1 A_∞ Algebra and Noncommutative Geometry

The notion of A_∞ algebras and A_∞ categories are the main tool used to encode the counting data from Lagrangian intersection. The A_∞ relation, in the simplest way, is usually written down in a rather combinatorial way. One way to understand this machinery, proposed by Kontsevich [KoSo1,2], is (the hope of) a translation of the theory of A_∞ categories into a geometric language of non-commutative geometry, for which a dictionary between algebra and geometry can be established like in algebraic geometry (in the "affine case").

maps $f : X \to Y$	morphisms of algebras $f^* : \mathscr{O}(Y) \to \mathscr{O}(X)$
points of X	homomorphisms $\mathrm{Hom}_{Alg_\Bbbk}(\mathscr{O}(X), \Bbbk)$
closed embedding $i : X \hookrightarrow Y$	epimorphism of algebras $i^* : \mathscr{O}(Y) \twoheadrightarrow \mathscr{O}(X)$
finite product $\prod_{i\in I} X_i$	tensor product $\bigotimes_{i\in I} \mathscr{O}(X_i)$
finite disjoint union $\bigsqcup_{i\in I} X_i$	direct sum $\bigoplus_{i\in I} \mathscr{O}(X_i)$
vector bundle E on X	finitely generated projective $\mathscr{O}(X)$-module $\Gamma(E)$

And we would also like some translation of notions in differential geometry, as we have done in algebraic geometry.

X is smooth	$\mathscr{O}(X)$ is formally smooth, i.e. for any $f : B \twoheadrightarrow B/I$ with $I^n = 0$ for some $n \geq 0$, the induced map $f^* : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B/I)$ is surjective
total space $totE$	symmetric algebra generated by $\Gamma(E^*)$, i.e. $Sym(\Gamma(E^*)) = \bigoplus_{n\geq 0} Sym^n(\Gamma(E^*))$
a vector field on X	a derivation of $\mathscr{O}(X)$
tangent bundle TX	$\mathscr{O}(X)$-module of all derivations $\mathrm{Der}(\mathscr{O}(X))$
tangent space T_pX	vector space $(m_{x_0}/m_{x_0}^2)^*$, where m_{x_0} is the kernel of $x_0 \in \mathrm{Hom}(\mathscr{O}(X), \Bbbk)$
cotangent bundle T^*X	$\mathscr{O}(X)$-module $\Omega^1(\mathscr{O}(X)) :=\mathrm{Hom}(\mathrm{Der}(\mathscr{O}(X)), \mathscr{O}(X))$
differential forms $\Omega_X^\bullet$	$\wedge^\bullet(\Omega^1(\mathscr{O}(X)))$

We start with the tensor category $\mathscr{C} = \mathrm{Vect}_\Bbbk^\mathbb{Z}$ or $\mathrm{Vect}_\Bbbk^{\mathbb{Z}_2}$ consists of $\mathbb{Z}$ or $\mathbb{Z}_2$-graded vector space, in which the unit object is given by the 1-dim vector space $\Bbbk$ with grading 0, and the tensor is the normal tensor product of vector spaces with added grading. An associative algebra over $\mathscr{C}$ is given by a vector space $A \in \mathrm{Ob}(\mathscr{C})$ and an associative product $A \otimes A \to A$ preserving the grading. And the coassociative coalgebras can be seen as the dual of associative algebras, with induced structure called the coproduct $\Delta : A^* \to A^* \otimes A^*$ (such that $\Delta(f)(a, b) = f(a)f(b)$ for $f \in A^*, a, b \in A$). Here we also assume our (co)algebra is (co)unital, and the counit is simply a linear map $e : A^* \to \Bbbk$. Notice under this duality, the category of (finite generated) coassociative coalgebras can be seen as the opposite of the category of (finite generated) associative algebras.

In algebraic geometry, the affine scheme associated to a commutative algebra R is given by a topological space $Spec(R)$ (the points are prime ideas), together with a structure sheaf induced by R. Due to the contrafunctoriality between the category of algebras and the category of affine schemes, it is reasonable to think a coalgebra B as our "scheme", and the dual algebra B^* as its "structure sheaf", which we call a formal graded manifold X_B and its algebra of functions $\mathscr{O}(X_B)$ (the non-commutative nature makes it hard to identify our formal manifold as a topological space). A point of X, following our dictionary above, can given by a morphsim of algebra $pt : \Bbbk \to B$, here k is the coalgebra by dualizing the algebra structure on k (with product $\Bbbk \otimes \Bbbk \to \Bbbk$ given by multiplication and grading 0), thus equipped with coproduct $\Delta(k) = 1 \otimes k + k \otimes 1, k \in \Bbbk$.

The simple objects in our category would be those cofree coalgebras (like $\mathbb{A}^n$ in algebraic geometry), which can be seen as tensor coalgebras over vector spaces.

Definition 1.1. A tensor coalgebra over vector space V is given by $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$, together with a coproduct

$$\Delta : T(V) \to T(V) \boxtimes T(V)$$

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \mapsto \sum_{0\leq i\leq n} (v_1 \otimes \cdots \otimes v_i) \boxtimes (v_{i+1} \otimes \cdots \otimes v_n).$$

Notice here we use $\boxtimes$ to distinguish the tensor product in $\mathscr{C}$ from the one in $T(V)$, and when $i = 0, n$ the corresponding term collapse to the identity $1 \in \Bbbk = V^{\otimes 0}$.

For the formal graded manifold $X_{T(V)}$ associated to $T(V)$, the structure algebra $\mathscr{O}(X) = \Bbbk\langle\langle V^* \rangle\rangle$ the non-commutative formal power series generated by V^*. There is a distinguished point on X given by the canonical isomorphism $id_{\Bbbk} : \Bbbk \to \Bbbk = V^{\otimes 0} \hookrightarrow T(V)$.

Now we want to formulate some differential geometry structures on X_B. The tangent space at a given point pt is defined to be the vector space

$$(m_{pt}/m_{pt}^2)^*, m_{pt} = \{a \in B^* | a \circ pt = 0\},$$

here m_{pt} is a well defined two-sided ideal of B^*. And a vector field on X is given by a coderivation d on the associated coalgebra B. Roughly speaking, it is the dual of a derivation d^* on the structure algebra $\mathscr{O}(X) = B^*$, and its evaluation at a given point $pt \circ d|_{m_{pt}}$ vanish on m_{pt}^2, thus gives a tangent vector in $(m_{pt}/m_{pt}^2)^*$. To be compatible with the graded structure on the algebra, we need some Koszul sign rule introduced below.

Definition 1.2. For any permutation σ and graded vector space V, we have a commutation morphism $\tau_\sigma : V^{\otimes n} \to V^{\otimes n}$ induced by the commutation morphism of swap $\tau(v_1 \otimes v_2) = (-1)^{\deg v_1 \deg v_2} v_2 \otimes v_1$. The Koszul sign of a permutation is the sign of τ_σ.

A coderivation of graded coalgebra B is a (not necessarily grading preserving) homomorphism $d : B \to B$ such that

$$\Delta \circ d = (d \boxtimes id_B + \tau \circ (d \boxtimes id_B) \circ \tau) \circ \Delta, e \circ d = 0.$$

For a formal graded manifold with point (X_B, pt), a differential is a coderivation d of degree 1, meaning it is a grading preserving map $B \to B[1]$, with zero square $d \circ d = 0$, and $d|_{pt} = d \circ pt = 0$.

The description of tangent space is quite simple when we consider the formal graded manifold $(X_{T(V)}, id_{\Bbbk})$, which is exactly the graded vector space V. In fact, the differential on X gives us an A_∞-structure on V, by following correspondence.

Proposition 1.3. The space of differential d on $T(V)$ is one-to-one corresponding to the space of the series of degree 1 maps $m_n : V^{\otimes n} \to V[1], n \geq 0$ that satisfy the shifted A_∞ relations, given by

$$\sum_{0 \leq k \leq k+l \leq n} (-1)^* m_{n-l+1}(v_1 \otimes \cdots \otimes v_k \otimes m_l(v_{k+1} \otimes \cdots \otimes v_{k+l}) \otimes v_{k+l+1} \otimes \cdots \otimes v_n) = 0$$

for all $v_i \in V$ and notice here m_0 naturally becomes a zero map.

The correspondence from left to right is given by $\mathrm{proj}_V \circ d$, for $\mathrm{proj}_V : T(V) \to V$ the normal projection, called the Taylor coefficients of d. And the other side is given by the augmentation $\{\hat{m_n}\}$ of $\{m_n\}$, defined by

$$\hat{m_n}(v_1 \otimes \cdots \otimes v_n) = \sum_{0 \leq k \leq k+l \leq n} (-1)^* v_1 \otimes \cdots \otimes v_k \otimes m_l(v_{k+1} \otimes \cdots \otimes v_{k+l}) \otimes v_{k+l+1} \otimes \cdots \otimes v_n.$$

By this construction, we can recover our original definition of A_∞-algebra A, i.e. a graded algebra equipped with differentials m_n satisfying the A_∞ relations above. The only difference is that our vector space V should be seen as $A[1]$, the A_∞-algebra shifted down by 1, which means that $m_n : A^{\otimes n} \to A[2-n]$ should be of degree $2-n$. In the following construction, we identify A_∞-algebra A and the corresponding formal pointed differential graded manifold $(X_{T(A[1])}, id_{\Bbbk})$ (X_A in short). And we restrict to this special case, although we can deal with more general coalgebras.

Definition 1.4. An A_∞-morphism between A_∞-algebra A_1 and A_2 is a grading preserving homomorphism $f : T(A_1[1]) \to T(A_2[1])$, commuting with differentials, i.e.

$$(f \boxtimes f) \circ \Delta_{A_1} = \Delta_{A_2} \circ f, f \circ d_{A_1} = d_{A_2} \circ f.$$

Again it is equivalent to consider its Taylor coefficients $f_n : A_1^{\otimes n} \to A_2$, which satisfy

$$\sum_{0 \leq k \leq k+l \leq n} (-1)^* f_{n-l+1}(v_1 \otimes \cdots \otimes v_k \otimes m_l(v_{k+1} \otimes \cdots \otimes v_{k+l}) \otimes v_{k+l+1} \otimes \cdots \otimes v_n)$$
$$= \sum_{\substack{1 \leq k \leq n \\ 1 = i_1 < \cdots < i_k \leq n}} (-1)^* m_k(f_{i_2 - i_1}(v_{i_1} \otimes \cdots) \otimes f_{i_3 - i_2}(v_{i_2} \otimes \cdots) \otimes \cdots \otimes f_{n - i_k + 1}(v_{i_k} \otimes \cdots \otimes v_n)).$$

For A_∞-algebra A, the A_∞-relation would imply $m_1 \circ m_1 = 0$, thus can be seen as a chain complex. Any A_∞-morphism $f : A_1 \to A_2$ is quasi-isomorphism if it induces the quasi-isomorphism between chain complexes $(A_1, m_{A_1,1})$ and $(A_2, m_{A_2,1})$.

A dg-vector bundle $\mathscr{E}$ over X_A is an free $\mathscr{O}(X_A)$-module $\Gamma(\mathscr{E})$ called the module of sections on $\mathscr{E}$, together with a differential $d_{\mathscr{E}} : \Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})[1]$ such that $d_{\mathscr{E}}^2 = 0, d_{\mathscr{E}}|_{\mathscr{E}_{x_0}} = 0$, which is compatible with the differential on $\mathscr{O}(X_A)$. Turning into the coalgebra side, we define an left A-module M as a graded vector space, with identification $\Gamma(\mathscr{E}) \xrightarrow{\sim} \mathscr{O}(X_A) \hat{\otimes} M^*$ for some dg-vector bundle $\mathscr{E}$, called a trivialization of $\mathscr{E}$ (here $\hat{\otimes}$ denotes the topologically completed tensor product, which can be thought as adding infinite sums). It is the same to say our left A-module M carries a differential $d_M : T(A[1]) \otimes M \to (T(A[1]) \otimes M)[1]$, making it a dg-comodule over the dg-coalgebra $T(A[1])$.

Again using $d_M^2 = 0$, the Taylor coefficients of d_M is a series of map $m_{M,n} : A[1]^{\otimes n} \otimes M \to M[1]$ satisfying

$$\sum_{0 \le k \le k+l \le n} (-1)^* m_{M,n-l+1}(v_1 \otimes \cdots \otimes v_k \otimes m_l(v_{k+1} \otimes \cdots \otimes v_{k+l}) \otimes v_{k+l+1} \otimes \cdots \otimes v_n \otimes y)$$
$$+ \sum_{0 \le k \le n} (-1)^* m_{M,k}(v_1 \otimes \cdots \otimes v_k \otimes m_{M,n-k}(v_{k+1} \otimes \cdots \otimes v_n \otimes y)) = 0.$$

The homomorphism space of the comodules M, N over $T(A[1])$, is a chain complex

$$\mathrm{Hom}(T(A[1]) \otimes M, T(A[1]) \otimes N), \delta(f) = d_N \circ f - (-1)^{\deg f + 1} f \circ d_M$$

where the grading given by the degree of the homomorphism. And we call f quasi-isomorphism if $\delta(f) = 0$. This gives a dg-category structure on the category of A-module. Similarly we can define the category of right A-module and the $A - B$ bimodule.

The tensor product of right and left A-modules M, N, is given by a dg $\Bbbk$-module

$$M \otimes T(A[1]) \otimes N, d \in \mathrm{Hom}(M \otimes T(A[1]) \otimes N, M \otimes T(A[1]) \otimes N)$$

where the differential d is induced by the ones on $M \otimes T(A[1])$, $T(A[1])$ and $T(A[1]) \otimes N$, and we can expect that a similar formula for such a coderivation to have a zero square can be deduced thereafter. Furthermore one can follow our construction to define the simultaneous tensor product between A-B bimodules and B-A bimodules.

2 Yoneda Homomorphism and Yoneda Embedding

Now we have developed the category of left (right) A_∞-module, so we expect to have an A_∞ version of Yoneda lemma. An analogous of commutative algebra indicates an important role of unital condition to get such a lemma.

Definition 2.1. An A_∞-algebra A is called strict unital if there exist an element $e \in A$ of degree zero (degree -1 in $A[1]$), such that

$$m_2(e, v) = m_2(v, e) = v, m_n(\cdots, e, \cdots) = 0, \forall n \neq 2.$$

It is called weakly (or homological) unital if the graded associative algebra $H^\bullet(A)$ has an unit.

Most of time, we can only get weakly unit when construct our A_∞-algebra. However the following proposition shows they are actually equivalent, which can be constructed geometrically.

Proposition 2.2. For any A_∞-algebra A with weakly unit e, there exist a quasi-isomorphism $A \hookrightarrow A'$, where $A' = A \oplus \Bbbk b[1] \oplus \Bbbk e^+$ is an A_∞-algebra satisfying $m_{A',1}(b) = e^+ - e$, so that e^+ is a strict unit of A'.

Also, given a non-unital A_∞-algebra A, we can forcely add strict unit by adding a formal dg-line, i.e. $X' = \mathbb{L} \times X_A$, where $\mathbb{L}$ is the formal dg-line corresponding to a 1-dim A_∞-algebra with $m_2 = id, m_n = 0, n \neq 2$.

Now we define the target of our Yoneda homomorphism, in terms of diagonal bimodule.

Definition 2.3. The dg-algebra A have a natural A-A bimodule structure, with differential given by d_A, called the diagonal bimodule of A. Notice its endomorphism algebra $End_{A\text{-}mod}(A)$ carries a dg-algebra structure.

Lemma 2.4 (Yoneda homomorphism)**.** *There exist a natural A_∞-morphism called left Yoneda homomorphism*

$$Y^l : A \to End_{A\text{-}mod}(A),$$

which has Taylor coefficient $Y_n^l : A^{\otimes n} \to End_{A\text{-}mod}(A)$ induced by the A_∞-relation

$$m_{A,n+m} \in \mathrm{Hom}(A^{\otimes n}, \mathrm{Hom}(A^{\otimes k} \otimes A, A)) = \mathrm{Hom}(A^{\otimes(n+k)} \otimes A, A),$$

and Y_l is a quasi-isomorphism if and only if A is weakly unital.

Now we can try to consider a categorical version of A_∞-algebra, like the generalize the group to the groupoid.

Definition 2.5. An A_∞ category $\mathscr{A}$ is a category, in which $\mathrm{Hom}(X, X'), X, X' \in Ob(\mathscr{A})$ are a graded vector spaces, equipped with products m_n, satisfying A_∞-relations

$$\sum_{0 \le k \le k+l \le n} (-1)^* m_{n-l+1}(v_1 \otimes \cdots \otimes v_k \otimes m_l(v_{k+1} \otimes \cdots \otimes v_{k+l}) \otimes v_{k+l+1} \otimes \cdots \otimes v_n) = 0$$

for any $v_i \in \mathrm{Hom}(X_i, X_{i-1})$.

It is helpful to connect it with an A_∞-algebra $A = \bigoplus_{X,X'} \mathrm{Hom}(X, X')$, for which the mismatched products are defined to be zero. So roughly speaking, A_∞ category is the A_∞-algebra equipped with labels. We can follow this idea to define the category of left (right, bi) $\mathscr{A}$-modules, and notice we say our A_∞ category $\mathscr{A}$ is weakly (strict) unital if every $\mathrm{Hom}(X, X)$ is weakly (strict) unital and they sum up to be a weak unit in the associated A_∞-algebra A. Following is a stronger version of Yoneda lemma.

Lemma 2.6 (Yoneda functor)**.** *There exist a natural A_∞-functor called left Yoneda functor*

$$\mathscr{Y}^l : \mathscr{A}^{op} \to \mathscr{A}\text{-}mod,$$

which maps each object X to its left Yoneda module $\mathscr{Y}^l(X) = \bigoplus_Y \mathrm{Hom}(X, Y)$. If $\mathscr{A}$ is weakly unital, then $\mathscr{Y}^l$ is a full functor and gives quasi-isomorphisms on the hom spaces.

3 Hochschild Cohomology and Homology

One interpretation of the Hochschild cochain is given by a shift of the DGLA (differential graded Lie algebra) of vector fields on X_A, i.e. the space of derivations. By the augmentation arguement we introduced before, it can be written as

$$CC^\bullet(A, A) = \mathrm{Der}(\mathscr{O}(X))[-1] = \mathrm{Hom}_{Vect}(T(A[1]), A),$$

and the differential is given by $\delta(\phi) = d_A \circ \phi - \phi \circ d_A, \phi \in CC^\bullet(A, A)$ (here we use the augmented differential).

Proposition 3.1. For A an A_∞-algebra, we have the following quasi-isomorphism of cochain complexes

$$CC^\bullet(A,A)[1] \simeq \mathrm{Der}(\mathscr{O}(X_A)) \simeq T_{id_{X_A}} Maps(X_A, X_A).$$

Here $Maps(-,-)$ is the internal hom in the category of dg-coalgebras, thus the tangent space at identity can be seen as the deformation of A_∞ structure.

Moreover, if A is weakly unital, it is quasi-isomorphic to $End_{A\text{-}mod\text{-}A}(A,A)$, which is a chain complex following our previous definitions.

Here we have a quick generalization, under the weakly unital assumption, to consider the Hochschild cochain with coefficients in M, given by

$$CC^\bullet(A,M) = \mathrm{Hom}_{A\text{-}mod\text{-}A}(A,M) = \mathrm{Hom}_{Vect}(T(A[1]),M),$$

and the differential $\delta(\phi) = d_M \circ \phi - \phi \circ d_A$.

To define Hochschild chain complex, we consider the tensor product of $A-A$ bimodule, which is a the chain complex

$$CC_\bullet(A,M) = A \otimes_{A\text{-}A} M = A \otimes T(A[1]) \otimes M \otimes T(A[1]),$$

and the differential $d_{A\otimes M}$ is given by d_A and the differential of bimodules A, M (notice we always think the bimoudle tensor product as a ”cyclic chain”).

Also we may construct the cap product map

$$\cap : CC^\bullet(A,A) \times CC_\bullet(A,M) \to CC_\bullet(A,M),$$

$$\phi \cap (b \otimes v_1 \otimes \cdots \otimes v_n) = \sum (-1)^* d_{M,*}(v_{j+1} \otimes \cdots \otimes v_n \otimes m \otimes v_1 \otimes \cdots \otimes \phi_*(v_i \otimes \cdots) \otimes \cdots \otimes v_s) \otimes \cdots \otimes v_j,$$

where $m \in M$.

Proposition 3.2. $CC^\bullet(A,A)$ naturally has a structure of dg-algebra. And the cap product gives $CC_\bullet(A,M)$ a structure of $CC^\bullet(A,A)$-module. In other words, we have

$$d_{A\otimes M}(\phi \cap \mu) = \delta(\phi) \cap \mu + \phi \cap d_{A\otimes M}(\mu),$$

$$(\phi_1 \cdot \phi_2) \cap \mu = \phi_1 \cap (\phi_2 \cap \mu),$$

for any $\phi_{1,2} \in CC^\bullet(A,A), \mu \in CC_\bullet(A,M)$.

We may generalize our discussion to an A_∞-category version, which turns out having no difference but only adding those labels.

4 Split-generation and Poincare Duality

Consider that $\mathscr{X}$ is a full subcategory of a triangulated category $\mathscr{C}$. We say $\mathscr{X}$ split-generates $\mathscr{C}$, if every element of $\mathscr{C}$ is isomorphic to a summand of a finite iterated cone of elements in X, i.e. the image of some idempotent maps on iterated mapping cones.

An A_∞-category $\mathscr{A}$ usually is not triangulated. However, the category of modules $\mathscr{A}$-mod is natrually pre-triangulated, i.e. its cohomology category $H^*(\mathscr{A}\text{-}mod)$ is triangulated, meaning we can do sums, shifts, mapping cones, etc. So we say any full subcategory $\mathscr{X}$ split-generates $\mathscr{A}$ if any object Z, $\mathscr{Y}^l(Z)$ is split-generated by objects $\mathscr{Y}^l(X), X \in Ob(\mathscr{X})$.

An important criterion, first introduced by Abouzaid, is to consider the tensor product $\mathscr{Y}^r(Z) \otimes_{\mathscr{X}} \mathscr{Y}^l(Z)$, which can be concretely given by

$$\bigoplus_{X_i} \mathrm{Hom}_{\mathscr{C}}(X_k, Z) \otimes \mathrm{Hom}_{\mathscr{C}}(X_{k-1}, X_k) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{C}}(Z, X_1).$$

And we have collapsing morphism $\mathscr{Y}^r(Z) \otimes_{\mathscr{X}} \mathscr{Y}^l(Z) \xrightarrow{\mu} \mathrm{Hom}_{\mathscr{C}}(Z, Z)$ given by

$$a \otimes x_k \otimes \cdots \otimes x_1 \otimes b \mapsto (-1)^* m_{\mathscr{A},k+2}(a \otimes x_k \otimes \cdots \otimes x_1 \otimes b),$$

which is a chain map, so having homology level morphism $[\mu]$.

Proposition 4.1. We have the following statements equivalent:

- The object Z is split-generated by $\mathscr{X}$.
- $[\mu]$ hits the identity.
- $[\mu]$ is an isomorphism.

We call an A_∞-category $\mathscr{A}$ (homologically) smooth, if the diagonal bimodule $\mathscr{A}_\Delta$ is split-generated by finitely many Yoneda bimodules $\mathscr{Y}^l(X) \otimes_{\mathbb{k}} \mathscr{Y}^r(Y)$ (we call it perfect object in $\mathscr{C}$-mod). This notation meets with the smoothness of (D^bCoh of) an algebraic variety.

It is natural to consider a possible Poincare duality for Hochschild (co)homology, which need two ingredient: a Calabi-Yau map $\mathcal{CY}_\# : HH_{*-n}(\mathscr{A}) \to HH^*(\mathscr{A}, \mathscr{A}^!)$ and the map $\bar{\mu} : HH^*(\mathscr{A}, \mathscr{A}^!) \to HH^*(\mathscr{A})$. $\mathscr{A}^!$ is the inverse dualizing bimodule of $\mathscr{A}$, and $\bar{\mu}$ is an isomorphism if $\mathscr{A}$ is smooth (using the split-generating criterion above).

A immediate result, given that our Calabi-Yau map is an isomorphism, is that we can now prove, for non-degenerate M ($\mathcal{OC}$ hits the identity), an isomorphism between symplectic cohomology $SH^*(M)$ and Hochschild (co)homology of wrapped Fukaya category $\mathcal{W}$. In fact we have maps

$$HH_{*-n}(\mathcal{W}) \xrightarrow{[\mathcal{OC}]} SH^*(M) \xrightarrow{[\mathcal{CO}]} HH^*(\mathcal{W})$$

where the closed-open map $\mathcal{CO}$ is a ring homomorphism and the open-closed map $\mathcal{OC}$ is a map of $SH^*(M)$-module. In fact, using cap product we can argue that $\mathcal{CO}$ is injective and $\mathcal{OC}$ is surjective in that case. We assume $\mathcal{OC}(\sigma) = 1 \in SH^*(M)$, then for any $s \in SH^*(M)$ we have

$$\mathcal{OC}(\mathcal{CO}(s) \cap \sigma) = s \cdot \mathcal{OC}(\sigma) = s$$

which implies that $\mathcal{CO}(s) \neq 0$. And we have our Poincare duality map

$$HH_{*-n}(\mathcal{W}) \xrightarrow{[\mathcal{CY}_\#]} HH^*(\mathcal{W}, \mathcal{W}^!) \xrightarrow{[\bar{\mu}]} HH^*(\mathcal{W}).$$

As long as we can prove the diagram (homotopy) commutes (called generalized Cardy condition), $\mathcal{OC}$ and $\mathcal{CO}$ naturally become isomorphisms.