Orbifolds as Stacks

Ex (Functor of points) Let $\mathcal{C}$ be a category of geom. obj's. By the Yoneda lemma, $\exists$ a fully faithful embedding

$$\mathcal{C} \hookrightarrow \underbrace{Set^{\mathcal{C}^{op}}}_{\text{"generalized objects"}}, \quad X \longmapsto Hom(-, X).$$

Ex ① $Mfld \to Set$, $U \mapsto \{$families of pseudo-hol curves in (X, J) over $U\}$ should be repr. by $\mathcal{M}(X, J)$

② $Mfld \to Set$, $U \mapsto \{$families of lines thru origin in $\mathbb{R}^n$ over $U\}$ is represented by $\mathbb{RP}^{n-1}$

③ $Top \to Set$, $U \longmapsto Vect_k(U)$.

Prob Isom. classes of vector bundles don't glue: "$Vect_k(-)$ is not a sheaf"

$$E_\alpha \in Vect_k(U_\alpha) \;+\; E_{\alpha\beta} \in Vect_k(U_{\alpha\beta}) \;\not\rightsquigarrow\; E \in Vect_k(X)$$

Idea Consider functors $\mathcal{C}^{op} \to Grpd$. Now vector bundles glue:

$$\begin{tikzcd}
E_\alpha \arrow[d] & & E_\beta \arrow[d] \\
U_\alpha & E_{\alpha\beta} \arrow[ul] \arrow[ur] \arrow[d] & U_\beta \\
& U_{\alpha\beta} \arrow[ul] \arrow[ur] &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
E \arrow[d] \\
U_\alpha \cap U_\beta
\end{tikzcd}$$

Formally, $F : \mathcal{C}^{op} \to Grpd$ should satisfy descent:

$$F(X) \xrightarrow{\simeq} \underbrace{\lim}_{\text{limit in 2-cat sense}} \left(\prod_\alpha F(U_\alpha) \rightrightarrows \prod_{\alpha\beta} F(U_\alpha \cap U_\beta) \right)$$

Descent Let $F : Top^{op} \to Grpd$ be a functor. Then a descent datum consists of:

① $P_\alpha \in F(U_\alpha)$ $\forall \alpha$

$\quad \downarrow$

$\quad P_\alpha|_{U_{\alpha\beta}} \in F(U_{\alpha\beta})$

② $P_\alpha|_{U_{\alpha\beta}} \xrightarrow{\phi_{\alpha\beta}} P_\beta|_{U_{\alpha\beta}}$ s.t. $\phi_{\alpha\gamma} = \phi_{\beta\gamma} \circ \phi_{\alpha\beta}$

"cocycle condition"

Let $\mathcal{D}(\{U_\alpha\})$ be the category of descent data. Then F satisfies descent if

$$F(X) \xrightarrow{\simeq} \mathcal{D}(\{U_\alpha\}).$$

In other words,

① Ess. surj. $\Rightarrow$ any descent datum glues,

② Faithful $\Rightarrow$ morphisms are uniquely determined by a cover,

③ Full $\Rightarrow$ morphisms glue

Def F is a stack if it satisfies descent.

A more convenient language for stacks:

Def $\mathcal{D} \to \mathcal{C}$ is a category fibered in groupoids if

①
$$\begin{tikzcd} \exists d' \arrow[r] \arrow[d] & d \arrow[d] \\ c' \arrow[r, "f"] & c \end{tikzcd} \qquad f^*d \cong d'$$

②
$$\begin{tikzcd} d'' \arrow[rr, dashed, "\exists!"] \arrow[dr] \arrow[dd] & & d' \arrow[dl] \arrow[dd] \\ & d \arrow[dd] & \\ c'' \arrow[rr] \arrow[dr] & & c' \arrow[dl] \\ & c & \end{tikzcd}$$

Rmk Given a CFG $\mathcal{D} \to \mathcal{C}$, we get a functor $\mathcal{C} \to \mathrm{Grpd}$.

Def Descent data is defined similarly: (when $\mathcal{C} = \mathrm{Top}$)

$$\begin{tikzcd} P_\alpha \arrow[d] & P_{\alpha\beta} \arrow[l] \arrow[r, leftrightarrow, "\sim"] \arrow[d] & P_{\beta\alpha} \arrow[r] \arrow[d] & P_\beta \arrow[d] \\ U_\alpha & U_{\alpha\beta} \arrow[l] \arrow[r, leftrightarrow, "\sim"] & U_{\beta\alpha} \arrow[r] & U_\beta \end{tikzcd} \quad + \text{ cocycle condition}$$

$\mathcal{D} \xrightarrow{\pi} \mathrm{Top}$ is a stack if

$$\pi^{-1}(X) \cong \text{descent cat for } \{U_\alpha\}$$

for any X and $\{U_\alpha\}$.

Ex Any $X \in \mathrm{Top}$ determines a stack $\underline{X}$ given by $Z \mapsto \mathrm{Top}(Z, X) \in \mathrm{Set} \subseteq \mathrm{Grpd}$.

Ex Given a Lie grpd G, the category of principal G-bundles is a stack/Mfld.
This stack is called BG.

Ex For $G = \{\Gamma \times M \rightrightarrows M\}$, this is the category where the objects are maps $N \to M/\Gamma$ given by gluing maps $U_\alpha \to M \twoheadrightarrow M/\Gamma$.

Ex By the Yoneda lemma, a map $\underline{X} \to \mathcal{X}$ is an obj. of $\mathcal{X}$ lying over X. This map sends a map $Z \to X$ to the pullback of the fixed object.

Ex Given a bibundle $P : G \to H$, define $BP : BG \to BH$ by composition of bibundles. If $F : BG \to BH$ is a morphism of stacks, take $P = F(G \to G_0)$.

Def (Fiber products) Given $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \rightrightarrows \mathcal{C}$ CFG's, and maps $\mathcal{X}, \mathcal{Y} \rightrightarrows \mathcal{Z}$, let $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ be the CFG

$$\mathrm{Ob} = \left\{ \begin{tikzcd} x \arrow[r, mapsto] & z \arrow[d] & \\ & w \arrow[d] & y \arrow[l, mapsto] \\ & I & \end{tikzcd} \right\}, \quad \mathrm{Mor} = \left\{ \begin{tikzcd} x \arrow[r, mapsto] \arrow[rrr] & z \arrow[d] \arrow[rrr] & & x' \arrow[r, mapsto] & z' \arrow[d] & \\ & w \arrow[rrr] & y \arrow[l, mapsto] \arrow[rrr] & & w' & y' \arrow[l, mapsto] \end{tikzcd} \right\}$$

Def (Representable maps) A morphism $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ of stacks is **representable** if $\forall Z \in \mathrm{Top}$, $\mathcal{X} \times_{\mathcal{Y}} Z \in \mathrm{Top}$.

For a property P satisfied by top'l spaces, f **satisfies P** if its pullback by every $Z \in \mathrm{Top}$ satisfies P.

Ex $\mathcal{U} \hookrightarrow \mathcal{X}$ is an open embedding if $\forall Z$, $Z \times_{\mathcal{X}} \mathcal{U} \to Z$ is an open embedding.

Ex (Surprising fact) The map $\underline{G_0} \to BG$ is a surjective submersion. In fact, if $\underline{M} \to BG$ is given by the principal G-bundle P,

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{G_0} \arrow[d] \\ \underline{M} \arrow[r] & BG \end{tikzcd}$$

Def An **atlas** for a stack $\mathcal{X}$ over Mfld is a surjective submersion

$$\underline{M} \longrightarrow \mathcal{X}.$$

A stack admitting an atlas is called a **geometric/differentiable stack**. **Topological stacks** are defined similarly.

Thm If $M \to \mathcal{X}$ is an atlas, then $\mathcal{X} \cong BG$ for some $G = \{ M \times_{\mathcal{X}} M \rightrightarrows M \}$.