INTRODUCTION TO FLOER HOMOTOPY THEORY

JOHAN ASPLUND

Abstract. These are (vastly expanded) notes for three talks given at the Symplectic Topology Student Seminar at Stony Brook University in Fall 2024. These notes are sketchy and may contain errors.

1. Overview

1.1. **Goals.** The three main goals of these talks are as follows:

- Give an introduction to Floer homotopy theory
- Define the wrapped Donaldson–Fukaya category of a stably polarized Liouville manifold with coefficients in a commutative ring spectrum R, denoted $\mathcal{W}(X; R)$.
- Discuss an outline of the proof of the following theorem.

Theorem 1.1 (A.–Deshmukh–Pieloch [ADP24])**.** *Let R be a commutative ring spectrum. Let Q be a closed manifold and $L \subset T^*Q$ a nearby Lagrangian. Then L is isomorphic to a "rank 1 local system" supported on Q in $\mathcal{W}(T^*Q; R)$.*

1.2. **Idea of Floer homotopy theory.** It was already known to Floer [Flo89, pp. 212–213] that Floer homology in favorable circumstances could be lifted to take values in appropriate generalized cohomology theories. Cohen–Jones–Segal [CJS95] formalized this vision and answered the following question in the affirmative.

Question 1.2. Is there a space (or spectrum) whose homology is equal to Floer homology?

In "good" situations Cohen–Jones–Segal showed that the Pontryagin–Thom construction can be used to build a spectrum whose homology is Floer homology. In these notes we are mainly interested in the situations of Morse theory or Floer theory of exact Lagrangians in Liouville manifolds, both of which are "good" situations.

The following philosophy summarizes the idea behind the construction of Floer homotopy theory.

Philosophy 1.3. Encapsulate all geometry of holomorphic curves in algebra.

Example 1.4. Suppose that M is a closed smooth manifold and $f\colon M \to \mathbb{R}$ is a smooth Morse–Smale function. To define Morse homology we consider the free module (over some commutative ring k) on the critical point set of f. Then we consider moduli spaces $\mathcal{M}(x, y)$ of unparametrized gradient flow lines from $x \in \operatorname{crit} f$ to $y \in \operatorname{crit} f$. Here are two facts:

- For a generically chosen metric $\mathcal{M}(x, y)$ is a smooth manifold with corners of dimension $|x| - |y| - 1$.
- The moduli spaces $\mathcal{M}(x, y)$ can be "coherently framed": for every $x, y \in \operatorname{crit} f$ there are canonical isomorphisms
$$TW^u(x) \xrightarrow{\cong} T\mathcal{M}(x, y) \oplus \underline{\mathbb{R}} \oplus TW^u(y),$$
of vector bundles over $\mathcal{M}(x, y)$.

To define Morse homology, throw away all $\mathcal{M}(x, y)$ of dimension > 1 and define a differential via $\partial x = \sum_{|x|-|y|=1} |\mathcal{M}(x, y)| y$.

Instead, we can in fact recover a space whose homology is Morse homology (of course, this is just a space homotopic to M itself) by not throwing away the high dimensional moduli spaces of gradient flow lines. Recording the information of the high dimensional moduli spaces of gradient flow lines and "orientation data" leads us to the notion of a "flow category."

2. Unoriented flow categories, bimodules and bordisms

We present a definition of flow categories that is in the flavor found in [AB21, Abo22, AB24] in a simplified setting that suffices for our main goals of discussing Floer homotopy theory in Liouville manifolds. The simplified setting also helps us to not stray too far away from the original definition of Cohen–Jones–Segal [CJS95], which is more concrete.

Definitions that are more closely related to the original definitions of Cohen–Jones–Segal may be found in [ADP24, Section 2].

2.1. **Manifolds with corners.** A smooth manifold with corners X determines a category $\mathcal{S}_X$ defined by

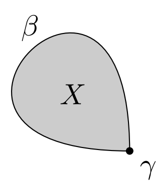
$$\operatorname{ob}(\mathcal{S}_X) = \{\text{components of corner strata } \partial^\sigma X\}$$

and

$$\mathcal{S}_X(\partial^\sigma X, \partial^\tau X) := \{\text{components of } \operatorname{int}(\partial^\sigma X) \cap N(\partial^\tau X)\} \text{ if } \partial^\sigma X \supset \partial^\tau X,$$

where N denotes a tubular neighborhood of the corner stratum $\partial^\tau X \subset X$. Assuming that X is connected, X is the initial object in $\mathcal{S}_X$. We furthermore have that $\mathcal{S}_X$ comes equipped with a functor $\operatorname{codim}\colon \mathcal{S}_X \to \mathbb{N}$ given by the codimension of a corner stratum.

Example 2.1.
(1) For any closed manifold M we have that $\operatorname{ob}(\mathcal{S}_M) = \{M\}$ and $\mathcal{S}_X(M, M) = \{M\}$.
(2) For $X = [0,1]$ we have $\operatorname{ob}(\mathcal{S}_{[0,1]}) = \{[0,1], \{0\}, \{1\}\}$ and morphisms $\{0\} \leftarrow [0,1] \to \{1\}$.
(3) If $X =$ teardrop



then we have $\operatorname{ob}(\mathcal{S}_X) = \{X, \beta, \gamma\}$ and morphisms generated by the arrows

$$X \longrightarrow \beta \rightrightarrows \gamma\ .$$

Definition 2.2. A category $\mathcal{P}$ equipped with a functor $\operatorname{codim}\colon \mathcal{P} \to \mathbb{N}$ is a *model for manifolds with corners* if for each $p \in \operatorname{ob}(\mathcal{P})$, there is an isomorphism of categories $\mathcal{P}_{/p} \cong 2^{\{1,\ldots,\operatorname{codim} p\}}$.

In the above definition $\mathcal{P}_{/p}$ denotes the overcategory of $\mathcal{P}$ over p. Furthermore $2^{\{1,\ldots,\operatorname{codim} p\}}$ denotes the power set of $\{1, \ldots, \operatorname{codim} p\}$ equipped with the partial order given by inclusion.

Definition 2.3. A *stratification of a smooth manifold with corners* X is a functor $s\colon \mathcal{S}_X \to \mathcal{P}$ where $\mathcal{P}$ is a model for manifolds with corners, such that
(1) $\operatorname{codim}_{\mathcal{P}} \circ s = \operatorname{codim}_{\mathcal{S}_X}$, and
(2) the induced functor $s_{/p}\colon (\mathcal{S}_X)_{/p} \to \mathcal{P}_{/s(p)} \cong 2^{\{1,\ldots,\operatorname{codim}_{\mathcal{P}} s(p)\}}$ is an isomorphism for any $p \in \operatorname{ob}(\mathcal{S}_X)$.

Example 2.4.
(1) Let M be a closed manifold. Then we have $\mathcal{S}_M \cong 2^{\varnothing} = \{\varnothing\}$, and so the functor $\mathcal{S}_M \to 2^{\varnothing}$ defined on objects by $\operatorname{int} M \mapsto \varnothing$ is a stratification, and is also an isomorphism of categories.

(2) For $X = [0,1]$ let $\mathcal{P} = \{0 \to 1\}$. We may define a stratification by $\mathcal{S}_X \to \mathcal{P}$ defined on objects by $[0,1] \mapsto 0$ and $\{0\}, \{1\} \mapsto 1$, and on morphisms by

$$([0,1] \to \{0\}), ([0,1] \to \{1\}) \longmapsto (0 \to 1).$$

It is clear that this functor preserves codimension. The induced functors on overcategories are

$$\begin{aligned} (\mathcal{S}_{[0,1]})_{/[0,1]} &\longrightarrow \mathcal{S}_{/0} \\ ([0,1] \to [0,1]) &\longmapsto (0 \to 0) \end{aligned}$$

and e.g.

$$\begin{aligned} (\mathcal{S}_{[0,1]})_{/\{0\}} &\longrightarrow \mathcal{S}_{/1} \\ ([0,1] \to \{0\}) &\longmapsto (0 \to 1) \\ (\{0\} \to \{0\}) &\longmapsto (1 \to 1), \end{aligned}$$

and correspondingly for the overcategory $(\mathcal{S}_{[0,1]})_{/[0,1]}$. It is clear that these are isomorphisms of categories.

(3) Let $X =$ teardrop. The existence of a stratification of X implies that the overcategory $(\mathcal{S}_X)_{/\gamma}$ is isomorphic to a power set, which is impossible because $|(\mathcal{S}_X)_{/\gamma}| = 5$ is not a power of 2.

Definition 2.5. Denote by $\mathbf{Man}_\partial$ the category of smooth manifolds with corners X that admit a stratification $\mathcal{S}_X \to \mathcal{P}$ where $\mathcal{P}$ is a model for manifolds with corners.

Remark 2.6. The teardrop is *not* an object in $\mathbf{Man}_\partial$ by Definition 2.5 as observed in Example 2.4(3).

Example 2.7. The notion of a $\langle k \rangle$-manifold introduced by Jänich [Jä68] (also see [Lau00]) is used in the original definition of a flow category due to Cohen–Jones–Segal. It is a fact that a $\langle k \rangle$-manifold X is equivalent to a smooth manifolds with corners such that $\mathcal{S}_X$ is isomorphic to the product poset $\{0 \to 1\}^k$ (it is related to the *cube category*, see e.g. [Jar06]).

A very useful point of view is to view $\{0 \to 1\}^k$ as a poset of rooted linear trees on $k+1$ vertices and k labeled internal edges, and with partial order generated by the opposite of edge contraction, see Figure 1.

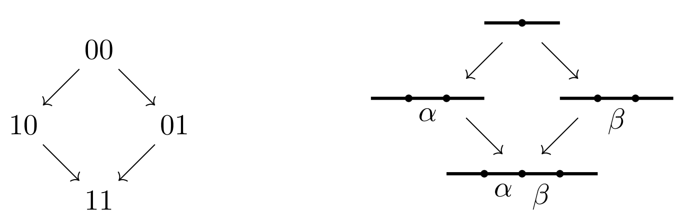


FIGURE 1. Interpreting $\{0 \to 1\}^2$ as a poset of rooted linear edge labeled trees.

Remark 2.8. An important fact that we will gloss over is that any object in $\mathbf{Man}_\partial$ admits a "coherent system of collars" of its corner strata. This will become important later, when we consider vector bundles over objects in $\mathbf{Man}_\partial$ that need to be suitably compatible with vector bundles over their corner strata.

A category C enriched in $\mathbf{Man}_\partial$ determines a 2-category $\mathcal{S}_C$ with the same objects as C, morphism categories $\mathcal{S}_C(x,y) := \mathcal{S}_{C(x,y)}$ and horizontal composition

$$\mathcal{S}_C(x,y) \times \mathcal{S}_C(y,z) \longrightarrow \mathcal{S}_C(x,z)$$

defined on objects by the composition in C using the fact that the product of corner strata of manifolds is a corner stratum of the product $\partial^\sigma C(x,y) \times \partial^\tau C(y,z) =: \partial^{\sigma\times\tau}(C(x,y)\times C(y,z))$, while respecting the codimension functors. We also equip $\mathcal{S}_C$ with a dimension 2-functor $\dim\colon \mathcal{S}_C \to \mathbb{N}$, where $\mathbb{N}$ is regarded as a 2-category with a single object.

Definition 2.9. Suppose that C is a category enriched in $\mathbf{Man}_\partial$ equipped with a dimension 2-functor $\dim_{\mathcal{S}_C}\colon \mathcal{S}_C \to \mathbb{N}$, and that P is a 2-category equipped with a dimension 2-functor $\dim_P\colon P \to \mathbb{N}$. A *stratification of C in P* is a 2-functor $s\colon \mathcal{S}_C \to P$ such that

(1) The stratification preserves the dimension 2-functors: $\dim_P \circ s = \dim_{\mathcal{S}_C}$.
(2) Each induced functor $\mathcal{S}_C(x,y) \to P(s(x), s(y))$ is a stratification.
(3) For every $x,y,z \in \mathrm{ob}(C)$, the following diagram commutes as a diagram of stratifications of smooth manifolds with corners

$$\begin{array}{ccc} \mathcal{S}_C(x,y)\times\mathcal{S}_C(y,z) & \xrightarrow{s(x,y)\times s(y,z)} & P(s(x),s(y))\times P(s(y),s(z)) \\ \downarrow & & \downarrow \\ \mathcal{S}_C(x,z) & \xrightarrow{\quad s(x,z)\quad} & P(s(x),s(z)) \end{array}.$$

2.2. **Flow categories.** Recall the definition of the poset $\{0\to 1\}^k$ from Example 2.7, and its interpretation as a poset of rooted linear trees. A rooted linear tree has one half-edge called a leaf, and one half-edge called the root; it will be useful to think about such a linear tree as an operation with "input" at the leaf and "output" at the root.

Let S be a $\mathbb{Z}$-graded set. For any $p,r\in S$ denote by $\square_S(p,r)$ the poset with objects being rooted linear trees with leaf labeled by p, root labeled by r and internal edges labeled by $q_1,\dots,q_k$ (in this order, according to the orientation from the leaf to the root) such that $|p|>|q_1|>\cdots>|q_k|>|r|$. The morphisms are induced by edge contraction, see Figure 2.

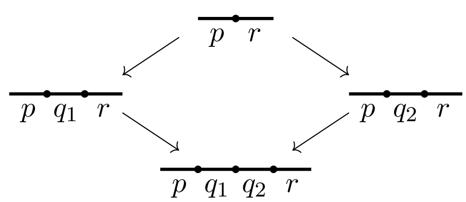


FIGURE 2. Four objects in the poset $\square_S(p,r)$ with four morphisms.

Given elements $p,q,r\in S$ with $|p|>|q|>|r|$ there is a natural functor defined on objects as follows:

$$\begin{gathered} \square_S(p,q)\times\square_S(q,r) \longrightarrow \square_S(p,r) \\ (\,\underset{p\ \ q}{\overline{\ \bullet\ }}\,,\,\underset{q\ \ r}{\overline{\ \bullet\ }}\,) \longmapsto \underset{p\ \ q\ \ r}{\overline{\ \bullet\ \bullet\ }} \end{gathered} \tag{2.1}$$

I.e. it is defined as grafting of trees along the leaf of one tree to the root of the other, creating a new internal edge with label q. It is clear that grafting is associative and is compatible with morphisms in the poset $\square_S(p,r)$. We equip $\square_S(p,r)$ with a codimension functor $\mathrm{codim}\colon \square_S(p,r)\to\mathbb{N}$ defined as the number of internal edges.

If S is a $\mathbb{Z}$-graded set, denote by $\square_S$ the 2-category with $\mathrm{ob}(\square_S)=S$ and morphism posets $\square_S(p,r)$ for every $p,r\in S$ with $|p|>|r|$. Vertical composition is induced by the composition in $\square_S(p,r)$ and horizontal composition is induced by grafting (2.1). We equip $\square_S$ with a dimension 2-functor $\dim\colon\square_S\to\mathbb{N}$ that is defined as follows: For any $p,r\in S$, an object $T\in\mathrm{ob}(\square_S(p,r))$ is defined to have dimension

$$\dim T := |p|-|r|-1-\mathrm{codim}\, T = |p|-|r|-1-\#\text{internal edges}.$$

Note that by definition there is an isomorphism of categories $\square_S(p,r) \cong \{0 \to 1\}^{M(p,r)}$ where

$$M(p,r) := \max\{k \mid \exists q_1, \ldots, q_k \; |p| > |q_1| > \cdots > |q_k| > |r|\}.$$

Definition 2.10. A *flow category* is a non-unital category $\mathcal{M}$ with a $\mathbb{Z}$-graded object set $\mathrm{ob}(\mathcal{M})$ enriched in $\mathbf{Man}_\partial$ that is equipped with a stratification $s_\mathcal{M} \colon \mathcal{S}_\mathcal{M} \to \square_{\mathrm{ob}(\mathcal{M})}$ that is an isomorphism of 2-categories.

Remark 2.11. Let us unpack Definition 2.10. Firstly, a flow category consists of a $\mathbb{Z}$-graded set of objects $\mathrm{ob}(\mathcal{M}) = \bigcup_{i\in\mathbb{Z}} \mathrm{ob}(\mathcal{M})_i$. Next, for every pair of objects $x, z \in \mathrm{ob}(\mathcal{M})$, there is a smooth manifold with corners $\mathcal{M}(x,z)$ that is stratified by $\square_{\mathrm{ob}(\mathcal{M})}(x,z) = \{\overline{x\;\bullet\;y_1}\;\cdots\;\overline{y_k\;\bullet\;z}\}$, compatible with grafting (2.1) (which is the horizontal composition in $\square_{\mathrm{ob}(\mathcal{M})}$). In particular there is a canonical identification between the codimension 0 stratum $\mathcal{M}(x,z)$ and the initial element $\overline{x\;\bullet\;z}$ in $\square_{\mathrm{ob}(\mathcal{M})}(x,z)$, and grafting (2.1) yields an identification of the disjoint union of the codimension 1 objects with the image of $\bigsqcup_{|p|>|q|>|r|} \overline{p\;\bullet\;q} \times \overline{q\;\bullet\;r}$ under (2.1). By the assumption that the stratification $s_\mathcal{M}$ preserves dimension 2-functors implies that $\dim \mathcal{M}(x,z) = |x| - |z| - 1$ for every $x, z \in \mathrm{ob}(\mathcal{M})$.

Therefore, the data of such stratifications (that also are isomorphisms) gives a bijection between codimension k strata $\partial^\sigma \mathcal{M}(x,z) \subset \mathcal{M}(x,z)$ and linear trees with exactly k internal edges. Moreover, the following diagram needs to commute

$$\begin{array}{ccc} \mathcal{S}_\mathcal{M}(x,y) \times \mathcal{S}_\mathcal{M}(y,z) & \longrightarrow & \mathcal{S}_\mathcal{M}(x,z) \\ \downarrow & & \downarrow \\ \square_{\mathrm{ob}(\mathcal{M})}(x,y) \times \square_{\mathrm{ob}(\mathcal{M})}(y,z) & \longrightarrow & \square_{\mathrm{ob}(\mathcal{M})}(x,z) \end{array},$$

and satisfies a suitable associativity diagram. This discussion implies that the union of the codimension 1 boundary strata of $\mathcal{M}(x,z)$ is identified with $\bigsqcup_{|x|>|y|>|z|} \mathcal{M}(x,y) \times \mathcal{M}(y,z)$. This coincides with the definition of an (unoriented) flow category as originally given by [CJS95].

We summarize the preceding remark as follows.

Proposition 2.12. *A flow category is equivalent to the following data:*

- *A $\mathbb{Z}$-graded object set* $\mathrm{ob}(\mathcal{M})$.
- *A $(|x| - |y| - 1)$-dimensional smooth manifold with corners $\mathcal{M}(x,y)$ for every pair of objects $x, y \in \mathrm{ob}(\mathcal{M})$ with $|x| > |y|$, such that each $\mathcal{M}(x,y)$ is equipped with a $\langle k \rangle$-manifold structure for some k.*
- *Composition maps*

$$\mu_{xyz} \colon \mathcal{M}(x,y) \times \mathcal{M}(y,z) \longrightarrow \mathcal{M}(x,z),$$

for all $x, y, z \in \mathrm{ob}(\mathcal{M})$ with $|x| > |y| > |z|$ that are diffeomorphisms onto codimension 1 corner strata of $\mathcal{M}(x,z)$ and required to be associative.

□

Example 2.13. Let M be a closed manifold, and let $f \colon M \to \mathbb{R}$ be a Morse–Smale function. We obtain a flow category $\mathcal{M}_f$ with object set $\mathrm{crit}(f)$ which is $\mathbb{Z}$-graded by the Morse index. The morphism spaces $\mathcal{M}_f(x,y)$ are defined to be the (compactification) of the moduli space of (unparametrized) gradient trajectories from x to y. By the compactification, the once broken flow lines provides us with the composition

$$\mathcal{M}_f(x,y) \times \mathcal{M}_f(y,z) \longrightarrow \mathcal{M}_f(x,z),$$

and the codimension 1 boundary of $\mathcal{M}_f(x,z)$ is identified with $\bigsqcup_y \mathcal{M}_f(x,y) \times \mathcal{M}_f(y,z)$. This yields the stratification $\mathcal{S}_{\mathcal{M}_f} \xrightarrow{\cong} \square_{\mathrm{ob}(\mathcal{M}_f)}$.

Example 2.14. Let X be a Liouville manifold and $L, K \subset X$ two transversely intersecting closed exact Lagrangians in X. Then we obtain a flow category $\mathcal{M}_{L,K}$ with object set $L \cap K$ which is $\mathbb{Z}$-graded by (the negative of) the Maslov index. The morphism spaces $\mathcal{M}_{L,K}(x, y)$ are defined by the (Gromov compactification) of the moduli space of (unparametrized) pseudoholomorphic strips from x to y with boundary conditions on L and K. It is a non-trivial fact that $\mathcal{M}_{L,K}(x, y)$ is a smooth manifold with corners; it follows from work by Large [Lar21] using work by Fukaya–Oh–Ohta–Ono [FOOO24]. By the Gromov compactification, the once broken pseudoholomorphic strips provides us with the composition

$$\mathcal{M}_{L,K}(x, y) \times \mathcal{M}_{L,K}(y, z) \longrightarrow \mathcal{M}_{L,K}(x, z),$$

and the codimension 1 boundary of $\mathcal{M}_{L,K}(x, z)$ is identified with $\bigsqcup_y \mathcal{M}_{L,K}(x, y) \times \mathcal{M}_{L,K}(y, z)$. This yields the stratification $\mathcal{S}_{\mathcal{M}_{L,K}} \xrightarrow{\cong} \square_{\mathrm{ob}(\mathcal{M}_{L,K})}$.

2.3. **Flow bimodules.** The correct notion of a "morphism between flow categories" is that of a *flow bimodule*.

Let S_1 and S_2 be two $\mathbb{Z}$-graded sets. For any $(p, r) \in S_1 \times S_2$, denote by $\square_{S_1,S_2}(p, r)$ the poset with objects being rooted linear trees with leaf labeled by p, root labeled by r and one distinguished vertex v. The distinguished vertex v determines a splitting of the internal edges (those on the "leaf side" of v and those on the "root side" of v). The internal edges are labeled by $q_1^1, \ldots, q_k^1, q_1^2, \ldots, q_\ell^2$, where $q_i^1 \in S_1$ are on the leaf side of v and $q_i^2 \in S_2$ are on the root side of v, such that

$$|p|_1 > |q_1^1|_1 > \cdots > |q_k^1|_1 \geq |q_1^2|_2 > \cdots > |q_\ell^2|_2 > |r|_2.$$

The morphisms are induced by (the opposite of) edge contraction, see Figure 3.

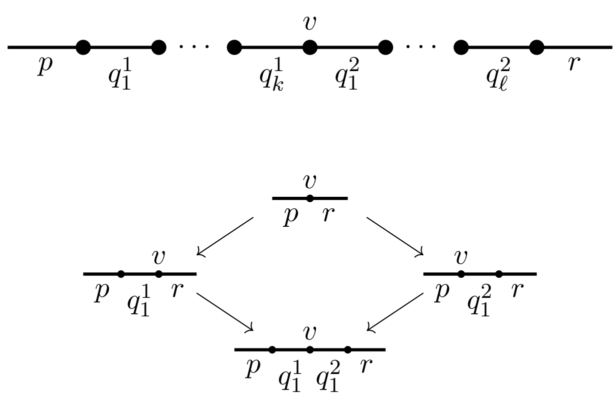


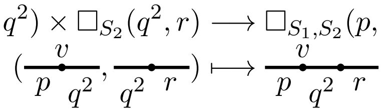
FIGURE 3. Top: An object in $\square_{S_1,S_2}(p, r)$. Bottom: Morphisms in $\square_{S_1,S_2}(p, r)$.

We equip each poset $\square_{S_1,S_2}(p, r)$ with a dimension functor defined on an object $T \in \mathrm{ob}(\square_{S_1,S_2}(p, r))$ as

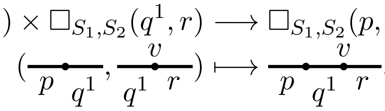
$$\dim T := |p|_1 - |r|_2 - \operatorname{codim} T := |p|_1 - |r|_2 - \#\text{internal edges}.$$

Given triples $(p, q^1, r) \in S_1^2 \times S_2$ and $(p, q^2, r) \in S_1 \times S_2^2$, respectively, we have grafting maps

$$\square_{S_1,S_2}(p, q^2) \times \square_{S_2}(q^2, r) \longrightarrow \square_{S_1,S_2}(p, r) \tag{2.2}$$



$$\square_{S_1}(p, q^1) \times \square_{S_1,S_2}(q^1, r) \longrightarrow \square_{S_1,S_2}(p, r) \tag{2.3}$$



which are both associative and compatible with morphisms in $\square_{S_1,S_2}(-,-)$ and $\square_{S_i}(-,-)$. Moreover the two grafting operations commute with each other. For technical reasons, adjoin identity morphisms to $\square_{S_1}$ and $\square_{S_2}$ by setting $\square_{S_1}(p,p) = \square_{S_2}(p,p) := \{\mathrm{pt}\}$. Then define $\square_{S_1,S_2}$ to be a 2-functor

$$\square_{S_1,S_2} \colon \square_{S_1}^{\mathrm{op}} \times \square_{S_2} \longrightarrow \mathbf{Cat},$$

i.e. a $\mathbf{Cat}$-enriched $(\square_{S_1}, \square_{S_2})$-bimodule (or "profunctor"), which on morphisms yields a map

$$(\square_{S_1}^{\mathrm{op}} \times \square_{S_2})((q^1,p),(q^2,r)) \to \mathbf{Cat}(\square_{S_1,S_2}(q^1,q^2), \square_{S_1,S_2}(p,r)),$$

which is equivalent to

$$\square_{S_1}(p,q^1) \times \square_{S_1,S_2}(q^1,q^2) \times \square_{S_2}(q^2,r) \longrightarrow \square_{S_1,S_2}(p,r),$$

which we define to be equal to the composition of (2.2) and (2.3). By setting $q^1 = p$ and $q^2 = r$ we recover the maps (2.2) and (2.3), respectively. These are colloquially called the "bimodule structure maps."

A $\mathbf{Man}_\partial$-enriched functor $\phi \colon C^{\mathrm{op}} \times D \to \mathbf{Man}_\partial$ determines a 2-functor

$$\begin{aligned} \mathcal{S}_\phi \colon \mathcal{S}_{C^{\mathrm{op}}} \times \mathcal{S}_D &\longrightarrow \mathbf{Cat} \\ (c,d) &\longmapsto \mathcal{S}_{\phi(c,d)}. \end{aligned}$$

It is defined on morphism categories by specifying bimodule structure maps (as explained above)

$$\begin{aligned} \mathcal{S}_{C(c,c')} \times \mathcal{S}_{\phi(c',d)} &\longrightarrow \mathcal{S}_{\phi(c,d)} \\ \mathcal{S}_{\phi(c,d')} \times \mathcal{S}_{D(d,d')} &\longrightarrow \mathcal{S}_{\phi(c,d)}, \end{aligned}$$

in such a way that they are induced from the bimodule structure maps on ϕ, similar to the definition of $\mathcal{S}_C$, namely the product of corner strata is a corner stratum of the product: $\partial^\sigma C(c,c') \times \partial^\tau \phi(c',d) = \partial^{\sigma\times\tau}(C(c,c') \times \phi(c',d))$, and now the bimodule structure maps on ϕ determine the bimodule structure maps above.

Assume that C and D are two $\mathbf{Man}_\partial$-enriched categories stratified by P_1 and P_2, respectively, in the sense of Definition 2.9, with stratifications $s_C \colon \mathcal{S}_C \to P_1$ and $s_D \colon \mathcal{S}_D \to P_2$, respectively. Let $\phi \colon C^{\mathrm{op}} \times D \to \mathbf{Man}_\partial$ be a $\mathbf{Man}_\partial$-enriched functor, and let $\mathcal{P} \colon P_1^{\mathrm{op}} \times P_2 \to \mathbf{Cat}$ be a 2-functor. We say that ϕ *is stratified by* $\mathcal{P}$ if there is a 2-natural isomorphism $F \colon \mathcal{S}_\phi \overset{\cong}{\Rightarrow} \mathcal{P} \circ (s_C^{\mathrm{op}} \times s_D)$.

For a similar reason to the above we now assume tacitly that flow categories are equipped with units, i.e requiring that $\mathcal{M}(x,x)$ is a point, this (without worrying about what possible unintended consequences this may have).

Definition 2.15 (Flow bimodule)**.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two flow categories. A *flow bimodule*, denoted by $\mathcal{N} \colon \mathcal{M}_1 \to \mathcal{M}_2$ is a functor

$$\mathcal{N} \colon \mathcal{M}_1^{\mathrm{op}} \times \mathcal{M}_2 \longrightarrow \mathbf{Man}_\partial,$$

that is stratified by $\square_{\mathrm{ob}(\mathcal{M}_1),\mathrm{ob}(\mathcal{M}_2)}$.

Remark 2.16. Again, we need to unpack Definition 2.15. The $\mathbf{Man}_\partial$-enriched functor $\mathcal{N} \colon \mathcal{M}_1^{\mathrm{op}} \times \mathcal{M}_2 \to \mathbf{Man}_\partial$ specifies for each pair of objects $(x,y) \in \mathrm{ob}(\mathcal{M}_1) \times \mathrm{ob}(\mathcal{M}_2)$ a smooth manifold with corners $\mathcal{N}(x,y)$. As described previously, by the definition of $\mathcal{N}$ specifies bimodule structure maps

$$\begin{aligned} \mathcal{M}_1(x,x') \times \mathcal{N}(x',y) &\longrightarrow \mathcal{N}(x,y) \\ \mathcal{N}(x,y') \times \mathcal{M}_2(y',y) &\longrightarrow \mathcal{N}(x,y), \end{aligned}$$

that are associative and also commute with each other.

By Proposition 2.12, recall that the fact that $\mathcal{M}_1$ and $\mathcal{M}_2$ admit stratifications by $\square_{\mathrm{ob}(\mathcal{M}_1)}$ and $\square_{\mathrm{ob}(\mathcal{M}_2)}$, respectively, equips $\mathcal{M}_1$ with the structure of a $\langle k_1 \rangle$-manifold and $\mathcal{M}_2$ with the structure of a $\langle k_2 \rangle$-manifold. That $\mathcal{N}$ is furthermore is stratified by $\square_{\mathrm{ob}(\mathcal{M}_1),\mathrm{ob}(\mathcal{M}_2)}$ means that $\mathcal{N}(x,y)$ is stratified by

$$\square_{\mathrm{ob}(\mathcal{M}_1),\mathrm{ob}(\mathcal{M}_2)}(x,y) = \left\{ \underset{x \quad x'_1 \qquad x'_k \; y'_1 \qquad y'_\ell \quad y}{\overset{v}{\text{—}\bullet\text{—}\bullet\cdots\bullet\text{—}\bullet\text{—}\bullet\cdots\bullet\text{—}\bullet\text{—}}} \right\},$$

compatible with the bimodule structure maps of $\square_{\mathrm{ob}(\mathcal{M}_1),\mathrm{ob}(\mathcal{M}_2)}(x,y)$. There is a canonical identification between the codimension 0 stratum $\mathcal{N}(x,y)$ and the initial element $\underset{x \quad y}{\overset{v}{\text{—}\bullet\text{—}}}$ in $\square_{\mathrm{ob}(\mathcal{M}_1),\mathrm{ob}(\mathcal{M}_2)}(x,y)$, and $\dim \mathcal{N}(x,y) = \dim \square_{\mathrm{ob}(\mathcal{M}_1),\mathrm{ob}(\mathcal{M}_2)}(x,y) = |x|_1 - |y|_2$. Via the grafting operations (2.2) and (2.3), we obtain an identification of the disjoint union of the codimension 1 strata of $\mathcal{N}(x,y)$ with the strata corresponding to image of

$$\bigsqcup_{|x|>|x'|\geq|y|} \underset{x \quad x'}{\text{—}\bullet\text{—}} \times \underset{x' \quad y}{\overset{v}{\text{—}\bullet\text{—}}} \cup \bigsqcup_{|x|\geq|y'|>|y|} \underset{x \quad y'}{\overset{v}{\text{—}\bullet\text{—}}} \times \underset{y' \quad y}{\text{—}\bullet\text{—}}$$

under (2.2) and (2.3).

Proposition 2.17. *A flow bimodule between two flow categories $\mathcal{M}_1$ and $\mathcal{M}_2$ is equivalent to the following data:*

- *An assignment of a $(|x|_1 - |y|_2)$-dimensional smooth manifold with corners $\mathcal{N}(x,y) \in \mathrm{ob}(\mathbf{Man}_\partial)$ for every pair of objects $(x,y) \in \mathrm{ob}(\mathcal{M}_1) \times \mathrm{ob}(\mathcal{M}_2)$ with $|x| \geq |y|$, such that each $\mathcal{N}(x,y)$ is equipped with a $\langle k \rangle$-manifold structure for some k.*
- *Bimodule structure maps*

$$\mathcal{M}_1(x,x') \times \mathcal{N}(x',y) \longrightarrow \mathcal{N}(x,y)$$
$$\mathcal{N}(x,y') \times \mathcal{M}_2(y',y) \longrightarrow \mathcal{N}(x,y)$$

for all triples $(x,x',y) \in \mathrm{ob}(\mathcal{M}_1)^2 \times \mathrm{ob}(\mathcal{M}_2)$ with $|x| > |x'| \geq |y|$, and triples $(x,y',y) \in \mathrm{ob}(\mathcal{M}_1) \times \mathrm{ob}(\mathcal{M}_2)^2$ with $|x| \geq |y'| > |y|$, respectively, that furthermore are diffeomorphisms onto codimension 1 corner strata of $\mathcal{N}(x,y)$. The bimodule structure maps are required to be associative and are required to commute with each other.

□

Example 2.18. Suppose that $f\colon M \to \mathbb{R}$ and $g\colon N \to \mathbb{R}$ are two Morse–Smale functions on two closed smooth manifolds, and $\varphi\colon M \to N$ is a smooth function such that $\varphi|_{W^u_{-\nabla f}(x)} \pitchfork W^s_{-\nabla g}(y)$ for every $x \in \mathrm{crit}\, f$ and $y \in \mathrm{crit}\, g$. Then we may define a flow bimodule by the assignment

$$(x,y) \longmapsto \mathcal{N}(x,y) := W^u_{-\nabla f}(x) \times_\varphi W^s_{-\nabla g}(y).$$

Example 2.19. Let X be a Liouville manifold and $\{L_t\}_{t\in[0,1]}, K \subset X$ closed exact Lagrangians in X such that $L_t \pitchfork K$ for every $t \in [0,1]$. Then we may define a flow bimodule by defining $\mathcal{N}_{L_t;K}(x,y)$ denote the Gromov compactification of the moduli space of pseudoholomorphic strips from $x \in L_0 \cap K$ to $y \in L_1 \cap K$, with boundary conditions on K and moving boundary conditions on L_t.

2.4. **Composition of flow bimodules.** Suppose that $\mathcal{N}_1\colon \mathcal{M}_1 \to \mathcal{M}_2$ and $\mathcal{N}_2\colon \mathcal{M}_2 \to \mathcal{M}_3$ are two flow bimodules. There is a natural way of defining their composition, and we discuss stratifying categories first.

Let S_1, S_2 and S_3 be three $\mathbb{Z}$-graded sets. Given elements $(p,r) \in S_1 \times S_3$, denote by $\square_{S_1,S_2,S_3}(p,r)$ the poset with objects being rooted linear trees with leaf labeled by p, root labeled by r and two distinguished vertices v_1 and v_2. The two distinguished vertices determines a splitting of the internal edges as before, and they are labeled by $q^1_1,\ldots,q^1_k$ near the leaf, by $q^2_1,\ldots,q^2_\ell$ between the two distinguished vertices, and by $q^3_1,\ldots,q^3_m$ near the root, see Figure 4.

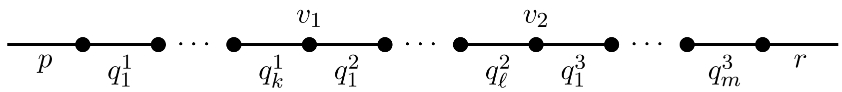


FIGURE 4. An object in $\square_{S_1,S_2,S_3}(p,r)$.

The degrees of the labels need to satisfy

$$|p|_1 > |q_1^1|_1 > \cdots > |q_k^1|_1 \geq |q_1^2|_2 > \cdots > |q_\ell^2|_2 \geq |q_1^3|_3 > \cdots > |q_m^3|_3 > |r|_3.$$

As before the morphisms are given by edge contraction with the exception that we do not allow edge contraction of an internal edge if both of its vertices are distinguished. Note that we now have a natural grafting operation:

$$\begin{aligned} \square_{S_1,S_2}(p,q) \times \square_{S_2,S_3}(q,r) &\longrightarrow \square_{S_1,S_2,S_3}(p,r) \\ \Big(\underset{p\quad q}{\overset{v_1}{\text{---}\bullet\text{---}}}, \underset{q\quad r}{\overset{v_2}{\text{---}\bullet\text{---}}}\Big) &\longmapsto \underset{p\quad q\quad r}{\overset{v_2\ \ v_2}{\text{---}\bullet\text{---}\bullet\text{---}}} \end{aligned} \tag{2.4}$$

The 2-category $\square_{S_1,S_2,S_3}$ also has a natural description in terms of the $(\square_{S_1},\square_{S_2})$-bimodule $\square_{S_1,S_2}$ and the $(\square_{S_2},\square_{S_3})$-bimodule $\square_{S_2,S_3}$. Namely, we have $\square_{S_1,S_2,S_3} = \square_{S_1,S_2} \times_{\square_{S_2}} \square_{S_2,S_3}$, where the 2-category $\square_{S_1,S_2} \times_{\square_{S_2}} \square_{S_2,S_3}$ is defined to have object set equal to $S_1 \times S_3$ and the morphisms are given by posets $(\square_{S_1,S_2} \times_{\square_{S_2}} \square_{S_2,S_3})(x,z)$ that is defined as the colimit of the following diagram

$$\coprod_{q,q' \in S_2} \square_{S_1,S_2}(p,q) \times \square_{S_2}(q,q') \times \square_{S_2,S_3}(q',r) \rightrightarrows \coprod_{q \in S_2} \square_{S_1,S_2}(p,q) \times \square_{S_2,S_3}(q',r) \ ,$$

where the two arrows are given by the bimodule grafting operations (2.2) and (2.3), respectively. There is furthermore a natural 2-functor $\sigma_{S_1,S_2,S_3} \colon \square_{S_1,S_2,S_3} \to \square_{S_1,S_3}$ defined on 1-morphisms by collapsing all internal edges between the two distinguished vertices.

Lemma 2.20. *The following diagram is commutative.*

$$\begin{array}{ccc} \square_{S_1,S_2} \times_{\square_{S_2}} \square_{S_2,S_3} \times_{\square_{S_3}} \times \square_{S_3,S_4} & \xrightarrow{\mathrm{id} \times_{\square_{S_2}} \sigma_{S_2,S_3,S_4}} & \square_{S_1,S_2} \times_{\square_{S_2}} \square_{S_2,S_4} \\ {\scriptstyle \sigma_{S_1,S_2,S_3} \times_{\square_{S_3}} \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \square_{S_1,S_2,S_4}} \\ \square_{S_1,S_3} \times_{\square_{S_3}} \times \square_{S_3,S_4} & \xrightarrow{\sigma_{S_1,S_3,S_4}} & \square_{S_1,S_4} \end{array}$$

□

Definition 2.21 (Composition of flow bimodules)**.** Let $\mathcal{N}_1 \colon \mathcal{M}_1 \to \mathcal{M}_2$ and $\mathcal{N}_2 \colon \mathcal{M}_2 \to \mathcal{M}_3$ be two flow bimodules. The composition $\mathcal{N}_2 \circ \mathcal{N}_1 \colon \mathcal{M}_1 \to \mathcal{M}_3$ is defined as the $\mathbf{Man}_\partial$-enriched functor

$$\mathcal{N}_2 \circ \mathcal{N}_1 \colon \mathcal{M}_1^{\mathrm{op}} \times \mathcal{M}_3 \longrightarrow \mathbf{Man}_\partial,$$

that is defined on objects by letting $(\mathcal{N}_2 \circ \mathcal{N}_2)(x,z)$ be the colimit of the following diagram

$$\coprod_{y,y' \in \mathrm{ob}(\mathcal{M}_2)} \mathcal{N}_1(x,y) \times \mathcal{M}_2(y,y') \times \mathcal{N}_2(y',z) \rightrightarrows \coprod_{y \in \mathrm{ob}(\mathcal{M}_2)} \mathcal{N}_1(x,y) \times \mathcal{N}_2(y,z) \ .$$

Lemma 2.22. *The stratifications of $\mathcal{N}_1$ by $\square_{\mathrm{ob}(\mathcal{M}_1),\mathrm{ob}(\mathcal{M}_2)}$ and $\mathcal{N}_2$ by $\square_{\mathrm{ob}(\mathcal{M}_2),\mathrm{ob}(\mathcal{M}_3)}$, respectively, induces a stratification of $\mathcal{N}_2 \circ \mathcal{N}_1$ by $\square_{\mathrm{ob}(\mathcal{M}_1),\mathrm{ob}(\mathcal{M}_2),\mathrm{ob}(\mathcal{M}_3)}$.* □

2.5. **Flow bordisms.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two flow categories, and $\mathcal{N}, \mathcal{N}' \colon \mathcal{M}_1 \to \mathcal{M}_2$ two flow bimodules. The flow categorical analog of a homotopy is that of a "flow bordism."

Let S_1 and S_2 be two $\mathbb{Z}$-graded sets, and let $M \in \mathrm{ob}(\mathbf{Man}_\partial)$ be a smooth manifold with corners. Given elements $(p, r, \partial^\sigma M) \in S_1 \times S_2 \times \mathrm{ob}(\mathcal{S}_M)$, define the poset $\square^{\partial^\sigma M}_{S_1,S_2}(p,r)$ to be an element of $\square_{S_1,S_2}(p,r)$ such that the distinguished vertex carries the label $\partial^\sigma M$, and

$$|p|_1 > |q_1^1|_1 > \cdots > |q_k^1|_1, \quad |q_1^2|_2 > \cdots > |q_\ell^2|_2 > |r|_2, \quad |q_k^1|_1 + \dim \partial^\sigma M \geq |q_1^2|_2.$$

The partial order on $\square^{\partial^\sigma M}_{S_1,S_2}(p,r)$ is the same as the one on $\square_{S_1,S_2}(p,r)$, and its dimension function is defined as $\dim \square^{\partial^\sigma M}_{S_1,S_2}(p,r) = |p|_1 - |r|_r + \dim \partial^\sigma M$. Similar to Section 2.3 we therefore obtain a 2-functor

$$\square^{\partial^\sigma M}_{S_1,S_2} \colon \square^{\mathrm{op}}_{S_1} \times \square_{S_2} \longrightarrow \mathbf{Cat}\,.$$

Define a functor on objects as follows

$$\begin{aligned} \square^{M}_{S_1,S_2} \colon \mathcal{S}_M^{\mathrm{op}} &\longrightarrow 2\text{-}\mathbf{Fun}(\square^{\mathrm{op}}_{S_1} \times \square_{S_2}, \mathbf{Cat}) \\ \partial^\sigma M &\longmapsto \square^{\partial^\sigma M}_{S_1,S_2}, \end{aligned}$$

i.e. a 2-$\mathbf{Fun}(\square^{\mathrm{op}}_{S_1} \times \square_{S_2}, \mathbf{Cat})$-presheaf on $\mathcal{S}_M$. The grafting operations (2.2) and (2.3) are defined on $\square^{\partial^\sigma M}_{S_1,S_2}$ for any $\partial^\sigma M \in \mathrm{ob}(\mathcal{S}_M)$. The functor $\square^{M}_{S_1,S_2}$ is defined on morphisms $\partial^\sigma M \to \partial^\tau M$ as the 2-natural transformation $\square^{\partial^\tau M}_{S_1,S_2} \Rightarrow \square^{\partial^\sigma M}_{S_1,S_2}$ that changes the label of the distinguished vertex. It is clear that the label change commutes with both edge contraction and grafting, see Figure 5 for the case $M = [0,1]$.

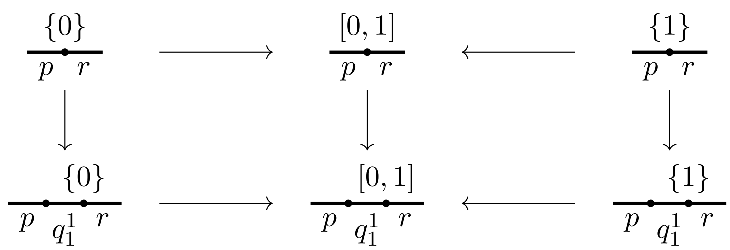


FIGURE 5. Label change and edge contraction commutes.

Assume C and D are two $\mathbf{Man}_\partial$-enriched categories equipped with stratifications $s_C \colon \mathcal{S}_C \to P_1$ and $s_D \colon \mathcal{S}_D \to P_2$, respectively. Assume $\phi_1, \phi_2 \colon C^{\mathrm{op}} \times D \to \mathbf{Man}_\partial$ are two $\mathbf{Man}_\partial$-enriched functors stratified by the 2-functors $\mathcal{P}_1, \mathcal{P}_2 \colon P_1^{\mathrm{op}} \times P_2 \to \mathbf{Cat}$ (recall that this means that there are 2-natural isomorphisms $\mathcal{S}_{\phi_1} \Rightarrow \mathcal{P}_1 \circ (s_C^{\mathrm{op}} \times s_D)$ and $\mathcal{S}_{\phi_2} \Rightarrow \mathcal{P}_2 \circ (s_C^{\mathrm{op}} \times s_D)$). Suppose that $\psi \colon \mathcal{S}^{\mathrm{op}}_{[0,1]} \to \mathbf{Fun}(C^{\mathrm{op}} \times D, \mathbf{Man}_\partial)$ is a functor such that $\psi(\{0\}) = \phi_1$ and $\psi(\{1\}) = \phi_2$. This determines a functor

$$\mathcal{S}_\psi \colon \mathcal{S}^{\mathrm{op}}_{[0,1]} \longrightarrow 2\text{-}\mathbf{Fun}(\mathcal{S}_{C^{\mathrm{op}}} \times \mathcal{S}_D, \mathbf{Cat}),$$

that satisfies $\mathcal{S}_\psi(\{0\}) = \mathcal{S}_{\phi_1}$ and $\mathcal{S}_\psi(\{1\}) = \mathcal{S}_{\phi_2}$. We say that ψ *is stratified by a functor* $\mathcal{Q} \colon \mathcal{S}^{\mathrm{op}}_{[0,1]} \to$ 2-$\mathbf{Fun}(P_1^{\mathrm{op}} \times P_2, \mathbf{Cat})$ satisfying $\mathcal{Q}(\{0\}) = \mathcal{P}_1$ and $\mathcal{Q}(\{1\}) = \mathcal{P}_2$ if there is a natural isomorphism $\mathcal{S}_\psi \overset{\cong}{\Rightarrow} \mathcal{Q}_{C,D}$, where $\mathcal{Q}_{C,D}$ is the induced functor

$$\mathcal{Q}_{C,D} \colon \mathcal{S}^{\mathrm{op}}_{[0,1]} \longrightarrow 2\text{-}\mathbf{Fun}(\mathcal{S}_{C^{\mathrm{op}}} \times \mathcal{S}_D, \mathbf{Cat}),$$

that satisfies $\mathcal{Q}_{C,D}(\{0\}) = \mathcal{P}_1 \circ (s_C^{\mathrm{op}} \times s_D)$ and $\mathcal{Q}_{C,D}(\{1\}) = \mathcal{P}_2 \circ (s_C^{\mathrm{op}} \times s_D)$.

Definition 2.23 (Flow bordism)**.** Let $\mathcal{N}_1, \mathcal{N}_2 \colon \mathcal{M}_1 \to \mathcal{M}_2$ be two flow bimodules. A *flow bordism from* $\mathcal{N}_1$ *to* $\mathcal{N}_2$, denoted by $\mathcal{B} \colon \mathcal{N}_1 \Rightarrow \mathcal{N}_2$, is a functor

$$\mathcal{B} \colon \mathcal{S}^{\mathrm{op}}_{[0,1]} \longrightarrow \mathbf{Fun}(\mathcal{M}_1^{\mathrm{op}} \times \mathcal{M}_2, \mathbf{Man}_\partial),$$

such that $\mathcal{B}(\{0\}) = \mathcal{N}_1$ and $\mathcal{B}(\{1\}) = \mathcal{N}_2$ and that is stratified by the functor $\square^{[0,1]}_{\mathrm{ob}(\mathcal{M}_1),\mathrm{ob}(\mathcal{M}_2)}$.

Remark 2.24. Note that $\mathcal{B}$ is simply a $\square^{[0,1]}_{\mathrm{ob}(\mathcal{M}_1),\mathrm{ob}(\mathcal{M}_2)}$-stratified $\mathbf{Fun}(\mathcal{M}_1^{\mathrm{op}} \times \mathcal{M}_2, \mathbf{Man}_\partial)$-presheaf on $\mathcal{S}_{[0,1]}$.

Remark 2.25. Let us unpack Definition 2.23. Since $\mathrm{ob}(\mathcal{S}^{\mathrm{op}}_{[0,1]}) = \{\{0\}, \{1\}, [0,1]\}$, and since $\mathcal{B}$ is already determined on objects $\{0\}$ and $\{1\}$, we have one additional functor

$$\mathcal{B}([0,1])\colon \mathcal{M}_1^{\mathrm{op}} \times \mathcal{M}_2 \longrightarrow \mathbf{Man}_\partial,$$

which by Remark 2.16 means an assignment of a smooth manifold with corners

$$\mathcal{B}(x,y) := \mathcal{B}([0,1])(x,y),$$

for each pair of objects $(x,y) \in \mathrm{ob}(\mathcal{M}_1) \times \mathrm{ob}(\mathcal{M}_2)$. This smooth manifold with corners comes with bimodule structure maps

$$\mathcal{M}_1(x,x') \times \mathcal{B}(x',y) \longrightarrow \mathcal{B}(x,y)$$
$$\mathcal{B}(x,y') \times \mathcal{M}_2(y',y) \longrightarrow \mathcal{B}(x,y),$$

that are associative and also commute with each other.

The functor $\mathcal{B}$ furthermore comes equipped with natural transformations

$$\mathcal{N}_1 \Rightarrow \mathcal{B}([0,1]) \Leftarrow \mathcal{N}_2,$$

which in particular yield maps

$$\mathcal{N}_1(x,y) \longrightarrow \mathcal{B}(x,y)$$
$$\mathcal{N}_2(x,y) \longrightarrow \mathcal{B}(x,y)$$

for all pairs of objects $(x,y) \in \mathrm{ob}(\mathcal{M}_1) \times \mathrm{ob}(\mathcal{M}_2)$. Similar to Remark 2.16, we obtain an identification of the disjoint union of the codimension 1 strata of $\mathcal{B}(x,y)$ with the strata corresponding to the image of

$$\coprod_{\substack{|x|>|x'|\\ |x'|+1\geq|y|}} \underset{x\quad x'}{-\!\!\!-\!\bullet\!-\!\!\!-} \times \overset{[0,1]}{\underset{x'\quad y}{-\!\!\!-\!\bullet\!-\!\!\!-}} \cup \coprod_{\substack{|x|+1\geq|y'|\\ |y'|>|y|}} \overset{[0,1]}{\underset{x\quad y'}{-\!\!\!-\!\bullet\!-\!\!\!-}} \times \underset{y'\quad y}{-\!\!\!-\!\bullet\!-\!\!\!-} \cup \coprod_{|x|\geq|y|} \overset{\{0\}}{\underset{x\quad y}{-\!\!\!-\!\bullet\!-\!\!\!-}} \cup \coprod_{|x|\geq|y|} \overset{\{1\}}{\underset{x\quad y}{-\!\!\!-\!\bullet\!-\!\!\!-}}$$

$$\cong \coprod_{\substack{|x|>|x'|\\ |x'|+1\geq|y|}} \mathcal{M}_1(x,x') \times \mathcal{B}(x',y) \cup \coprod_{\substack{|x|+1\geq|y'|\\ |y'|>|y|}} \mathcal{B}(x,y') \times \mathcal{M}_2(y',y) \cup \coprod_{|x|\geq|y|} \mathcal{N}_1(x,y) \cup \coprod_{|x|\geq|y|} \mathcal{N}_2(x,y)$$

in the image of the grafting and label change operations, see e.g. Figure 5.

Proposition 2.26. *A flow bordism between two flow bimodules $\mathcal{N}_1, \mathcal{N}_2\colon \mathcal{M}_1 \to \mathcal{M}_2$ is equivalent to the following data:*

- *An assignment of a $(|x|_1 - |y|_2 + 1)$-dimensional smooth manifold with corners $\mathcal{B}(x,y) \in \mathrm{ob}(\mathbf{Man}_\partial)$ for every pair of objects $(x,y) \in \mathrm{ob}(\mathcal{M}_1) \times \mathrm{ob}(\mathcal{M}_2)$ with $|x| + 1 \geq |y|$, such that each $\mathcal{B}(x,y)$ is equipped with a $\langle k \rangle$-manifold structure for some k.*
- *Bimodule structure maps*

$$\mathcal{M}_1(x,x') \times \mathcal{B}(x',y) \longrightarrow \mathcal{B}(x,y)$$
$$\mathcal{B}(x,y') \times \mathcal{M}_2(y',y) \longrightarrow \mathcal{B}(x,y)$$
$$\mathcal{N}_1(x,y) \longrightarrow \mathcal{B}(x,y)$$
$$\mathcal{N}_2(x,y) \longrightarrow \mathcal{B}(x,y)$$

for all triples $(x,x',y) \in \mathrm{ob}(\mathcal{M}_1)^2 \times \mathrm{ob}(\mathcal{M}_2)$ with $|x| > |x'|$ and $|x'| + 1 \geq |y|$, triples $(x,y',y) \in \mathrm{ob}(\mathcal{M}_1) \times \mathrm{ob}(\mathcal{M}_2)^2$ with $|x| + 1 \geq |y'| > |y|$ and tuples $(x,y) \in \mathrm{ob}(\mathcal{M}_1) \times \mathrm{ob}(\mathcal{M}_2)$ with $|x| \geq |y|$, respectively. These maps are furthermore required to be diffeomorphisms *onto codimension* 1 *corner strata of* $\mathcal{B}(x,y)$*, and they are required to be associative and commutative with each other.*

Remark 2.27. (1) By recent major technical advancements made by Abouzaid–Blumberg [AB24], there is a construction of a stable ∞-category of (framed) flow categories, that is equivalent to the stable ∞-category of spectra.

(2) The definitions of flow categories, flow bimodules and flow bordisms in these notes indicates a technically different but equivalent route to defining such an ∞-category (in our simplified setting, which would suffice for Floer theory in the exact setting for instance). Namely, we consider an object set consisting of flow categories. Letting Δ^k denote the standard k-simplex. For any pair of flow categories $(\mathcal{M}_1, \mathcal{M}_2)$ we define a simplicial set

$$[k] \longmapsto \mathbf{Fun}_{\square_{\Delta^k}}(\mathcal{S}_{\Delta^k}^{\mathrm{op}}, \mathbf{Fun}(\mathcal{M}_1^{\mathrm{op}} \times \mathcal{M}_2, \mathbf{Man}_\partial)),$$

where $\mathbf{Fun}_{\square_{\Delta^k}}$ denotes functors that are stratified by $\square^{\Delta^k}_{\mathrm{ob}(\mathcal{M}_1),\mathrm{ob}(\mathcal{M}_2)}$, similar to Definition 2.23. In other words, the k-simplices in the simplicial set that we call $\mathbf{Flow}(\mathcal{M}_1, \mathcal{M}_2)$ are $\square^{\Delta^k}_{\mathrm{ob}(\mathcal{M}_1),\mathrm{ob}(\mathcal{M}_2)}$-stratified $\mathbf{Fun}(\mathcal{M}_1^{\mathrm{op}} \times \mathcal{M}_2, \mathbf{Man}_\partial)$-presheaves on $\mathcal{S}_{\Delta^k}$. It turns out that this simplicial set is in fact an ∞-groupoid, and this data (objects the set of flow categories and morphism ∞-groupoids $\mathbf{Flow}(\mathcal{M}_1, \mathcal{M}_2)$) is equivalent to the data of an ∞-category with 0-cells being flow categories.

(3) It is possible to define a "flow multimodule" which would be interpreted as a flow bimodule with several inputs. This is necessary to consider naturally occurring operations in Floer theory such as the triangle product and higher A_∞-operations appearing in the Fukaya category. This naturally leads to a "multicategory of flow categories," or more precisely its ∞-categorical analog: an ∞-operad of flow categories.

2.6. **Forgetful functor.** In order to make the notion of "throwing away" the high dimensional moduli spaces as described in Example 1.4 more precise, there is in fact a *forgetful functor*, which takes as input a flow category and has as output a chain complex. Given a flow category $\mathcal{M}$, define a chain complex over $\mathbb{Z}/2$ $C\mathcal{M}_k$ defined by

$$C\mathcal{M}_k := \mathbb{Z}/2 \left\langle \mu^{-1}(k) \right\rangle,$$

where $\mu \colon \mathrm{ob}(\mathcal{M}) \to \mathbb{Z}$ denotes the grading function. We may define a differential $\partial \colon C\mathcal{M}_* \to C\mathcal{M}_{*-1}$ by

$$\partial x := \sum_{|x|=|y|+1} (\#_2 \mathcal{M}(x,y)) y,$$

where $\#_2\mathcal{M}(x,y)$ denotes the mod 2 count of the elements in $\mathcal{M}(x,y)$. The proof that $\partial^2 = 0$ follows from the fact that contributions to $\partial^2 x$ counts elements in $\bigsqcup_y \mathcal{M}(x,y) \times \mathcal{M}(y,z)$ of dimension 0, which are precisely all the boundary strata of the 1-dimensional moduli space $\mathcal{M}(x,z)$ by construction.

Given a flow bimodule $\mathcal{N} \colon \mathcal{M} \to \mathcal{M}'$, we may define a chain map $C\mathcal{N} \colon C\mathcal{M}_* \to C\mathcal{M}'_*$ by

$$C\mathcal{N}(x) := \sum_{|x|=|y|'} (\#_2 \mathcal{N}(x,y)) y.$$

The proof that $\partial' \circ C\mathcal{N}(x) = C\mathcal{N}(x) \circ \partial$ is similar to the above; it follows from the fact that contributions to $\partial' \circ C\mathcal{N}(x) + C\mathcal{N}(x) \circ \partial$ counts elements in

$$\bigsqcup_{x' \in \mathrm{ob}(\mathcal{M}_1)} (\mathcal{M}_1(x,x') \times \mathcal{N}(x',y)) \sqcup \bigsqcup_{y' \in \mathrm{ob}(\mathcal{M}_2)} (\mathcal{N}(x,y') \times \mathcal{M}_2(y',y)),$$

of dimension 0, which are precisely all the boundary strata of the 1-dimensional moduli space $\mathcal{N}(x,y)$ by construction.

Given another flow bimodule $\mathcal{N}'\colon \mathcal{M} \to \mathcal{M}'$ and a flow bordism $\mathcal{B}\colon \mathcal{N} \Rightarrow \mathcal{N}'$, we may define a chain homotopy $C\mathcal{B}\colon C\mathcal{M}_* \to C\mathcal{M}'_{*+1}$ by

$$C\mathcal{B}(x) := \sum_{|x|+1=|y|'} (\#_2\mathcal{B}(x,y))y.$$

The proof that $\partial' \circ C\mathcal{B}(x) + C\mathcal{B}(x) \circ \partial = C\mathcal{N}(x) + C\mathcal{N}'(x)$ is proved in a similar fashion as above.

3. Spectra and orientations on flow categories

In Floer homology, when working with integer coefficients we must orient moduli spaces of pseudoholomorphic curves in a coherent fashion. Part of the idea of Floer homotopy theory is to make sense of using "coefficients in a ring spectrum." To make this precise we give a brief account for what notions from stable homotopy theory we require.

3.1. Spectra.

Definition 3.1. A *spectrum* is a sequence $\{Z_k\}_{k=0}^{\infty}$ of pointed topological spaces with pointed maps $\Sigma Z_k \to Z_{k+1}$.

Example 3.2. Given a pointed topological space we can consider its suspension spectrum $\Sigma^\infty X$ which is defined as

$$(\Sigma^\infty X)_k := \Sigma^k X,$$

and structure maps $\Sigma\Sigma^k X \xrightarrow{\simeq} \Sigma^{k+1} X$.

Example 3.3. The *sphere spectrum* $\mathbb{S}$ is defined as $\mathbb{S}_k := S^k$ with structure maps $\Sigma S^k \xrightarrow{\simeq} S^{k+1}$.

Example 3.4. If A is an Abelian group, the *Eilenberg–MacLane spectrum* HA is defined as $(HA)_k := K(A,k)$ which is the k-th Eilenberg–MacLane space of A. The structure maps $\Sigma K(a,k) \to K(A,k+1)$ are obtained from the homotopy equivalences $K(A,k) \xrightarrow{\simeq} \Omega K(A,k+1)$ via the adjunction $[\Sigma X, Y] \simeq [X, \Omega Y]$.

Remark 3.5. We work in a certain (hard to construct) category of spectra called "EKMM spectra" or "$\mathbb{S}$-modules." It is defined in [EKMM97]. The category of EKMM spectra is equipped with a symmetric monoidal smash product, and we are not going to discuss the details of this construction.

Definition 3.6. If Z is a spectrum, we define its *homotopy groups* by

$$\pi_n Z := \operatorname*{colim}_{k\to\infty} \pi_{n+k} Z_k.$$

Definition 3.7. A *ring spectrum* R is a spectrum R equipped with maps $\mu\colon R \wedge R \to R$ and $\eta\colon \mathbb{S} \to R$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathbb{S} \wedge R & \xrightarrow{\eta\times\mathrm{id}} & R \wedge R & \xleftarrow{\mathrm{id}\times\eta} & R \wedge \mathbb{S} \\ & \searrow^{\simeq} & \downarrow{\scriptstyle\mu} & {}^{\simeq}\swarrow & \\ & & R & & \end{array}$$

Remark 3.8. For every ring spectrum R there is a unique ring map $\mathbb{S} \to R$. This makes $\mathbb{S}$ the initial object in the category of ring spectra.

Remark 3.9. If E is a spectrum, and X is a space, then

$$X \longmapsto \pi_n(E \wedge X) =; H_n(X; E),$$

is a generalized homology theory, and

$$X \longmapsto [\Sigma^{-n}X, E] =: H^n(X; E)$$

is a generalized cohomology theory. (If E is a ring spectrum it follows that the cohomology theory $H^n(-;E)$ is multiplicative.)

3.2. Orientations.

Definition 3.10. Let R be a spectrum. The *space of units* of R is defined as the following homotopy limit

$$GL_1(R) := \operatorname{holim} \left(\begin{array}{ccc} & & \Omega^\infty R \\ & & \downarrow \\ (\pi_0 R)^\times & \longrightarrow & \pi_0 R \end{array} \right),$$

where the vertical map is the composition $\Omega^\infty R \to \pi_0 \Omega^\infty R \xrightarrow{\cong} \pi_0 R$. It is always true that $GL_1(R)$ is an infinite loop space, so that we can always deloop (as many times as we wish.

Example 3.11. The space of units of the Eilenberg–MacLane spectrum Hk for some discrete ring is $k^\times$, i.e., $GL_1(Hk) \simeq k^\times$.

We denote by $G := GL_1(\mathbb{S})$, and this space is a classical object in homotopy theory. If $G(n)$ denotes the set of homotopy equivalences $S^n \to S^n$ that fix the point ∞, then we have $G = \operatorname{colim}_n G(n)$, and for any $n \in \mathbb{Z}_{\geq 0}$ there is a map

$$O(n) \longrightarrow G(n),$$

where $O(n)$ denotes the orthogonal group. It is defined by extending an orthogonal linear transformation of $\mathbb{R}^n$ to a homeomorphism of its one-point compactification, fixing ∞. By passing to the colimit as $n \to \infty$ we obtain the so-called J-homomorphism $J \colon O \to G$, and it induces a map between the deloopings: $BJ \colon BO \to BG$. Recall that a (stable) vector bundle on a space X is classified by a map

$$X \longrightarrow BO.$$

Now, maps $X \to BG$ turn out to classify "stable spherical fibrations" over X. Associated to any rank n vector bundle $\xi \colon X \to BO$ there is an associated stable spherical fibration $\xi_{\mathbb{S}} = BJ \circ \xi \colon X \to BO \to BG$, which is the principal S^n-fibration obtained from ξ via fiberwise one-point compactification.

Definition 3.12. A vector bundle $X \to BO$ is said to be $\mathbb{S}$-*orientable* if and only if $X \to BO \to BG$ is null-homotopic (i.e., homotopic to the constant map at the basepoints).

Remark 3.13. The definition of R-orientability of a vector bundle is similar: Since $\mathbb{S}$ is the initial ring spectrum, the unit map $\mathbb{S} \to R$ induces a map $BG \to BGL_1(R)$, and a vector being R-orientable is equivalent (by definition) to the following composition being null-homotopic:

$$X \longrightarrow BO \longrightarrow BG \longrightarrow BGL_1(R).$$

Definition 3.14 (R-line bundle)**.** An *R-line bundle* is a homotopy class of a map $X \to BGL_1(R)$.

Definition 3.15. Given two R-line bundles $f \colon X \to BGL_1(R)$ and $g \colon Y \to BGL_1(R)$, we define their tensor product $f \otimes_R g$ as the product map

$$f \otimes_R g \colon X \wedge Y \xrightarrow{f \wedge g} BGL_1(R) \wedge BGL_1(R) \longrightarrow BGL_1(R)$$

where the second map is the ring structure on $BGL_1(R)$.

Example 3.16. Classical orientability of vector bundles is equivalent to $H\mathbb{Z}$-orientability. Namely, by our definition above, $H\mathbb{Z}$-orientability of a vector bundle is equivalent to the following map being null-homotopic:

$$X \xrightarrow{\xi} BO \longrightarrow BG \longrightarrow BGL_1(H\mathbb{Z}) \cong K(\mathbb{Z}/2, 1) \cong \mathbb{R}\mathrm{P}^\infty.$$

Now, it turns out that the latter part of this composition $BO \to \mathbb{R}\mathrm{P}^\infty$ is representing the (universal) first Stiefel–Whitney class w_1, so that the composition $X \to \mathbb{R}\mathrm{P}^\infty$ represents $w_1(\xi)$, the vanishing of which is equivalent to classical orientability.

Example 3.17. If $\xi\colon X \to BO$ is an $\mathbb{S}$-orientable vector bundle, then it is equivalent (by definition) to its associated stable spherical fibration being trivial. If σ^k is the trivial principal S^k-bundle over X, which means that there exists some $N > 0$ such that

$$\xi_{\mathbb{S}} \wedge \sigma^N \cong \sigma^{k+N},$$

where $\wedge$ denotes fiberwise smash product. This happens in fact if and only if ξ is a stably trivial vector bundle.

Definition 3.18. A *free rank one* $\mathbb{S}$-*module* is defined as a map $S^0 \to BG$.

Remark 3.19.
(1) A free rank one R-module is defined as a map $S^0 \to BGL_1(R)$.
(2) A free rank one $\mathbb{S}$-module is by definition equivalently an $\mathbb{S}$-line bundle over a point.

3.3. **Oriented flow categories and flow bimodules.** For this section fix a commutative ring spectrum R. We are now going to incorporate orientations in our definitions of flow categories. Recall from Example 1.4 that in Morse theory, we have a vector bundle $TW^u(x)$ associated to each critical point, which is trivial because the unstable manifold is contractible, and we also have canonical isomorphisms of vector bundles

$$TW^u(x) \xrightarrow{\cong} T\mathcal{M}(x,y) \oplus \underline{\mathbb{R}} \oplus TW^u(y).$$

We will require a flow category to carry this extra coherent orientation structure. Before getting too abstract it is useful to give the concrete definitions first.

Definition 3.20 (R-oriented flow category)**.** An *R-orientation* on a flow category $\mathcal{M}$ is a choice of a free rank one R-module $\mathfrak{o}(x)$ for every object $x \in \mathrm{ob}(\mathcal{M})$ together with isomorphisms of R-line bundles over $\mathcal{M}(x,y)$

$$\mathfrak{o}(x,y)\colon \mathfrak{o}(x) \xrightarrow{\simeq} (T\mathcal{M}(x,y) \oplus \underline{\mathbb{R}})_R \otimes_R \mathfrak{o}(y),$$

for every $x, y \in \mathrm{ob}(\mathcal{M})$, that are compatible with the composition maps in $\mathcal{M}$, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{o}(x) & \longrightarrow & (T\mathcal{M}(x,y) \oplus \underline{\mathbb{R}})_R \otimes_R \mathfrak{o}(y) \\ \downarrow & & \downarrow \\ (T\mathcal{M}(x,z) \oplus \underline{\mathbb{R}})_R \otimes_R \mathfrak{o}(z) & \longrightarrow & (T\mathcal{M}(x,y) \oplus \underline{\mathbb{R}})_R \otimes_R (T\mathcal{M}(y,z) \oplus \underline{\mathbb{R}})_R \otimes_R \mathfrak{o}(z) \end{array}.$$

Example 3.21. If $f\colon M \to \mathbb{R}$ is a Morse–Smale function on a closed smooth manifold, recall from Example 2.13 that we obtain a flow category $\mathcal{M}_f$. This flow category admits an R-orientation for any commutative ring spectrum R. Namely, let $\mathfrak{o}(x) := (TW^u(x))_R$ for every $x \in \mathrm{ob}(\mathcal{M}_f) = \mathrm{crit}\, f$. Next, we have the following exact sequences in Morse theory

$$0 \longrightarrow T_p\widetilde{\mathcal{M}}_f(x,y) \longrightarrow T_pW^u(x) \longrightarrow N_pW^s(y) \longrightarrow 0$$

and

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow T_p\widetilde{\mathcal{M}}_f(a,b) \longrightarrow T_p\mathcal{M}_f(a,b) \longrightarrow 0$$

for any $p \in \mathcal{M}(x,y)$, where $\widetilde{\mathcal{M}}(x,y)$ is the moduli space of *parametrized* gradient flow lines (i.e. $\mathcal{M}$ is the quotient of $\widetilde{\mathcal{M}}$ with the $\mathbb{R}$-action given by translation). These exact sequences of vector spaces gives canonical isomorphisms

$$(TW^u(x))_R \xrightarrow{\simeq} (T\mathcal{M}_f(x,y) \oplus \underline{\mathbb{R}})_R \otimes_R (TW^u(y))_R,$$

which is precisely the data of an R-orientation on the flow category $\mathcal{M}_f$.

Example 3.22. Let $L, K \subset X$ be two transversely intersecting exact Lagrangians in a (stably polarized) Liouville manifold X. Under certain assumptions (a sufficient assumption is that the "stable Lagrangian Gauss maps" $L \to U/O$ and $K \to U/O$ are null-homotopic; see Theorem 4.5 for a weaker sufficient condition), the flow category $\mathcal{M}_{L,K}$ defined in Example 2.14 admits an R-orientation.

For any object $x \in \mathrm{ob}(\mathcal{M}_{L,K})$, define $\mathcal{V}(x)$ to be the index bundle of the linearized Cauchy–Riemann operator on a space of disks $\mathcal{D}(x)$ with a negative boundary puncture asymptotic to x. So the associated R-line bundle of $\mathcal{V}(x)$ is viewed as a map $\mathcal{V}(x)_R \colon \mathcal{D}(x) \to BGL_1(R)$. A technical point here is that $\mathcal{V}(x)_R$ is R-orientable under the above technical assumptions on L, K and X, which means that the map $\mathcal{V}(x)$ factors as $\mathcal{D}(x) \to S^0 \to BGL_1(R)$. The latter map is by definition a free rank one R-module, which we also denote by $\mathcal{V}(x)_R$. We define

$$\mathfrak{o}(x) := (-\mathcal{V}(x))_R \tag{3.1}$$

(The reason we invert this vector bundle before taking its associated R-line bundle is technical.)

Next it is a classical theorem going back to at least Floer–Hofer [FH93] that the index bundles respect gluing in the sense that there are canonical isomorphisms of R-line bundles over $\mathcal{M}_{L,K}(x, y)$

$$\mathcal{V}(y)_R \xrightarrow{\simeq} (T\mathcal{M}_{L,K}(x, y) \oplus \underline{\mathbb{R}})_R \otimes_R \mathcal{V}(x)_R.$$

that are associative. Therefore, since we tacitly assume that $\mathcal{M}_{L,K}(x, y)$ are transversely cut out, definition (3.1), we have

$$\mathfrak{o}(x) \xrightarrow{\simeq} (T\mathcal{M}_{L,K}(x, y) \oplus \underline{\mathbb{R}})_R \otimes_R \mathfrak{o}(y),$$

which means that $\mathcal{M}_{L,K}$ is R-oriented. (In fact, by our assumptions, the above discussion goes through for any R and in particular the sphere spectrum $\mathbb{S}$; the assumption that the stable Lagrangian Gauss maps of L and K are null-homotopic is quite restrictive.)

Definition 3.23 (R-oriented flow bimodule)**.** If $(\mathcal{M}_1, \mathfrak{o}_1)$ and $(\mathcal{M}_2, \mathfrak{o}_2)$ are two R-oriented flow categories, and $\mathcal{N} \colon \mathcal{M}_1 \to \mathcal{M}_2$ is a flow bimodule, then we say that $\mathcal{N}$ *is R-oriented* if there are canonical isomorphisms of R-line bundles over $\mathcal{N}(x, y)$ for every $(x, y) \in \mathrm{ob}(\mathcal{M}_1) \times \mathrm{ob}(\mathcal{M}_2)$

$$\mathfrak{o}_1(x) \xrightarrow{\simeq} (T\mathcal{N}(x, y))_R \otimes_R \mathfrak{o}_2(y).$$

Definition 3.24 (R-oriented flow bordism)**.** Let $(\mathcal{M}_1, \mathfrak{o}_1)$ and $(\mathcal{M}_2, \mathfrak{o}_2)$ be two R-oriented flow categories and let $\mathcal{N}_1, \mathcal{N}_2 \colon \mathcal{M}_1 \to \mathcal{M}_2$ be two R-oriented flow bimodules. If $\mathcal{B} \colon \mathcal{N}_1 \Rightarrow \mathcal{N}_2$ is a flow bordism, we say that $\mathcal{B}$ *is R-oriented* if there are canonical isomorphisms of R-line bundles over $\mathcal{B}(x, y)$ for every $(x, y) \in \mathrm{ob}(\mathcal{M}_1) \times \mathrm{ob}(\mathcal{M}_2)$

$$\mathfrak{o}_1(x) \xrightarrow{\simeq} (T\mathcal{B}(x, y) \ominus \underline{\mathbb{R}})_R \otimes_R \mathfrak{o}_2(y).$$

3.3.1. *Alternative definition.* With the concrete definition, and examples out of the way, we will give a more abstract definition that is more closely related to the definition given in [AB24].

Recall the definition of the category $\mathbf{Man}_\partial$ from Definition 2.5. We upgrade this to a bicategory as follows. Recall that R still denotes a commutative ring spectrum.

Definition 3.25. Define $\mathbf{Man}_\partial^R$ to be the bicategory given by the following data:

- The set of objects consists of free rank one R-modules $\mathfrak{o}$.
- 1-morphisms from $\mathfrak{o}_1$ to $\mathfrak{o}_2$ consists of an object $X \in \mathbf{Man}_\partial$ together with an isomorphism of R-line bundles
$$\mathfrak{o}_X \colon \mathfrak{o}_1 \xrightarrow{\simeq} (TX)_R \otimes_R \mathfrak{o}_2.$$
- 2-morphisms from X to X' consists of a morphism $\varphi \colon X \to X'$ in $\mathbf{Man}_\partial$ together with an isomorphism $(TX)_R \cong (TX')_R$ of associated R-line bundles of their tangent bundles, such

that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{o}_1 & \xrightarrow{\mathfrak{o}_X} & (TX)_R \otimes_R \mathfrak{o}_2 \\ & \searrow^{\mathfrak{o}_{X'}} & \simeq \downarrow (T\varphi)_R \otimes_R \mathrm{id} \\ & & (TX')_R \otimes_R \mathfrak{o}_2 \end{array} .$$

- The horizontal composition is defined on objects (meaning on 1-morphisms) as follows:

$$\begin{aligned} \mathbf{Man}_\partial^R(\mathfrak{o}_1, \mathfrak{o}_2) \times \mathbf{Man}_\partial^R(\mathfrak{o}_2, \mathfrak{o}_3) &\longrightarrow \mathbf{Man}_\partial^R(\mathfrak{o}_1, \mathfrak{o}_3) \\ ((X_{12}, \mathfrak{o}_{12}), (X_{23}, \mathfrak{o}_{23})) &\longmapsto (X_{12} \times X_{23}, (\mathrm{id} \otimes_R \mathfrak{o}_{23}) \circ \mathfrak{o}_{12}), \end{aligned}$$

Remark 3.26. The reason $\mathbf{Man}_\partial^R$ is only a bicategory as opposed to a strict 2-category is simple. Namely, the associativity of the horizontal composition is only up to natural isomorphism. Even when ignoring the isomorphisms of R-line bundles, the smooth manifolds with corners $(X_{12} \times X_{23}) \times X_{34}$ and $X_{12} \times (X_{23} \times X_{34})$ are not *equal* (indeed, their underlying sets are not equal!), but only naturally isomorphic.

Now, we may upgrade the definitions of R-oriented flow category, R-oriented flow bimodule and R-oriented flow bordism by using $\mathbf{Man}_\partial^R$ instead of $\mathbf{Man}_\partial$; we omit the details in these notes.

3.4. **Cohen–Jones–Segal geometric realization.** Given an R-oriented flow category, there is a procedure known as *Cohen–Jones–Segal (CJS) geometric realization*, which gives an R-module spectrum. We will not discuss this process in detail, but will give an overview sketch of the idea of the construction. More can be found in the original work of Cohen–Jones–Segal [CJS95], see also [ADP24, Section 3].

An R-orientation on a flow category $\mathcal{M}$ consists of associative maps

$$\mathfrak{o}(x) \xrightarrow{\simeq} I_R(x, y) \otimes_R \mathfrak{o}(y), \tag{3.2}$$

where $I(x, y) := T\mathcal{M}(x, y) \oplus \underline{\mathbb{R}}$ and $I_R(x, y) := I(x, y)_R$. Recall that $\mathfrak{o}(x), \mathfrak{o}(y) \colon S^0 \to BGL_1(R)$ are R-line bundles. As such we may consider the Thom spectra $(S^0)^{\mathfrak{o}(x)}$ and $(S^0)^{\mathfrak{o}(y)}$, that we by abuse of notation still denote by $\mathfrak{o}(x)$ and $\mathfrak{o}(y)$, respectively. From (3.2) we get isomorphisms of R-modules

$$\mathfrak{o}(x) \wedge \mathcal{M}^{-I}(x, y) \xrightarrow{\simeq} \mathfrak{o}(y), \tag{3.3}$$

where $\mathcal{M}^{-I}(x, y)$ denotes the suspension spectrum of the Thom space $(\mathcal{M}(x, y))^{-I(x,y)}$. The rough idea is to construct a CW spectrum by considering cells (of dimension m)

$$\bigvee_{\substack{x \in \mathrm{ob}(\mathcal{M}) \\ |x| = m}} \mathfrak{o}(x),$$

and use (3.3) as "gluing maps," via the Pontryagin–Thom construction.

4. The spectral wrapped Donaldson–Fukaya category

Recall from Example 2.14 that we already defined an unoriented flow category associated to two transversely intersecting closed exact Lagrangian submanifolds in a Liouville manifold. We now discuss details on how to orient this flow category, expanding the discussion from Example 3.22.

4.1. **Stable polarizations and the Lagrangian Gauss map.** First recall that the tangent bundle of any symplectic manifold X is canonically almost complex, and we may therefore represent it by a map $X \to BU$. There is a map $c\colon BO \to BU$ called complexification that is understood as follows: If $\xi\colon X \to BO$ is a vector bundle, then $c \circ \xi\colon X \to BU$ is the vector bundle $\xi \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 4.1 (Stable polarization)**.** A *stable polarization* of a symplectic manifold X is a lift of its tangent bundle $X \to BU$ through the complexification map $c\colon BO \to BU$.

Remark 4.2.
(1) In more concrete terms, the existence of a stable polarization on X is the requirement that $TX \oplus \mathbb{C}^N$ is the complexification of a real vector bundle on X for some $N > 0$.
(2) Since we have a fiber sequence $BO \xrightarrow{c} BU \to B(U/O)$, a lift of $X \to BO$ through c is equivalent to the composition $X \to BU \to B(U/O)$ being null-homotopic. This assertion is equivalent to the existence of a Lagrangian subbundle of the vector bundle $TX \oplus \mathbb{C}^N$ for some $N > 0$, i.e., a global Lagrangian distribution of $TX \oplus \mathbb{C}^N$.

From now on we assume that any symplectic manifold that appears is equipped with a choice of stable polarization, namely let $TX \oplus \mathbb{C}^N \cong (TX)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

Given a Lagrangian submanifold $L \subset X$ in a symplectic manifold, the "Lagrangian Gauss map" is a measure of the "difference" between the subbundle $TL \oplus \mathbb{R}^N \subset TX \oplus \mathbb{C}^N$ and the global Lagrangian distribution of $TX \oplus \mathbb{C}^N$ determined by the stable polarization. We then have the two maps $TL\colon L \to BO$ and $(TX)_{\mathbb{R}}|_L\colon L \to BO$ that satisfies

$$c \circ TL = c \circ (TX)_{\mathbb{R}}|_L = TX|_L\colon L \longrightarrow BU \tag{4.1}$$

which determines a map

$$\mathcal{G}_L\colon L \to U/O,$$

since we have the fiber sequence

$$U/O \longrightarrow BO \xrightarrow{c} BU.$$

This lift is called the *stable Lagrangian Gauss map*. More precisely, it is constructed as follows: The two equal (and thus homotopic) maps in (4.1) are used to produce a null-homotopic map $c \circ (TL - (TX)_{\mathbb{R}}|_L)\colon L \to BU$ where $TL - (TX)_{\mathbb{R}}|_L\colon L \to BO$ is the map classifying the virtual vector bundle $TL - (TX)_{\mathbb{R}}|_L$. Since $c \circ (TL - (TX)_{\mathbb{R}}|_L)$ is null-homotopic, the map $TL - (TX)_{\mathbb{R}}|_L$ admits a lift to the homotopy fiber of $c\colon BO \to BU$, which is U/O.

Bott periodicity gives a homotopy equivalence $\Omega(U/O) \simeq \mathbb{Z} \times BO$ which yields a map $\Omega(U/O) \to BO$ via the projection. This map admits a delooping $U/O \to B^2O$.

Definition 4.3. Define $(U/O)^{\#}$ to be the homotopy fiber of the following composition

$$U/O \longrightarrow B^2O \xrightarrow{B^2J} B^2G \longrightarrow B^2GL_1(R).$$

Definition 4.4 (Lagrangian R-brane)**.** Let $L \subset X$ be a Lagrangian submanifold. A *Lagrangian R-brane structure* is a lift of the stable Lagrangian Gauss map $L \to U/O$ to a map $L \to (U/O)^{\#}$. In other words, a choice of null-homotopy of the following composition

$$L \xrightarrow{\mathcal{G}_L} U/O \longrightarrow B^2O \xrightarrow{B^2J} B^2G \longrightarrow B^2GL_1(R).$$

Theorem 4.5. *Let X be a stably polarized Liouville manifold, and let $L, K \subset X$ two exact Lagrangian R-brane. The flow category $\mathcal{M}_{L,K}$ is R-orientable.*

Proof. Briefly, the idea is the same as the classical proof involving classical orientation. The key is that the assumption L and K carry R-brane structures yields that the index bundle of the linearized Cauchy–Riemann operator on a space of caps (disks with one negative boundary puncture) is R-orientable. □

Proposition 4.6 ([ADP24, Lemma 5.17])**.** *A choice of $H\mathbb{Z}$-brane structure on $L \subset X$ is equivalent to a relative pin structure on L, relative to the background class $w_2((TX)_{\mathbb{R}}) \in H^2(X;\mathbb{Z}/2)$.* □

Definition 4.7. Let $L, K \subset X$ be two Lagrangian R-branes in a stably polarized Liouville manifold. Define

$$HW(L,K;R) := |\mathcal{M}_{L,K}, \mathfrak{o}|,$$

where $\mathfrak{o}$ is the R-orientation defined in Example 3.22.

Proposition 4.8. *Given two Lagrangian R-branes $L, K \subset X$ in a stably polarized Liouville manifold, the homotopy groups of $HW(L,K;H\mathbb{Z})$ are given by the wrapped Floer cohomology (with integer coefficients) with reversed grading:*

$$\pi_*(HW(L,K;H\mathbb{Z})) \cong HW^{-*}(L,K;\mathbb{Z}).$$

□

Remark 4.9. The above also holds true for Hk, where k is any commutative ring.

4.2. **Flow multimodules.** Before defining the wrapped Donaldson–Fukaya category, we digress and give a sketch of the definition of *flow multimodules.* Given flow categories $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_k$ a flow multimodule

$$\mathcal{N}\colon \mathcal{M}_1, \ldots, \mathcal{M}_k \longrightarrow \mathcal{M}_0$$

is an assignment $(a_1, \ldots, a_k; a_0) \mapsto \mathcal{N}(a_1, \ldots, a_k; a_0)$ of an object in $\mathbf{Man}_\partial$ that enjoys a similar decomposition of the union of its codimension 1 corner strata to that of a flow bimodule Definition 2.15. Namely, there are structure maps

$$\mathcal{M}_i(a_i; a_i') \times \mathcal{N}(a_1, \ldots, a_i', \ldots, a_k; a_0) \longrightarrow \mathcal{N}(a_1, \ldots, a_k; a_0),\ i \in \{1, \ldots, k\}$$
$$\mathcal{N}(a_1, \ldots, a_k; a_0') \times \mathcal{M}_0(a_0', a_0) \longrightarrow \mathcal{N}(a_1, \ldots, a_k; a_0)$$

that are suitably associative and commutative. An R-orientation on a flow multimodule consists of isomorphisms of R-line bundles over $\mathcal{N}(a_1, \ldots, a_k; a_0)$

$$\mathfrak{o}_1(a_1) \otimes_R \cdots \otimes_R \mathfrak{o}_k(a_k) \xrightarrow{\simeq} (T\mathcal{N}(a_1, \ldots, a_k; a_0))_R \otimes_R {}_0(a_0)$$

for all tuples of objects $(a_0, \ldots, a_k) \in \prod_{i=0}^k \mathrm{ob}(\mathcal{M}_i)$, that satisfy suitable compatibilities with respect to the bimodule structure on $\mathcal{N}$. The CJS realization is furthermore functorial with respect to flow multimodules in the sense that it yields a map of R-modules

$$|\mathcal{N}, \mathfrak{m}|\colon |\mathcal{M}_1, \mathfrak{o}_1| \wedge_R \cdots \wedge_R |\mathcal{M}_k, \mathfrak{o}_k| \longrightarrow |\mathcal{M}_0, \mathfrak{o}_0|.$$

4.3. **The wrapped Donaldson–Fukaya category.** Given three Lagrangian R-branes $L_0, L_1, L_2 \subset X$ in a stably polarized Liouville sector, we define a flow multimodule

$$\mathcal{N}_\mu\colon \mathcal{M}_{L_0,L_1}, \mathcal{M}_{L_1,L_2} \longrightarrow \mathcal{M}_{L_0,L_2}$$

that is defined by letting $\mathcal{N}_\mu(a_1, a_2; a_0)$ be the moduli space of pseudoholomorphic triangles, responsible for defining the product in wrapped Floer cohomology. By the general gluing results in Floer theory, we equip $\mathcal{N}_\mu$ with an R-orientation. Its CJS realization yields a map

$$\mu^2\colon HW(L_0, L_1; R) \wedge_R HW(L_1, L_2; R) \longrightarrow HW(L_0, L_2; R),$$

Proposition 4.10. *The map μ^2 of R-modules is homotopy associative. I.e., the following diagram is commutative up to homotopy:*

$$\begin{array}{ccc} HW(L_0,L_1;R) \wedge_R HW(L_1,L_2;R) \wedge_R HW(L_2,L_3;R) & \xrightarrow{\mu^2 \wedge_R \mathrm{id}} & HW(L_0,L_2;R) \wedge_R HW(L_2,L_3;R) \\ \big\downarrow \mathrm{id} \wedge_R \mu^2 & & \big\downarrow \mu^2 \\ HW(L_0,L_1;R) \wedge_R HW(L_1,L_3;R) & \xrightarrow{\mu^2} & HW(L_0,L_3;R) \end{array}.$$

Proof. The homotopy between the two compositions is defined by a flow bordism between two flow multimodules $\mathcal{M}_{L_0,L_1}, \mathcal{M}_{L_1,L_2}, \mathcal{M}_{L_2,L_3} \longrightarrow \mathcal{M}_{L_0,L_3}$, which is defined by moduli spaces of pseudo-holomorphic pairs of pants with three inputs. (These moduli spaces define a flow bordism and *not* a flow multimodule as there are boundary strata of this moduli space that looks like two pairs of pants composed in two different ways, which is precisely the two compositions in the diagram.) □

Remark 4.11. The unit in the wrapped Fukaya category obtained by counting pseudoholomorphic disks with one negative boundary puncture may be used to define the unit in $HW(L, L; R)$, too.

Definition 4.12. The wrapped Donaldson–Fukaya category with coefficients in R of X is the category $\mathcal{W}(X; R)$ enriched over the homotopy category of R-modules whose objects are given by the set of Lagrangian R-branes in X, the morphisms from L_0 to L_1 are given by

$$\mathcal{W}(X; R)(L_0, L_1) := HW(L_0, L_1; R),$$

and the composition maps are given by μ^2.

Remark 4.13. This category is enriched in the homotopy category of R-modules, since the product is only homotopy associative.

The wrapped Donaldson–Fukaya category enjoys the following properties. A similar version was also recently proved by Porcelli–Smith [PS24].

Theorem 4.14 (A.–Deshmukh–Pieloch)**.** *Let $L \cong K$ in $\mathcal{W}(X; R)$.*

(1) There is an isomorphism of R-modules $\Sigma^\infty L \wedge R \simeq \Sigma^\infty K \wedge R$.
(2) If L is R-oriented, then K is R-orientable, and there exists an R-orientation on K such that $[L]_R = [K]_R \in H_n(X; R)$.

5. Spectral equivalence of nearby Lagrangians

5.1. **Floer homotopy type of the cotangent fiber.** In this section we focus on $X = T^*Q$ for a closed manifold Q. By classical work of Milnor [Mil63], we use a Morse theoretic description of the based loop space ΩQ to define a flow category whose CJS realization is $\Sigma^\infty \Omega Q \wedge R$.

Namely, there is a homotopy equivalent space $\Omega Q \simeq BQ$ consisting of piecewise geodesic loops in Q. This space admits an increasing filtration

$$BQ(1) \subset BQ(2) \subset \cdots \subset BQ(n) \subset \cdots,$$

such that $BQ = \operatorname{colim}_n BQ(n)$, and each inclusion $BQ(n) \subset BQ(n+1)$ is an inclusion as a submanifold with boundary. Moreover, each $BQ(n)$ is a finite dimensional smooth manifold with corners and for a generic choice of Riemannian metric on Q, the energy functional $E(\gamma) := \int_0^1 |\dot\gamma(t)|^2 dt$ is a Morse–Smale function. For each $n \in \mathbb{Z}_{\geq 1}$ we therefore obtain a Morse–Smale flow category $\mathcal{M}_{BQ(n),E|_{BQ(1)}}$, and this produces a sequence of flow categories

$$\mathcal{M}_{BQ(1),E|_{BQ(1)}} \subset \mathcal{M}_{BQ(2),E|_{BQ(2)}} \subset \cdots \subset \mathcal{M}_{BQ(n),E|_{BQ(n)}} \subset \cdots,$$

where each inclusion is an inclusion as a full subcategory. We define $\mathcal{M}_{\Omega Q}$ to be the flow category with

- $\operatorname{ob}(\mathcal{M}_{\Omega Q}) = \bigcup_{n=1}^\infty \operatorname{ob}(\mathcal{M}_{BQ(n),E|_{BQ(n)}})$
- The grading is given by the Morse index.
- $\mathcal{M}_{\Omega Q}(x, y)$ is defined as $\mathcal{M}_{BQ(n),E|_{BQ(n)}}(x, y)$ for some large enough n.
- The R-orientation
$$\mathfrak{o}(a) \xrightarrow{\simeq} (T\mathcal{M}_{\Omega Q} \oplus \mathbb{R})_R \otimes_R \mathfrak{o}(b),$$
is given by the R-orientation on $\mathcal{M}_{BQ(n),E|_{BQ(n)}}$ for some large enough n.

Proposition 5.1. *The CJS realization of $(\mathcal{M}_{\Omega Q}, \mathfrak{o})$ is equal to $\Sigma^\infty \Omega Q \wedge R$.* □

The Pontryagin product $P\colon \Omega Q \times \Omega Q \to \Omega Q$ given by concatenation of loops, induces a flow multimodule

$$\mathcal{P}\colon \mathcal{M}_{\Omega Q}, \mathcal{M}_{\Omega Q} \longrightarrow \mathcal{M}_{\Omega Q}$$

by

$$\mathcal{P}(\gamma_1, \gamma_2; \gamma) := (\overline{W}^u_{-\nabla E}(\gamma_1) \times \overline{W}^u_{-\nabla E}(\gamma_2)) \times_P \overline{W}^s_{-\nabla E}(\gamma),$$

there is a natural "intersection orientation" that we equip this flow multimodule with; the CJS realization of $\mathcal{P}$ coincides with the Pontryagin product on $\Sigma^\infty \Omega Q \wedge R$ up to homotopy.

Next, we consider a cotangent fiber $F = T^*_\xi Q \subset T^*Q$ at the basepoint $\xi \in Q$ that we implicitly used for our based loop space above. Let $a \in \mathrm{ob}(\mathcal{M}_{F,F})$ be a Hamiltonian chord. We pick some Hamiltonian that vanishes near the zero section $Q \subset T^*Q$. Since $F \cap Q = \{\xi\}$, there is a constant Hamiltonian chord at ξ, which we also denote by ξ. Considering the moduli space $\mathcal{N}_\mu(a, \xi; \xi)$ defined in Section 4.3, we have an evaluation map

$$\mathrm{ev}\colon \mathcal{N}_\mu(a, \xi; \xi) \longrightarrow \Omega Q,$$

by restricting a map u to the boundary component along which u maps to Q. We then define a flow bimodule

$$\mathcal{N}\colon \mathcal{M}_{F,F} \longrightarrow \mathcal{M}_{\Omega Q},$$

via the assignment

$$\mathcal{N}(a; \gamma) := \mathcal{N}_\mu(a, \xi; \xi) \times_{\mathrm{ev}} \overline{W}^s_{-\nabla E}(\gamma).$$

This flow bimodule is equipped with the "intersection R-orientation."

Lemma 5.2. *The following diagram is homotopy commutative*

$$\begin{array}{ccc} HW(F,F;R) \wedge_R HW(F,F;R) & \xrightarrow{|\mathcal{N}| \wedge_R |\mathcal{N}|} & (\Sigma^\infty \Omega Q \wedge R) \wedge_R (\Sigma^\infty \Omega Q \wedge R) \\ \big\downarrow \mu^2 & & \big\downarrow |\mathcal{P}| \\ HW(F,F;R) & \xrightarrow{\quad |\mathcal{N}| \quad} & \Sigma^\infty \Omega Q \wedge R \end{array}.$$

Proof. This proof is the flow categorical upgrade of the classical proof that the triangle product (and indeed the entire A_∞-structure if it had been defined) in $HW(F,F)$ is transferred to the Pontryagin product on the based loop space, see [Abo12]. ☐

Remark 5.3. We now comment on a technical point regarding the homotopy commutativity which now becomes important. The suspension spectrum $\Sigma^\infty \Omega Q \wedge R$ is an R-module and a ring spectrum, so it is an R-algebra. A *ring spectrum* in our sense is the same as an "A_∞-ring spectrum" in the classical sense. This means that the ring structure is associative up to *coherent* homotopy. This is stronger than the ring structure being associative up to homotopy. (This is the distinction between an A_2-algebra and an A_∞-algebra in algebra or between an H-space and an A_∞-space in topology.)

Next, we have that $HW(F,F;R)$ is a *homotopy R-algebra*, because we have only proved that μ^2 defines a homotopy ring structure. Moreover Lemma 5.2 shows that $|\mathcal{N}|$ defines a map of homotopy R-algebras.

We now assume that R is connective, meaning $\pi_k R = 0$ for $k < 0$. For connective spectra there is a version of Whitehead's theorem that holds true: If M and N are connective R-modules, and $\varphi\colon M \to N$ is a map of R-modules that induces and equivalence $\varphi_*\colon \pi_*(M \wedge_R Hk) \xrightarrow{\cong} \pi_*(N \wedge_R Hk)$ where $k := \pi_0 R$, then φ is an equivalence of R-modules.

Lemma 5.4. *The CJS realization*

$$|\mathcal{N}|\colon HW(F,F;R) \longrightarrow \Sigma^\infty \Omega Q \wedge R$$

defines an equivalence of homotopy R-algebras.

Proof. It is clear that $\Sigma^\infty \Omega Q \wedge R$ is connective. Moreover, there is a grading preserving bijection $\mathrm{ob}(HW(F,F;R)) \cong \mathrm{ob}(\mathcal{M}_{\Omega Q})$ which implies that $HW(F,F;R)$ must be connective. It follows from Proposition 4.8 that $\pi_*(HW(F,F;R) \wedge_R Hk) \cong HW^{-*}(F,F;k)$, and we know from [Abo12, Theorem 1.1] that this is equivalent to $\pi_*((\Sigma^\infty \Omega Q \wedge R) \wedge_R Hk) \cong H_*(\Omega Q; k)$. This equivalence is recovered from $|\mathcal{N}|$, which by Whitehead means that $|\mathcal{N}|$ is an equivalence of R-modules. Since Lemma 5.2 shows that μ^2 and the Pontryagin product are intertwined up to homotopy, $|\mathcal{N}|$ is an equivalence of homotopy R-algebras. □

Now, we equip $HW(F,F;R)$ with the ring structure pulled back from $\Sigma^\infty \Omega Q \wedge R$ via the equivalence of R-modules $|\mathcal{N}|$, which automatically upgrades $|\mathcal{N}|$ to an equivalence of R-algebras, with respect to this ring structure.

5.2. **Spectral equivalence of nearby Lagrangians.** Let us first show that an R-brane structure on a Lagrangian is equivalent to a "rank 1 local system." First recall that given a null-homotopy H of a (homotopy class of a) map $f\colon X \to Y$ between two spaces, any other null-homotopy $g\colon X \to Y$ is equivalent to a homotopy class of a map $X \to \Omega Y$.

Lemma 5.5. *An R-brane structure on Q is equivalent to a $(\Sigma^\infty \Omega L \wedge R)$-module structure on R.*

Proof sketch. By definition an R-brane structure on L is equivalent to a null-homotopic map $L \to B^2 GL_1(R)$. Given such, any other R-brane structure is equivalent to a map $L \to BGL_1(R)$ which in turn is equivalent to a ring map $\Omega L \to GL_1(R)$. Such a map defines a $(\Sigma^\infty \Omega L \wedge R)$-module structure on R by the following composition

$$(\Sigma^\infty \Omega L \wedge R) \wedge_R R \simeq \Sigma^\infty \Omega L \wedge R \longrightarrow \Sigma^\infty GL_1(R) \wedge R \longrightarrow \Sigma^\infty \Omega^\infty R \wedge R \longrightarrow R \wedge R \longrightarrow R.$$

□

Remark 5.6. For $R = H\mathbb{Z}$ this lemma says that a relative pin structure on L (relative to the background class that is the second Stiefel–Whitney class of the stable polarization of the Liouville manifold) corresponds to a rank 1 local system on L, meaning a ring map $\Omega L \to GL_1(\mathbb{Z}) \cong \mathbb{Z}^\times = \mathbb{Z}/2$.

We now restate our main theorem.

Theorem 5.7 (A.–Deshmukh–Pieloch [ADP24])**.** *Let R be a commutative ring spectrum. Let Q be a closed manifold and $L \subset T^*Q$ a nearby Lagrangian R-brane. There exists an R-brane structure on Q such that $L \cong Q$ in $\mathcal{W}(X;R)$.*

Outline of proof.

(1) Let $\mathcal{F}(X;R) \subset \mathcal{W}(X;R)$ denote the full subcategory consisting of closed Lagrangian R-branes. Let $k := \pi_0 R$. Using Whiteheads theorem one can show that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(T^*Q;R) & \xrightarrow{\ \mathcal{Y}\ } & (\Sigma^\infty \Omega Q \wedge R)\textbf{-mod} \\ \downarrow & & \downarrow \\ \mathcal{F}(T^*Q;Hk) & \xrightarrow{\mathcal{Y}\wedge_R Hk} & (\Sigma^\infty \Omega Q \wedge Hk)\textbf{-mod} \end{array},$$

with the property that if $\mathcal{Y} \wedge_R Hk$ is fully faithful, so is $\mathcal{Y}$.

(2) The functor $\mathcal{Y} \wedge_R Hk$ is fully faithful, by the classical result that the cotangent fiber generates the wrapped Fukaya category of T^*Q [Abo11]. Therefore $\mathcal{Y}$ is fully faithful. Fully faithful functors reflect isomorphisms, and therefore to prove $L \cong Q$ (for some R-brane structure on Q) in $\mathcal{F}(T^*Q;R)$ it suffices to prove $\mathcal{Y}(L) \simeq \mathcal{Y}(Q)$ in $(\Sigma^\infty \Omega Q \wedge R)$**-mod**. (Of course, $L \cong Q$ in $\mathcal{F}(T^*Q;R)$ implies $L \cong Q$ in $\mathcal{W}(T^*Q;R)$.)

(3) In reality there is a commutative diagram

$$\begin{array}{ccc} & & (\Sigma^\infty \Omega Q \wedge R)\textbf{-mod} \\ & \overset{\mathcal{Y}}{\nearrow} & \big\downarrow{\scriptstyle \text{forget}} \\ \mathcal{F}(T^*Q;R) & \xrightarrow[\mathcal{Y}_F]{} & (\Sigma^\infty \Omega Q \wedge R)\textbf{-homod} \end{array},$$

where $\mathcal{Y}_F$ is defined on objects by $\mathcal{Y}_F(L) := HW(F, L; R)$ and induced by the Hurewicz map on morphisms. This Yoneda functor $\mathcal{Y}_F$ a priori lands in the category of *homotopy modules* over $\Sigma^\infty \Omega Q \wedge R$, since $HW(F, F)$ only has been equipped with a homotopy ring structure μ^2 (as opposed to a highly structured (meaning A_∞) one). The functor $\mathcal{Y}$ is roughly defined on the object L by considering (the CJS realization of) $\mathcal{M}_{Q,L}$ equipped with a "local system" that for each element in $\mathcal{M}_{Q,L}(x, y)$ records the boundary component of the holomorphic strip that is mapped to Q, as an element of the path space of Q from x to y.

For the rest of the proof we assume for simplicity that $\mathcal{Y} = \mathcal{Y}_F$ and that every homotopy module is a highly structured one. This is in order to focus on the high level idea of the proof.

(4) By Whitehead's theorem we obtain $HW(F, L; R) \simeq \Sigma^\ell R$ for some $\ell \in \mathbb{Z}$. This is because from classical Floer theory we have $\pi_*(HW(F, L; R) \wedge_R Hk) \cong HW^{-*}(F, L; k) \cong k[\ell]$ for some $\ell \in \mathbb{Z}$.

(5) Since $F \cap Q = \{\xi\}$ it follows from the definition of the CJS construction that $HW(F, Q; R) \simeq \Sigma^m R$ for some $m \in \mathbb{Z}$; by choosing appropriate grading data we may assume $m = \ell$.

(6) Now, $HW(F, L; R) \simeq \Sigma^\ell R$ is equipped with the $HW(F, F; R) \simeq (\Sigma^\infty \Omega Q \wedge R)$-module structure obtained from the composition μ^2 in the wrapped Donaldson–Fukaya category.

(7) Finally one can show that we can pick some R-brane structure on Q such that the $(\Sigma^\infty \Omega Q \wedge R)$-module structure on $HW(Q, L; R)$ becomes equivalent to the given $(\Sigma^\infty \Omega Q \wedge R)$-module structure on $HW(F, L; R) \simeq \Sigma^\ell R$. (Recall that an R-brane structure on Q corresponds by Lemma 5.5 to a rank one $(\Sigma^\infty \Omega Q \wedge R)$-module.)

□

References

[AB21] Mohammed Abouzaid and Andrew J Blumberg. Arnold conjecture and Morava K-theory. *arXiv:2103.01507*, 2021.

[AB24] Mohammed Abouzaid and Andrew J Blumberg. Foundation of Floer homotopy theory I: Flow categories. *arXiv:2404.03193*, 2024.

[Abo11] Mohammed Abouzaid. A cotangent fibre generates the Fukaya category. *Adv. Math.*, 228(2):894–939, 2011.

[Abo12] Mohammed Abouzaid. On the wrapped Fukaya category and based loops. *J. Symplectic Geom.*, 10(1):27–79, 2012.

[Abo22] Mohammed Abouzaid. An axiomatic approach to virtual chains. *arXiv:2201.02911*, 2022.

[ADP24] Johan Asplund, Yash Deshmukh, and Alex Pieloch. Spectral equivalence of nearby Lagrangians. *arXiv:2411.08841*, 2024.

[CJS95] R. L. Cohen, J. D. S. Jones, and G. B. Segal. Floer's infinite-dimensional Morse theory and homotopy theory. In *The Floer memorial volume*, volume 133 of *Progr. Math.*, pages 297–325. Birkhäuser, Basel, 1995.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[FH93] A. Floer and H. Hofer. Coherent orientations for periodic orbit problems in symplectic geometry. *Math. Z.*, 212(1):13–38, 1993.

[Flo89] Andreas Floer. Witten's complex and infinite-dimensional Morse theory. *J. Differential Geom.*, 30(1):207–221, 1989.

[FOOO24] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Exponential decay estimates and smoothness of the moduli space of pseudoholomorphic curves. *Mem. Amer. Math. Soc.*, 299(1500):v+139, 2024.

[Jar06] J. F. Jardine. Categorical homotopy theory. *Homology Homotopy Appl.*, 8(1):71–144, 2006.

[Jä68] Klaus Jänich. On the classification of $O(n)$-manifolds. *Math. Ann.*, 176:53–76, 1968.

[Lar21] Tim Large. *Spectral Fukaya Categories for Liouville Manifolds*. PhD thesis, Massachusetts Institute of Technology, 2021.

[Lau00] Gerd Laures. On cobordism of manifolds with corners. *Trans. Amer. Math. Soc.*, 352(12):5667–5688, 2000.

[Mil63] J. Milnor. *Morse theory*, volume No. 51 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1963. Based on lecture notes by M. Spivak and R. Wells.

[PS24] Noah Porcelli and Ivan Smith. Bordism of flow modules and exact Lagrangians. *arXiv:2401.11766*, 2024.