic. So choosing a homotopy gives a map $[0, 1] \times X \to X \times X$. Again, flipping this map by τ gives another such homotopy, and thus an $\mathbb{F}_2$-invariant map
$$S^1 \times X \to X \times X$$
where the $\mathbb{F}_2$-action on S^1 is the antipodal action. Continuing this procedure gives us an $\mathbb{F}_2$-invariant map
$$S^\infty \times X \to X \times X$$
and passing to $\mathbb{F}_2$-equivariant cohomology in $\mathbb{F}_2$-coefficient gives us a map
$$H^*(X; \mathbb{F}_2) \to H^*_{\mathbb{F}_2}(X \times X; \mathbb{F}_2) \to H^*(X; \mathbb{F}_2) \otimes H^*(B\mathbb{F}_2; \mathbb{F}_2) \cong H^*(X; \mathbb{F}_2)[x]$$
where $|x| = 1$, and we define the x^k-coefficient of this map to be Sq^k.

Definition 5.1. A *(primary) cohomology operation of type* (n, m, G, H) is a collection of
$$\phi \colon H^n(X, Y; G) \to H^m(X, Y; H)$$
which are natural with respect to mappings of pairs.

A *stable (primary) cohomology operation of degree* i is a collection of
$$\phi_n \colon H^n(X, Y; G) \to H^{n+i}(X, Y; G)$$
defined for each n and each pair (X, Y), so that each ϕ_n is natural with respect to mappings of pairs, and that $\phi_{n+1}\delta = \delta\phi_n$ where δ is the coboundary homomorphism.

Definition 5.2. Define the set $\mathcal{A}$ to be the set of stable primary operations for $G = \mathbb{F}_2$.

The set $\mathcal{A}$ is obviously a graded associative algebra (by addition and composition).

Theorem 5.3 (Serre)**.** *The algebra $\mathcal{A}$ is generated by the Steenrod squares Sq^i. More precisely, $\mathcal{A}$ has a $\mathbb{Z}_2$-basis of operations*

$$\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_t}$$

where $i_1, \ldots, i_t$ take all values such that

$$i_r \geq 2i_{r+1} \ (1 \leq r < t), \quad i_t > 0,$$

and the empty product is admitted and interpreted as the identity operation.

These products are called *admissible monomials*. The reason of the $i_r \geq 2i_{r+1}$ is the Adem relations. This theorem is why $\mathcal{A}$ is called the *Steenrod algebra*.

Proof sketch. The linear independence can be seen by looking at the action of $\mathcal{A}$ on specific spaces. Since this is a useful computation we do it in some detail.

Let us first calculate the action of $\mathcal{A}$ on $H^*\mathbb{RP}^\infty = \mathbb{F}_2[x]$, where $x = w_1(\xi_1) \in H^1(\mathbb{RP}^\infty; \mathbb{F}_2)$ is the 1st Stiefel-Whitney class of the universal line bundle ξ_1 over $\mathbb{RP}^\infty \cong K(\mathbb{F}_2, 1)$. By property (3a) and (4a), $\mathrm{Sq}(x) = x + x^2$, and thus by the Cartan formula,

$$\mathrm{Sq}(x^k) = (\mathrm{Sq}(x))^k = x^k(1+x)^k.$$

Thus

$$\mathrm{Sq}^i x^k = \binom{k}{i} x^{k+i}.$$

Now let $X := \prod_{i=1}^n \mathbb{RP}^\infty$. The cohomology ring $H^*(X;\mathbb{F}_2)$ is the polynomial ring $\mathbb{F}_2[x_1, \ldots, x_n]$ where $x_i \in H^1(X;\mathbb{F}_2)$ comes from the i-th $\mathbb{RP}^\infty$ factor. We compute that the action of $\mathrm{Sq}^I := \mathrm{Sq}^{i_1}\cdots\mathrm{Sq}^{i_k}$ (where $I = (i_1, \ldots, i_k)$ is admissible: $i_{j-1} \geq 2i_j$ for all j) on $x_1 \ldots x_n \in H^*(X;\mathbb{F}_2)$. This is done by using our previous calculation and the Cartan formula repeatedly. This will result in a huge sum. To deal with it, we order the monomials $x_1^{i_1} \ldots x_n^{i_n}$ lexicographically. Then:

- For $k = 1$, we have
$$\mathrm{Sq}^i(x_1 \cdots x_n) = x_1^2 \cdots x_i^2 x_{i+1} \cdots x_n + \text{smaller terms};$$
- For $k = 2$, we have, for $i_1 \geq 2i_2$ (admissibility),
$$\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}(x_1 \cdots x_n) = x_1^4 \cdots x_{i_2}^4 x_{i_2+1}^2 \cdots x_{i_1-2i_2}^2 x_{i_1-2i_2+1} \cdots x_n + \text{smaller terms};$$
- In general, for admissible $\mathrm{Sq}^I := \mathrm{Sq}^{i_1}\cdots\mathrm{Sq}^{i_k}$,
$$\mathrm{Sq}^I(x_1 \cdots x_n) = x_1^{2^k} \cdots x_{e_k}^{2^k} x_{e_k+1}^{2^{k-1}} \cdots x_{e_k+e_{k-1}}^{2^{k-1}} x_{e_k+e_{k-1}+1}^{2^{k-2}} \cdots x_n$$
where $(e_1, \ldots, e_k)$ are defined by
$$e_s := \begin{cases} i_2 - 2i_{s+1}, & 0 < s < k \\ i_k, & s = k \end{cases}.$$

Noticing that we can recover the sequence $I = (i_1, \ldots, i_k)$ from $(e_1, \ldots, e_k)$ gives the linear independence statement.

The part of the theorem that says Sq^I spans $\mathcal{A}$ is harder. We only give a very brief sketch, and leave details to Adams's Berkeley notes, Chapter 2. The key geometric step is to notice that

Lemma 5.4. *There is a 1-1 correspondence between stable primary operations of degree n and sequences of elements $e^{q+n} \in H^{q+n}(K(\mathbb{F}_2, q); \mathbb{F}_2)$ satisfying $\Omega e^{q+n} = e^{q+n-1}$, where we use the desuspension map*

$$H^*(K(\mathbb{F}_2, q); \mathbb{F}_2) \cong [K(\mathbb{F}_2, q), K(\mathbb{F}_2, *)] \xrightarrow{\Omega} [K(\mathbb{F}_2, q-1), K(\mathbb{F}_2, *-1)] \cong H^{*-1}(K(\mathbb{F}_2, q-1); \mathbb{F}_2).$$

That is:

$$\mathcal{A}^n \cong \varprojlim_q H^{q+n}(K(\mathbb{F}_2, q); \mathbb{F}_2) \cong H^{q+n}(K(\mathbb{F}_2, q); \mathbb{F}_2)$$

for $q \geq n$.

Thus we need to compute the cohomologies of $K(\mathbb{F}_2, q)$. We of now do induction on q by studying the Serre spectral sequence associated to the path fibration of these spaces, which look like $K(\mathbb{F}_2, q-1) \hookrightarrow * \to K(\mathbb{F}_2, q)$ (also notice that $K(\mathbb{F}_2, 1)$ has a model $\mathbb{RP}^\infty$ whose cohomology we understand well). □

5.2. **The Steenrod algebra as a Hopf algebra.** This part will be very brief, and we refer to

- Milnor, *The Steenrod Algebra and Its Dual*;
- Milnor-Moore, *On the Structure of Hopf Algebras*.

The Steenrod algebra $\mathcal{A}$ has a coproduct $\psi\colon \mathcal{A} \to \mathcal{A}$ that is coassociative and (crucially) cocommutative, defined by (the Cartan formula)

$$\mathrm{Sq}^k \mapsto \sum_{i+j=k} \mathrm{Sq}^i \otimes \mathrm{Sq}^j.$$

In fact this makes $\mathcal{A}$ into a *Hopf algebra*. The fact that ψ is cocommutative means that the dual $\mathcal{A}^*$ is a Hopf algebra with a commutative product, which makes it much easier to study, and Milnor has worked out the structure of the Hopf algebra $\mathcal{A}^*$ which is surprisingly clear. However we will not use this strong result. What we do need is a general fact about Hopf algebra actions.

Let X be a space and we now consider the action of $\mathcal{A}$ on $H^*(X; \mathbb{F}_2)$. The observation is that the coproduct ψ on $\mathcal{A}$ is compatible with the cup product $\cup$ on $H^*(X; \mathbb{F}_2)$, in that the following two maps coincide:

(1) First take $\cup$ and then act by $\mathcal{A}$:

$$\mathcal{A} \otimes H^*(X; \mathbb{F}_2) \otimes H^*(X; \mathbb{F}_2) \xrightarrow{1 \otimes \cup} \mathcal{A} \otimes H^*(X; \mathbb{F}_2) \to H^*(X; \mathbb{F}_2);$$

(2) First take the ψ, then act by $\mathcal{A}$ component-wise, then take $\cup$:

$$\begin{tikzcd}
\mathcal{A} \otimes H^*(X;\mathbb{F}_2) \otimes H^*(X;\mathbb{F}_2) \arrow[rr] \arrow[d, "\psi\otimes 1\otimes 1"'] & & H^*(X;\mathbb{F}_2) \\
\mathcal{A} \otimes \mathcal{A} \otimes H^*(X;\mathbb{F}_2) \otimes H^*(X;\mathbb{F}_2) \arrow[r, "1\otimes T\otimes 1"'] & \mathcal{A} \otimes H^*(X;\mathbb{F}_2) \otimes \mathcal{A} \otimes H^*(X;\mathbb{F}_2) \arrow[r] & H^*(X;\mathbb{F}_2) \otimes H^*(X;\mathbb{F}_2) \arrow[u, "\cup"']
\end{tikzcd}$$

where the map T is simply switching factors.

This is simply a consequence of the Cartan formula and the very definition of ψ.

Let us abstractly formulate this situation. Let A be a Hopf algebra over $\mathbb{K}$, and let M, N be A-modules (using only the algebra structure of A). Then $M \otimes_{\mathbb{K}} N$ is an A-module using the diagonal map ψ:

$$\begin{tikzcd} A \otimes (M \otimes N) \arrow[rr] \arrow[d, "\psi\otimes 1"'] & & M \otimes N \\ (A \otimes A) \otimes (M \otimes N) \arrow[rr, "1\otimes T\otimes 1"'] & & (A \otimes M) \otimes (A \otimes N) \arrow[u] \end{tikzcd} .$$

We say an A-module M is an A-*module algebra* if M is an algebra such that the algebra structure map

$$M \otimes_{\mathbb{K}} M \to M$$

is an A-module map. For example, $H^*(X)$ is an $\mathcal{A}$-module algebra.

Similarly, we say an A-module M is an A-*module coalgebra* if M is a coalgebra such that the coalgebra structure map

$$M \to M \otimes_{\mathbb{K}} M$$

is an A-module map.

Recall that a coalgebra A is called *connected* if the counit $\epsilon\colon A \to \mathbb{K}$ is an isomorphism in degree ≤ 0.

Proposition 5.5 (Milnor-Moore)**.** *Let $\mathbb{K}$ be a field, let A be a connected Hopf algebra over $\mathbb{K}$, and let M be a connected A-module coalgebra with counit $1 \in M_0$. Assume that $i\colon A \to M;\ a \mapsto a \cdot 1$ is monic. Then M is a free A-module.*

5.3. **The Steenrod algebra action on $H^*(MO)$.** We will adopt the following abuse of notation in this section: we write $H^*(MO)$ as the $H\mathbb{F}_2$-cohomology of the Thom spectrum MO. More explicitly (using Proposition 2.17),

$$H^k(MO) := [MO, H\mathbb{F}_2]_{-k} \cong \varinjlim_{n\to\infty} [MO(n), K(\mathbb{F}_2, n+k)] \cong \varinjlim_{n\to\infty} H^{n+k}(MO(n); \mathbb{F}_2).$$

Thus the Thom class $U \in H^n(MO(n); \mathbb{F}_2)$ gives a stable cohomology class $U \in H^0 MO$.

Proposition 5.6. *The Steenrod algebra $\mathcal{A}$ acts freely through degree n on the Thom class U of $MO(n)$. Thus $\mathcal{A}$ acts freely on $U \in H^0 MO$.*

Proof sketch. Use our computation of the action of $\mathcal{A}$ on $x_1 \cdots x_n \in H^*\left(\prod_{i=1}^n \mathbb{RP}^\infty; \mathbb{F}_2\right)$, and the looking at

$$H^*(MO(n); \mathbb{F}_2) \xrightarrow{s^*} H^*(BO(n); \mathbb{F}_2) \xrightarrow{f^*} H^*\left(\prod_{i=1}^n \mathbb{RP}^\infty; \mathbb{F}_2\right)$$

where

1. s is the zero section inclusion map, and maps U to w_n (this is exactly the same as the proof that the restriction of the Thom class to the zero section is the Euler class, for oriented bundles – the point here is that w_n is the "Euler class" for $\mathbb{F}_2$-oriented vector bundles, i.e. all real vector bundles);
2. f is the classifying map for $\xi_1 \times \cdots \times \xi_1$, and maps w_n to
$$w_n(\xi_1 \times \cdots \times \xi_1) = x_1 \cdots x_n.$$

□

Theorem 5.7. *$H^* MO$ is free as an $\mathcal{A}$-module.*

Proof. We just need to demonstrate that H^*MO is an $\mathcal{A}$-module coalgebra, which is connected with counit $U \in H^0MO$. The theorem is then implied by Proposition 5.6 using Milnor-Moore's Proposition 5.5.

That H^*MO is a coalgebra is a result of MO being a ring spectrum:

$$H^*MO \to H^*(MO \wedge MO) \cong H^*MO \otimes_{\mathbb{F}_2} H^*MO.$$

One can show that all maps here respect Steenrod operations, so H^*MO is an $\mathcal{A}$-module coalgebra. Moreover, that H^*MO is connected with counit U simply follows from Thom isomorphism. $\square$

6. Concluding the proof

We put everything together and sketch the final few steps of the proof.

Recall that we have two spectra Ω^O and $H\mathbb{F}_2$ together with a map $\Omega^O \to H\mathbb{F}_2$. We would like to construct a splitting $H\mathbb{F}_2 \to \Omega^O$ so that the composition $H\mathbb{F}_2 \to \Omega^O \to H\mathbb{F}_2$ is the identity. By the Pontryagin-Thom construction, we have identified Ω^O with the Thom spectrum MO, and thus we can replace $\Omega^O \to H\mathbb{F}_2$ with $MO \to H\mathbb{F}_2$ given by the Thom class

$$U \in H^0MO \cong [MO, H\mathbb{F}_2]_0.$$

Exercise 6.1. Why?

To summarize what we have achieved in studying H^*MO, we have demonstrated that H^*MO is an $\mathcal{A}$-module coalgebra in which the Thom class $U \in H^0MO$ is the counit, and we have showed that H^*MO is free as an $\mathcal{A}$-module.

This means that we can choose a basis $\{v_\alpha\}$ for H^*MO as a free $\mathcal{A}$-module, and thus obtain maps $v_\alpha \colon MO \to \Sigma^{|\alpha|}H\mathbb{F}_2$ (where $\Sigma^{|\alpha|}$ means suspending by the degree $|\alpha|$ of v_α). Thus we obtain a map

$$v \colon MO \to \bigvee_\alpha \Sigma^{|\alpha|}H\mathbb{F}_2,$$

which is by construction an isomorphism on $H^*(-;\mathbb{F}_2)$ (both sides are free as $\mathcal{A}$-modules with generators labeled by $\{\alpha\}$).

The final insight required is that one could "break the homotopy type into rational and p-adic parts"[4]:

Theorem 6.2. *In fact v is a weak homotopy equivalence.*

Proof sketch. We indicate roughly how the proof works.

For each abelian group G, there exists a spectrum $\mathbb{S}G$, unique up to homotopy equivalence, called the *Moore spectrum* of G, such that

$$H_*(\mathbb{S}G;\mathbb{Z}) \cong \begin{cases} G, & * = 0 \\ 0, & * \neq 0 \end{cases}.$$

Define the *stable homotopy group of X with coefficient in G* to be

$$\pi_n(X;G) := [\mathbb{S}, \mathbb{S}G \wedge X]_n.$$

Given a prime p and a morphism $f \colon X \to Y$ of spectra

[4]See https://ncatlab.org/nlab/show/fracture+theorem.

(1) If f is an isomorphism in mod-p homotopy, then f is an isomorphism in mod p^k-homotopy for all k by the exact sequence
$$\cdots \to \pi_n(X; \mathbb{Z}/p^{k+1}\mathbb{Z}) \to \pi_n(X; \mathbb{Z}/p^k\mathbb{Z}) \to \pi_n(X; \mathbb{Z}/p\mathbb{Z}) \to \cdots$$
and is therefore an isomorphism in $\mathbb{Z}/p^\infty\mathbb{Z} := \lim \mathbb{Z}/p^k$ (p-complete) homotopy;

(2) We have the Whitehead theorem that if $f\colon X \to Y$ is an isomorphism in mod-p homology and X, Y are connected, then f is an isomorphism in mod-p homotopy;

Thus v is an isomorphism in mod-2 homotopy.

On the other hand, since every element in Ω_*^O has order 2, the rational and p-adic homotopy type for any odd prime p vanishes. Therefore v is an isomorphism in integral homotopy. $\square$

Thus we can conclude the proof: from the weak equivalence we can construct a splitting $H\mathbb{F}_2 \to \bigvee_\alpha \Sigma^{|\alpha|} H\mathbb{F}_2 \cong MO$, so that its composition with $MO \xrightarrow{U} H\mathbb{F}_2$ is the identity. Combining with the fact that $MO \xrightarrow{U} H\mathbb{F}_2$ coincides with the fundamental class map $\Omega^O \to H\mathbb{F}_2$ shows that every mod-2 cycle is represented by a cycle in Ω^O.