The homotopy theory of topological spaces

In the previous talk, we saw the idea of the "homotopy hypothesis," which (roughly) states that

$$\text{"homotopy types of spaces"} \simeq \text{"weak } \infty\text{-groupoids."}$$

Today, we will formulate a 1-categorical version of the homotopy hypothesis, realized as an equivalence of 1-categories

$$\mathrm{Ho}(\mathrm{Top}) \simeq \mathrm{Ho}(\mathrm{sSet}).$$

We will then go over some important constructions from classical homotopy theory, which will hopefully highlight some limitations of working in this 1-categorical setting.

1. The homotopy category

1.1. The homotopy category of topological spaces

A preliminary definition of the homotopy category of spaces might be:

Definition 1.1. Let hTop be the category whose objects are topological spaces and whose morphisms are homotopy classes of continuous maps.

Isomorphisms in hTop are precisely homotopy equivalences. However, it turns out homotopy equivalence is too strong a notion, as hTop contains a lot of pathological spaces.

Example 1.2. The *pseudocircle* is a space consisting of four points which is not homotopy equivalent to any CW complex.

Instead, we want to consider spaces up to weak equivalence. Recall that a map $X \to Y$ is a *weak equivalence* if it induces isomorphisms on all homotopy groups. For most purposes, weak equivalence is suitable because of the following famous result.

Theorem 1.3 (Whitehead)**.** A weak equivalence of CW complexes is a homotopy equivalence.

The problem is that weak equivalence is not an equivalence relation. Thus, to construct a category of spaces up to weak equivalence, we need the following construction.

Definition 1.4. Let $\mathcal{C}$ be a category and W a set of morphisms. The *localization* of $\mathcal{C}$ at W is a category $\mathcal{C}[W^{-1}]$ equipped with a functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ which sends morphisms in W to isomorphisms. Furthermore, for any other functor $\mathcal{C} \to \mathcal{D}$ satisfying this property, there exists a unique functor $\mathcal{C}[W^{-1}] \to \mathcal{D}$ so that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \\ \mathcal{C}[W^{-1}] \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Remark 1.5. The definition of localization given here is easy to state, but it is not quite the correct definition. Saying $Q : \mathcal{C} \to \mathcal{C}[W^{-1}]$ is a localization functor as per the above definition is the same as saying that the pullback map

$$Q^* : \mathrm{Fun}(\mathcal{C}[W^{-1}], D) \to \mathrm{Fun}^W(\mathcal{C}, \mathcal{D})$$

is a bijection, where $\mathrm{Fun}^W(\mathcal{C}, \mathcal{D})$ is the set of functors $\mathcal{C} \to \mathcal{D}$ which send morphisms in W to isomorphisms. But both the domain and codomain of Q^* are naturally thought of as categories, not sets. Thus, the correct definition of localization should require that Q is an *equivalence* of categories, instead of a bijection of sets.

Definition 1.6. Let Ho(Top) be the localization of Top at the set of weak equivalences.

Although this definition is natural, it is not easy to work with directly. Fortunately, there is a much more concrete way to think of Ho(Top).

Definition 1.7. Let hCW be the full subcategory of hTop spanned by spaces which are homotopy equivalent to a CW complex.

Theorem 1.8. The categories Ho(Top) and hCW are equivalent.

The key ingredient needed to prove Theorem 1.8 is the following.

Theorem 1.9 (CW approximation)**.** There exists a functor Γ : hTop $\to$ hTop and a natural transformation $\gamma : \Gamma \Rightarrow \mathrm{id}$ that assigns to each space X a CW complex ΓX and a weak equivalence $\gamma_X : \Gamma X \to X$.

To prove Theorem 1.8, one can show that the composition of functors

$$\mathrm{Top} \to \mathrm{hTop} \xrightarrow{\Gamma} \mathrm{hCW}$$

satisfies the universal property of localization (cf. Remark 1.5).

1.2. The homotopy category of simplicial sets

Definition 1.10. Let $\mathbf{\Delta}$ be the category whose objects are the ordered sets

$$[n] = \{0 < 1 < \cdots < n-1 < n\}$$

and whose morphisms are nondecreasing maps $[m] \to [n]$. A *simplicial set* is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$. A *morphism* of simplicial sets is a natural transformation. The category of simplicial sets is denoted sSet.

Here is a more down-to-earth description of a simplicial set. A simplicial set is a collection of sets $S_\bullet = (S_n)_{n \geq 0}$, where S_n should be thought of as the set of n-simplices, equipped with *face operators*

$$d_i^n : S_n \to S_{n-1}$$

and *degeneracy operators*

$$s_i^n : S_n \to S_{n+1}$$

for $0 \leq i \leq n$.

Example 1.11. Take a (topological) 2-simplex and identify two edges to form a cone. This can be thought of as a simplicial set where

- S_0 consists of two vertices x, y,
- S_1 consists of two edges a, b and a degenerate edge with image x,
- S_2 consists of a simplex U, two degenerate simplices with image a, two degenerate simplices with image b, one degenerate simplex with image x, and one degenerate simplex with image y,

and so on.

Example 1.12. The singular complex $\mathrm{Sing}_\bullet(X)$ of a topological space X is defined to be the simplicial set

$$\mathrm{Sing}_n(X) = \mathrm{Top}(\Delta^n, X),$$

where Δ^n is the topological n-simplex. The collection $\mathrm{Sing}_\bullet(X)$ naturally admits the structure of a simplicial set.

Any simplicial set $S_\bullet$ can be realized as a topological space $|S_\bullet|$, called the *geometric realization* of $S_\bullet$.

Definition 1.13. A morphism $S_\bullet \to T_\bullet$ of simplicial sets is a *weak equivalence* if the induced map $|S_\bullet| \to |T_\bullet|$ is a weak equivalence of topological spaces. Let Ho(sSet) be the localization of sSet at the set of weak equivalences.

1.3. Comparison of the two categories

Observe that the functors defined in the previous section

$$\mathrm{sSet} \underset{\mathrm{Sing}_\bullet}{\overset{|\cdot|}{\rightleftarrows}} \mathrm{Top}$$

form an adjoint pair, i.e., there are natural isomorphisms

$$\mathrm{Top}(|S_\bullet|, X) \cong \mathrm{sSet}(S_\bullet, \mathrm{Sing}_\bullet(X)).$$

Theorem 1.14 (Milnor 1957)**.** For a topological space X, the counit map $|\mathrm{Sing}_\bullet(X)| \to X$ is a weak equivalence of topological spaces.

Corollary 1.15. For a simplicial set $S_\bullet$, the unit map $S_\bullet \to \mathrm{Sing}_\bullet(|S_\bullet|)$ of the above adjunction is a weak equivalence of simplicial sets.

Proof. The composition

$$|S_\bullet| \to |\mathrm{Sing}_\bullet(|S_\bullet|)| \to |S_\bullet|$$

is the identity, and the second map is a weak equivalence by Milnor's theorem. Hence, the first map is a weak equivalence, as required. □

Corollary 1.16. The categories Ho(Top) and Ho(sSet) are equivalent.

2. Some classical homotopy theory

2.1. Fibrations and cofibrations

Definition 2.1. A *Hurewicz fibration* is a map $E \to B$ satisfying the *homotopy lifting property*:

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d] & E \arrow[d] \\ X \times I \arrow[r] \arrow[ur, dashed, "\exists"] & B \end{tikzcd}$$

A *Serre fibration* is a map $E \to B$ satisfying the homotopy lifting property whenever $X = D^n$ (equivalently, when X is a CW complex).

Example 2.2. Every (sufficiently nice) covering space is a fibration; more generally, every (sufficiently nice) fiber bundle is a fibration. For a counterexample, consider the projection $\mathbb{R}^2 - \{0\} \to \mathbb{R}$.

Definition 2.3. A *Hurewicz cofibration* is a map $A \to X$ satisfying the *homotopy* *extension property*:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y^I \arrow[d, "p_0"] \\ X \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

Example 2.4. If A is a subcomplex of a CW complex X, then the inclusion map $A \to X$ is a cofibration. More generally, if a subspace $A \subseteq X$ is the deformation retract of some neighborhood, then $A \to X$ is a cofibration. In fact, under nice conditions (e.g., all spaces are Hausdorff), a cofibration $A \to X$ is always an inclusion of a closed subspace.

Proposition 2.5. Any map $f : X \to Y$ can be factored as the composition of

(i) a map $X \to Z$ which is both a homotopy equivalence and a cofibration and a fibration $Z \to Y$,

(ii) a cofibration $X \to W$ a map $W \to Y$ which is both a homotopy equivalence and a fibration.

Proof. For (i), take Z to be the *mapping path space*

$$E_f = X \times_Y Y^I = \{(x, \gamma) \in X \times Y^I \mid f(x) = \gamma(0)\},$$

and for (ii), take W to be the *mapping cylinder*

$$M_f = (X \times I) \cup_X Y = ((X \times I) \amalg Y)/((x, 1) \sim f(x)). \qquad \square$$

2.2. Homotopy limits and colimits

One of the issues with the category Top is that limits and colimits do not behave nicely with respect to homotopy.

Example 2.6. Consider the morphism of spans:

$$\begin{tikzcd} D^2 \arrow[d] & S^1 \arrow[l] \arrow[r] \arrow[d] & D^2 \arrow[d] \\ * & S^1 \arrow[l] \arrow[r] & * \end{tikzcd}$$

The vertical maps are all homotopy equivalences, but the pushout of the two spans (S^2 for the top span and $*$ for the bottom) are not homotopy equivalent.

Definition 2.7. The *homotopy pushout* of two maps $f : A \to X$ and $g : A \to Y$ is the *double mapping cylinder*

$$M(f, g) = (X \amalg (A \times I) \amalg Y)/{\sim},$$

where $\sim$ identifies $(a,0) \sim f(a)$ and $(a,1) \sim g(a)$.

Proposition 2.8. Given a diagram in Ho(Top)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \arrow[ddr, bend left] & \\
Y \arrow[r] \arrow[drr, bend right] & M(f,g) \arrow[dr, dashed, "\exists"] & \\
& & Z
\end{tikzcd}$$

there exists a (not necessarily unique) arrow $M(f,g) \to Z$ such that the diagram commutes.

Proposition 2.9. If we have a morphism of spans

$$\begin{tikzcd}
X \arrow[d] & A \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\
X' & A' \arrow[l, "f'"] \arrow[r, "g'"'] & Y'
\end{tikzcd}$$

where the vertical maps are weak equivalences, then $M(f,g)$ and $M(f',g')$ are weak equivalent.

Proof. The natural map $M(f,g) \to M(f',g')$ induces an isomorphism on fundamental groups by van Kampen's theorem, as well as an isomorphism on homology groups by the Mayer–Vietoris sequence. By Whitehead's theorem, this implies the map is a weak equivalence. $\square$

Homotopy pushouts are special cases of *homotopy colimits*. We will not go over the general construction of homotopy limits and colimits, but we will give several more examples.

Example 2.10. Given a map $f : X \to Y$, its *homotopy fiber* F_f is the fiber of the map $E_f \to Y$ (see Proposition 2.5) over any basepoint y_0, i.e.,

$$F_f = \{(x,\gamma) \mid \gamma(0) = f(x), \gamma(1) = y_0\}.$$

As evidence for F_f being the correct homotopical notion of fiber, one can show that there is a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F_f) \to \pi_n(X) \to \pi_n(Y) \to \pi_{n-1}(F_f) \to \cdots \to \cdots \pi_0(X) \to \pi_0(Y).$$

Example 2.11. The *homotopy pullback* of two maps $f : X \to A$ and $g : Y \to A$ is the *double mapping space*

$$N(f, g) = X \times_A A^I \times_A Y = \{(x, \gamma, y) \in X \times A^I \times Y \mid \gamma(0) = f(x), \gamma(1) = g(y)\}.$$

Example 2.12. Let G be a topological group and X be a G-space. The *homotopy quotient* is the space

$$X_G = X \times_G EG = (X \times EG)/G,$$

where EG is a weakly contractible space on which G acts freely.

These constructions may seem fairly ad hoc, but note that they are obtained from replacing maps in the relevant diagram with fibrations or cofibrations, and then taking a limit or colimit. As motivation for ∞-categories, it will turn out that homotopy limits and colimits coincide with ordinary limits and colimits in the ∞-category of spaces.

2.3. Classifying spaces

In this section, we construct the classifying space BG of a group G using the language of simplicial sets. We will assume G is discrete, but a similar construction works for any topological group.

Definition 2.13. Given a category $\mathcal{C}$, let $N(\mathcal{C})_n$ be the set of all n-tuples of composable morphisms

$$c_0 \to c_1 \to \cdots \to c_{n-1} \to c_n$$

in $\mathcal{C}$. This defines a simplicial set $N(\mathcal{C})_\bullet$ called the *nerve* of $\mathcal{C}$.

Definition 2.14. Let $\mathbf{B}G$ be the category with one object and automorphism group G. The *classifying space BG* is defined to be the geometric realization of $N(\mathbf{B}G)$.

Theorem 2.15. For a CW complex X, there is a natural bijection

$$[X, BG] \cong G\mathrm{Bund}(X)$$

between the the set of homotopy classes of maps from X to BG and the set of isomorphism classes of principal G-bundles on X.

Proof. The theorem holds if we can show that BG is the quotient of a contractible space by a free G-action. Let $\mathbf{E}G$ be the groupoid with objects the elements of G and morphisms $h : g \to gh$, and set $EG = |N(\mathbf{E}G)|$. Since $\mathbf{E}G$ is equivalent to the trivial category, EG is contractible. Moreover, the category $\mathbf{E}G$ has a natural free G-action whose quotient is the category $\mathbf{B}G$. This induces a free G-action on EG whose quotient is BG, as desired. $\square$