ORBIFOLDS & LIE GROUPOIDS

§1. The classical definition. (Satake, Thurston)

Def An orbifold chart is $(\tilde{U}, \Gamma, \phi)$ s.t.

① $\tilde{U} \subseteq \mathbb{R}^n$ open

② Γ finite group $\curvearrowright \tilde{U}$

③ $\phi : \tilde{U}/\Gamma \xrightarrow{\cong} U \subseteq X$

Def An embedding of orbifold charts $(\tilde{U}_\alpha, \Gamma_\alpha, \phi_\alpha) \hookrightarrow (\tilde{U}_\beta, \Gamma_\beta, \phi_\beta)$ consists of

① $\rho : \Gamma_\alpha \hookrightarrow \Gamma_\beta$

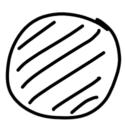
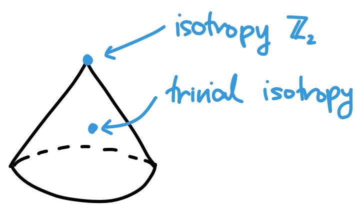
② $\begin{tikzcd} \tilde{U}_\alpha \arrow[r, hook] \arrow[d] & \tilde{U}_\beta \arrow[d] \\ U_\alpha \arrow[r, hook] & U_\beta \end{tikzcd}$ ρ-equivariant

Def An orbifold is a space X w/ an atlas $\{(U_\alpha, \Gamma_\alpha, \phi_\alpha)\}$ s.t.

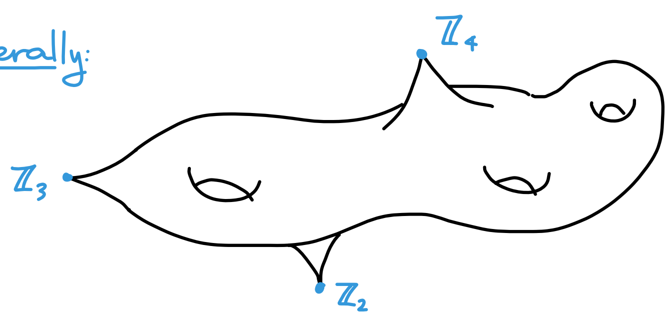
$x \in U_\alpha \cap U_\beta \Rightarrow \exists\, \gamma$ and $(\tilde{U}_\gamma, \Gamma_\gamma, \phi_\gamma) \begin{array}{l} \hookrightarrow (\tilde{U}_\alpha, \Gamma_\alpha, \phi_\alpha) \\ \hookrightarrow (\tilde{U}_\beta, \Gamma_\beta, \phi_\beta) \end{array}$ s.t. $x \in U_\gamma$

Def For $x \in X$ in a chart $(\tilde{U}_\alpha, \Gamma_\alpha, \phi_\alpha)$, the isotropy group is $\mathrm{Stab}_{\Gamma_\alpha}(\tilde{x})$

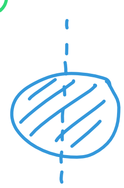
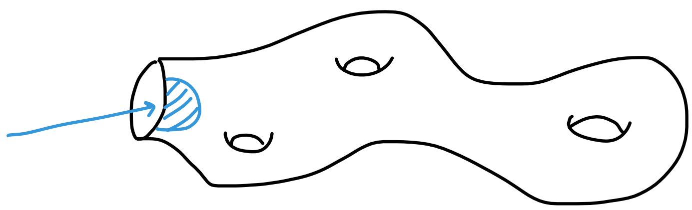
Ex (Cones)

$\mathbb{Z}_2 \curvearrowright$  by rotation $\rightsquigarrow$ 

More generally:

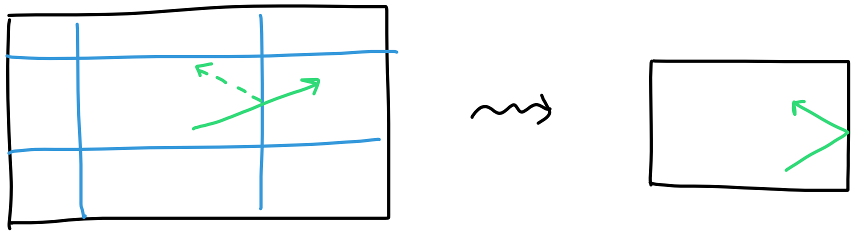


Ex (Manifolds w/ ∂)

$\mathbb{Z}_2 \curvearrowright$  

Ex (Billiards table)

$D_\infty \times D_\infty \circlearrowright$



Ex (Quotients) If $G \circlearrowright M$ is effective, proper, almost free $\rightsquigarrow$ orbifold $[X/G]$

Ex $[\mathbb{R}^3/\mathbb{Z}_2]$ (by reflection) is not a topological mfld

Ex (Moduli spaces of J-hol curves) $\mathcal{M}^{reg}_{g,m}(M,J)$ is an orbifold

Problem How do we define morphisms, vector bundles, etc.?

Ex Let V be a Γ-rep, $W = V^\gamma$ for some $\gamma \in \Gamma$. Then $C_G(g) \circlearrowright W$. If $K = \mathrm{Stab}_G(W)$, $C_G(g)/K \circlearrowright W$. How do we define a morphism $[W/(C_G(g)/K)] \hookrightarrow [V/G]$?

§2. Lie groupoids

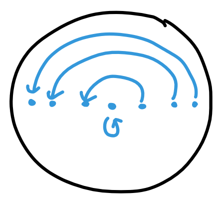
Def A Lie groupoid is a groupoid $G_1 \rightrightarrows G_0$ where

morphisms: G_1, objects: G_0

① The structure maps are smooth

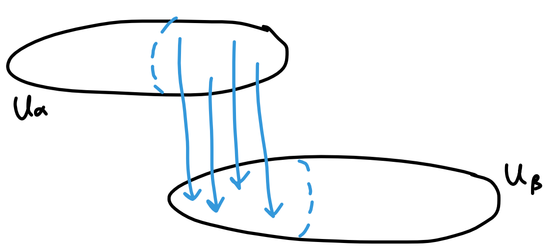
② The source & target maps $s, t : G_1 \rightrightarrows G_0$ are submersions

Ex (Manifolds) $M \rightrightarrows M$ for any mfld M

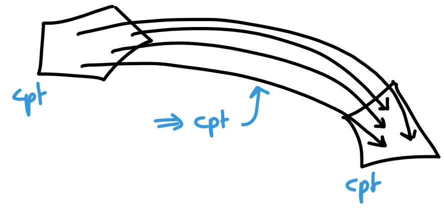


Ex (Action groupoids) $\Gamma \circlearrowright M \rightsquigarrow \Gamma \times M \rightrightarrows M$

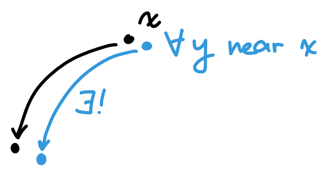
Ex (Cover groupoids) $\{U_\alpha\}$ open cover of M $\rightsquigarrow$ $\coprod U_{\alpha\beta} \rightrightarrows \coprod U_\alpha$



Def G is *proper* if $(s,t): G_1 \to G_0 \times G_0$ is proper



Def G is *étale* if s,t are local diffeos



Ex $\Gamma \times M \rightrightarrows M$ proper & étale $\iff$ Γ is finite

Ex $\mathbb{R} \curvearrowright \mathbb{R}^2$ by translation is not étale:

Prop Γ proper, étale $\Rightarrow$ $\forall x \in G_0$, $\exists$ nbhd U st. $G|_U \cong [U/\Gamma]$

Def $G \xrightarrow{f} H$ is a *weak equivalence* if

① (Fully faithful)

$$\begin{tikzcd} G_1 \arrow[r, "f"] \arrow[d, "(s{,}t)"'] & H_1 \arrow[d, "(s{,}t)"] \\ G_0 \times G_0 \arrow[r] & H_0 \times H_0 \end{tikzcd}$$

② (Essentially surj) $G_0 \times_{H_0} H_1 \longrightarrow H_0$ is a surj. submersion
$(x, \gamma) \longmapsto t(f)$

Ex $(\coprod U_{\alpha\beta} \rightrightarrows \coprod U_\alpha) \longrightarrow (M \rightrightarrows M)$ is a weak equiv.

Def G, H are *Morita equiv.* if $\exists\, G \xleftarrow[\text{w.e.}]{} K \xrightarrow[\text{w.e.}]{} H$

Def An *orbifold* is a groupoid Morita equiv. to a proper étale grpd

§3 Bibundles

Idea Category of orbifolds = invert weak equivalences in cat. of Lie grpds

Def A left action $G \curvearrowright M$ consists of

① An anchor map $a : M \longrightarrow G_0$

② An action $G_1 \times_{s,G_0,a} M \longrightarrow M,\quad (\gamma, m) \longmapsto \gamma \cdot m$

$$\begin{tikzcd} m \arrow[r] \arrow[d] & \gamma\cdot m \arrow[d] \\ a(m) \arrow[r, "\gamma"] & \bullet \end{tikzcd}$$

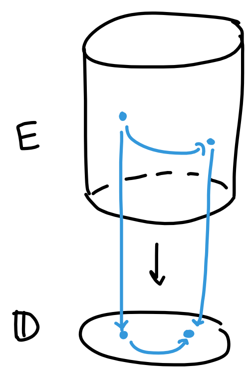
s.t.

① $\mathbb{1}_{a(x)} \cdot x = x$

② $a(\gamma \cdot x) = t(\gamma)$

③ $\gamma_2 \cdot (\gamma_1 \cdot x) = (\gamma_2 \gamma_1) \cdot x$

Ex An action of $(\Gamma \rightrightarrows *)$ is a Γ-action



Ex $\Pi_1(X) \curvearrowright$ universal cover $\tilde{X}$

Ex If $E \to \mathbb{D}$ is a $\mathbb{Z}_2$-equiv. vector bundle $\rightsquigarrow$ $(\mathbb{Z}_2 \times \mathbb{D} \rightrightarrows \mathbb{D}) \curvearrowright E$

Ex An action $(\Gamma \times N \rightrightarrows N) \curvearrowright M$ is a Γ-equiv. map $M \longrightarrow N$

Def A right action is defined similarly:

$$\begin{tikzcd} p\cdot\gamma \arrow[r] \arrow[d] & p \arrow[d] \\ \bullet \arrow[r, "\gamma"] & a(p) \end{tikzcd}$$

Ex $H_1 \circlearrowleft H$

$$\begin{tikzcd} z \arrow[r] \arrow[d] & z \arrow[d] \\ x \arrow[r] \arrow[d, "\gamma"'] & a(\gamma) = x \\ y & \end{tikzcd}$$

Def A principal H-bundle is a surj. subm. $\pi : P \longrightarrow B$ s.t. $P \circlearrowleft H$ and

① π is H-invt

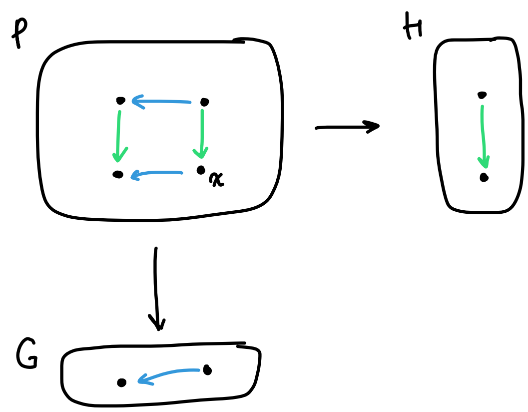
② H acts freely & transitively on the fibers of M.

$\Big\} \Longleftrightarrow$ $P \circlearrowleft H$ is free & orbits = fibers

Ex A principal $(\Gamma \rightrightarrows *)$ is a principal Γ-bundle

Def A (G,H)-bibundle is a mfld P w/

$$\begin{cases} G \curvearrowright P \text{ given by } a_L, \\ P \curvearrowleft H \text{ given by } a_R, \text{ s.t.} \end{cases}$$



① $a_L : P \to G_0$ is a principal H-bundle

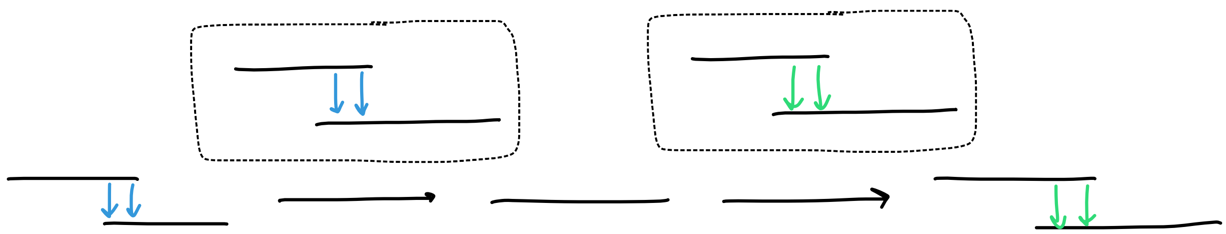
② a_R is G-invt

③ the actions commute

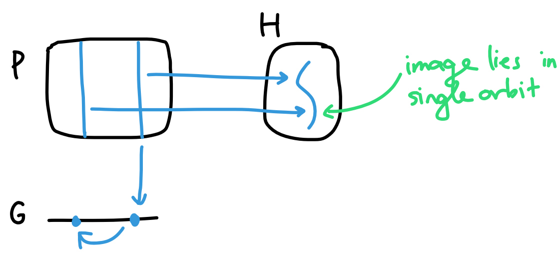
Ex H_1 is a (H,H)-bibundle. If $f : G \to H$,

$$f^*H_1 = \left\{ (x,\gamma) \;\middle|\; x \longrightarrow f(x),\ \downarrow\gamma \right\} \text{ is a } (G,H)\text{-bibundle.}$$

Ex



Fact Bibundles induce maps on underlying spaces:



Def Given $G \xrightarrow{P} H \xrightarrow{Q} K$,

$$Q \circ P = P \times_{H_0} Q / H = \left\{ \begin{array}{c} p \searrow \bullet \swarrow q \\ a_R^P(p) = a_L^Q(q) \end{array} \right\} / H$$

Def (Hilsum-Skandalis)

① Bi = weak 2-category of Lie grpds & bibundles

② $HL = \mathrm{Ho}(Bi$