INTERSECTION THEORY IN ALGEBRAIC GEOMETRY

1. 1/27/20 - Introduction

Announcements. The class will meet on MW in general, but not this Wednesday. The references are:

- *3264 and all that* by Joe Harris and David Eisenbud.
- *Intersection Theory* by William Fulton.

Let X be a set. A basic observation about subsets $A, B \subset X$ is that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

So $\cap$ distributes over $\cup$ just as $\times$ distributes over $+$. Can we convert this observation into a ring structure? One way to make this work is by forming the Boolean algebra:

$$(\mathbb{F}_2)^X.$$

The product corresponds to $\cap$ and the sum corresponds to symmetric difference (union excluding intersection). The multiplicative identity is X itself. Unfortunately this ring is enormous, and it does not see any geometric structures on X.

Next assume that X is a smooth manifold, and that $A, B \subset X$ are closed submanifolds. If A and B intersect transversely (which can be arranged by perturbing them slightly), then

$$\operatorname{codim}(A \cap B) = \operatorname{codim}(A) + \operatorname{codim}(B).$$

So if we restrict our attention to submanifolds of X, then we should be able to define a *graded* ring structure, graded by codimension. To accommodate small perturbations, we need to consider equivalence classes instead of literal submanifolds.

This idea can be formalized using de Rham cohomology. Let X be an oriented manifold of dimension n. To any closed oriented submanifold $A \subset X$ of codimension a, we can define its Poincaré dual

$$[\omega_A] \in H^a(X, \mathbb{R}),$$

where ω_A is a closed a-form supported on a small tubular neighborhood of A with the property that the integral of ω_A over any transverse cross-section of the tube is 1.

Exercise. If $[\omega_A]$ and $[\omega_B]$ are the Poincaré duals of closed oriented submanifolds $A, B \subset X$ meeting transversely, then

$$\omega_A \wedge \omega_B = [\omega_{A\cap B}].$$

The de Rham cohomology ring gives us a model for transverse intersections:

$$H^*(X, \mathbb{R}) = \bigoplus_{a=0}^{n} H^a(X, \mathbb{R}).$$

If we assume that X is compact, then the isomorphism $\int_X : H^n(X, \mathbb{R}) \to \mathbb{R}$ sends the Poincaré dual of a point to 1. This allows us to count the (oriented) number of intersection points between A and B when $\mathrm{codim}(A) + \mathrm{codim}(B) = n$.

Poincaré duality takes submanifolds to integral cohomology classes, and in fact the wedge product can be defined directly on the integral cohomology groups, where it is called the cup product.

Example. Let X be a closed Riemann surface of genus g. Then $H^0(X, \mathbb{Z}) \simeq H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$, and $H^1(X, \mathbb{Z}) = \mathbb{Z}^{2g}$, with a basis given by the duals of the standard loops $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$. Because 1 is odd, the wedge product pairing

$$H^1(X, \mathbb{Z}) \otimes H^1(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$$

is skew-symmetric. The product counts intersections of two loops intersecting transversely *with orientations.*

This can give funny answers. For example, in a genus 4 surface (with the holes arranged like a cross), you can draw a pair of loops which meet twice, but their oriented intersection number is 0. These loops cannot be deformed to disjoint loops.

One way to avoid the orientation issue is to study complex manifolds and their holomorphic submanifolds. A complex vector space induces a natural orientation on the underlying real vector space such that any two complementary complex subspaces have *positive* intersection. Furthermore, the real codimension of a complex submanifold is always even, so the wedge product is *commutative.*

Definition. A projective manifold is a closed complex submanifold of $\mathbb{CP}^N$.

Theorem. (Chow) Any projective manifold is cut out by polynomial equations.

As a consequence, if X is a projective variety, then any complex submanifold $A \subset X$ is a projective variety too. Perhaps there is a way to define an intersection ring for algebraic subvarieties of X purely in terms of algebraic geometry, without using smooth topology. This will be the main subject of the course.

Proposition. The cohomology ring of $\mathbb{CP}^N$ is given by

$$H^*(\mathbb{CP}^N, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd} \end{cases}$$

Proof. Breaking $\mathbb{CP}^N$ into $\mathbb{A}^N \cup \mathbb{A}^{N-1} \cup \cdots \cup \mathbb{A}^0$ gives a cell decomposition which yields a chain complex with a single generator in even degrees and 0 differential. This complex computes cellular homology or cohomology. $\square$

Each generator is represented by the Poincaré dual of a projective subspace $\mathbb{P}^k$ for $0 \leq k \leq N$. It is easy to see the ring structure in terms of these generators. Since any two linear subspaces can be translated so that they intersect transversely,

$$[\mathbb{P}^k] \cdot [\mathbb{P}^l] = [\mathbb{P}^{k+l-N}].$$

Let $A \subset \mathbb{P}^N$ be a complex submanifold of (complex) codimension a. What is the class $[A] \in H^{2a}(\mathbb{P}^N, \mathbb{Z})$? Well we know that $[A] = d[\mathbb{P}^{N-a}]$. To find d, we just need to intersect $[A]$ with $[\mathbb{P}^a]$:

$$[A] \cdot [\mathbb{P}^a] = d[\mathbb{P}^0].$$

Definition. The integer $d > 0$ is called the *degree* of A.

From the multiplication law on cohomology, we see that degree is multiplicative under transverse intersections. If $A \subset \mathbb{P}^N$ is a *hypersurface*, that is a submanifold cut out by a single homogeneous polynomial, then d is the degree of the polynomial.

Theorem. (Bézout) Let $C, D \subset \mathbb{P}^2$ be two curves in $\mathbb{P}^2$ defined by irreducible polynomials of degree d and e. As long as they are not the same curve, they will intersect at de points, counted with multiplicity.

If C meets D transversely, then this theorem follows immediately from the discussion above. If they have non-transverse intersection points, then there is a multiplicity involved.

Definition. Suppose that $0 \in \mathbb{C}^2$ is an intersection point of C and D, defined by polynomials $f(x, y) = 0$ and $g(x, y) = 0$. The intersection multiplicity at 0 is:

$$\mathrm{mult}_0(C, D) = \dim_{\mathbb{C}} \mathbb{C}[x, y]/(f, g).$$

We will prove Bézout's Theorem later on. To finish up, let us work out a sample application of intersection theory.

Question. Fix 4 general lines in $\mathbb{C}^3$. How many lines in $\mathbb{C}^3$ meet all 4?

Answer. 2! And the answer is the same for lines in $\mathbb{R}^3$ (easier to visualize).

To get this answer, we consider the moduli space of all lines in $\mathbb{C}^3$. This space is non-compact, so to use an intersection product, we compactify it by considering the space of projective lines in $\mathbb{CP}^3$. The result is a Grassmannian:

$$\mathbb{G}(1, 3) = G(2, 4) = \{\text{2-dimensional subspaces } U \subset \mathbb{C}^4\}$$

Now $G(2, 4)$ is itself a projective manifold. In fact it is a hypersurface in $\mathbb{P}^5$. To see this, we define a map from $G(2, 4)$ into $\mathbb{P}(\wedge^2\mathbb{C}^4)$ sending $U = \mathrm{span}(v, w)$ to $[v \wedge w]$. Choosing a different basis for U changes the image by a determinant, which is why we must projectivize $\wedge^2\mathbb{C}^4$.

Exercise. The subset of $\mathbb{P}^5 = \mathbb{P}(\wedge^2\mathbb{C}^4)$ consisting of "pure wedges" is cut out by the equation $\alpha \wedge \alpha = 0$, where $\alpha \in \wedge^2\mathbb{C}^4$, so $G(2, 4)$ is a quadric hypersurface.

Now that we have a nice moduli space, we need to impose the condition of meeting the fixed lines $\ell_1, \ell_2, \ell_3, \ell_4 \subset \mathbb{P}^3$. Let's define

$$\Sigma(\ell) := \{\text{lines } \mathbb{P}^1 \subset \mathbb{P}^3 \text{ which meet } \ell\} \subset \mathbb{G}(1, 3) = G(2, 4)$$

The count we are after is the size of $\Sigma(\ell_1) \cap \Sigma(\ell_2) \cap \Sigma(\ell_3) \cap \Sigma(\ell_4)$. A dimension count reveals that each $\Sigma(\ell_i)$ is a codimension 1 subvariety[1] of $\mathbb{G}(1,3)$. In fact, it is cut out by the equation $\ell_i \wedge \alpha = 0$. The count we want is the intersection number of a quadric hypersurface with 4 hyperplanes in $\mathbb{P}^5$, which is 2 by the multiplicative property of degree.

To generalize this example, we can study the cohomology ring of the Grassmannian:

$$H^{2*}(\mathbb{G}(1,3),\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} \\ \mathbb{Z} \\ \mathbb{Z}^2 \\ \mathbb{Z} \\ \mathbb{Z}. \end{cases}$$

The generator of $H^2(\mathbb{G}(1,3),\mathbb{Z})$ is called σ_1, and it is the class of $\Sigma(\ell)$ for any line $\ell \subset \mathbb{P}^3$. The observations above imply that

$$\int_{\mathbb{G}(1,3)} (\sigma_1)^4 = 2.$$

Question. What are the two generators of $H^4(\mathbb{G}(1,3),\mathbb{Z})$? To be continued.

More generally, the Grassmannian $\mathbb{G}(1,n)$ has dimension $2n-2$. There is an analogous class σ_1 defined as the set of lines meeting a fixed codimension 2 linear subspace $\Lambda \subset \mathbb{P}^n$. Another natural question is then:

$$\int_{\mathbb{G}(1,n)} (\sigma_1)^{2n-2} = \,?$$

Alternative approaches to the 4 lines problem:

1. The lines meeting ℓ_1, ℓ_2, ℓ_3 sweep out a quadric surface $Q \subset \mathbb{P}^3$. The last line ℓ_4 meets that surface in two points.

2. Assume that ℓ_1 and ℓ_2 intersect at p (so they are not in general position). There are two ways for L to meet both ℓ_1 and ℓ_2: either L lies in the plane H spanned by ℓ_1 and ℓ_2, or L contains p. In the first case, L must be the line between $\ell_3 \cap H$ and $\ell_4 \cap H$. In the second case, projecting from p sends ℓ_3 and ℓ_4 to a pair of lines on a screen which meet at a single point q, so L must be $\overline{pq}$.

[1] $\Sigma(\ell)$ is a singular subvariety because it is the intersection of $\mathbb{G}(1,3)$ with the hyperplane tangent to $\mathbb{G}(1,3)$ at $[\ell]$.

2. 2/3/20 - Chow Groups

2.1. **Basic Properties.** Let X be a scheme (often a smooth quasi-projective variety if intersections are involved) over a field k. We would like to define a theory of algebraic cycles on X. Start with a free abelian group:

$$Z(X) = \mathbb{Z}\{\text{algebraic subvarieties of } X\}$$

By variety, we mean a reduced, irreducible subscheme (separatedness is automatic). A cycle $\sum n_i Z_i \in Z(X)$ is *effective* if all the $n_i > 0$. If Y is a subscheme, its associated cycle $[Y]$ is a sum over the irreducible components weighted by their *multiplicity*. For an irreducible component, $Y_i \subset Y_{red}$, its multiplicity is defined as the length of the local ring $\mathcal{O}_{Y,Y_i}$ over itself. X is Noetherian, so any subscheme has finite length over its local ring.

Let $\mathrm{Rat}(X) \subset Z(X)$ be the subgroup generated by

$$W \cap (\{0\} \times X) - W \cap (\{\infty\} \times X)$$

for $W \subset \mathbb{P}^1 \times X$ a subvariety not contained in any fiber of $\mathbb{P}^1 \times X \to \mathbb{P}^1$.

Definition. The Chow group $A(X)$ is the quotient $Z(X)/\mathrm{Rat}(X)$.

The Chow group is graded by dimension:

$$A(X) = \bigoplus_k A_k(X).$$

Theorem. There exists a product structure on $A(X)$ satisfying the condition that if $A, B \subset X$ are generically transverse subvarieties, then

$$[A] \cdot [B] = [A \cap B].$$

Generically here means in $A \cap B$. The most intuitive definition of the intersection product goes through the Moving Lemma.

Lemma. For every subvarieties $A, B \subset X$, there is a cycle $\alpha \in Z(X)$ rationally equivalent to $[A]$ such that every component of α is generically transverse to B. Furthermore, the class $[\alpha \cap B]$ is independent of α. We will prove this on Wednesday. NB: this can fail when X is singular.

Theorem. (Kleiman): Suppose that G is an algebraic group acting transitively on X (characteristic 0), and let $A, B \subset X$ be subvarieties. Then for generic $g \in G$, gA is generically transverse to B.

Theorem. (Bertini): A general hyperplane section of a smooth projective variety X is smooth. More generally, a general member of a linear system of divisors on X is smooth away from its base locus.

How do we compute the Chow group of X? If $\dim(X) = n$, then $A_n(X) = \mathbb{Z}[X]$ because $\mathbb{P}^1 \times X$ is irreducible. If X is reducible but equidimensional, then $A_n(X) = \mathbb{Z}^c$. Furthermore $A(X_{red}) = A(X)$.

Affine space: $A(\mathbb{A}^n) = \mathbb{Z}$. Indeed, let $Y \subset \mathbb{A}^n$ be any subvariety not containing the origin. Consider $W^\circ \subset (\mathbb{A}^1)^\times \times \mathbb{A}^n$ be the set of (t, ty) for $y \in Y$, and let W be its closure in $\mathbb{P}^1 \times \mathbb{A}^n$. Precisely, it is defined by the polynomials $f(y/t)$ for all $f \in I(Y)$. The fiber over $1 \in \mathbb{P}^1$ is Y, and the fiber over $\infty \in \mathbb{P}^1$ is empty (there is polynomial f with nonzero constant term, and it will not vanish at ∞).

Exercise. The fiber over 0 is the cone over $\overline{Y} \cap H_\infty$.

Theorem. Let X be a scheme. For any subschemes $X_1, X_2 \subset X$, we have an exact sequence

$$A(X_1 \cap X_2) \to A(X_1) \oplus A(X_2) \to A(X_1 \cup X_2) \to 0.$$

For $Y \subset X$ a closed subscheme, we have an exact sequence

$$A(Y) \to A(X) \to A(U) \to 0$$

Proof. Both theorems are proved the same way.

$$\begin{tikzcd}
0 \arrow[r] & Z(Y \times \mathbb{P}^1) \arrow[r] \arrow[d] & Z(X \times \mathbb{P}^1) \arrow[r] \arrow[d] & Z(U \times \mathbb{P}^1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z(Y) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & Z(U) \arrow[r] \arrow[d] & 0 \\
 & A(Y) \arrow[r] \arrow[d] & A(X) \arrow[r] \arrow[d] & A(U) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

the first vertical maps are ∂, defined to be zero on cycles contained in fibers, and otherwise the difference between the fibers at $\{0\}$ and $\{\infty\}$. The theorem follows from a diagram chase. The Mayer-Vietoris is similar. $\square$

Corollary. If $U \subset \mathbb{A}^n$ is a nonempty open set, then $A(U) = A_n(U) = \mathbb{Z}$. (The first map is 0, and the second is an isomorphism.)

Remark. The right exact sequences above can be extended to the left using the higher Chow groups of Bloch, which are a special case of motivic cohomology.

We say that X is *stratified* if it is a union of locally closed irreducible subschemes U_i, and the closure of any U_i is a union of U_j. We say that a stratification is *affine* if every U_i is isomorphic to an affine space. We say it is *quasi-affine* if every U_i is isomorphic to an open subset of affine space.

Proposition. If X admits a quasi-affine stratification, then $A(X)$ is generated by the classes of closed strata $\overline{U_i}$.

Proof. Induct on the number of strata. If U_0 is a minimal stratum, it is closed.

$$\mathbb{Z}[U_0] = A(U_0) \to A(X) \to A(X \smallsetminus U_0) \to 0.$$

By induction, the last piece is generated by closed strata classes. □

Theorem. (Totaro 2014) The classes of closed strata in an affine stratification form a basis of $A(X)$.

Remark. In practice, we can avoid appealing to this theorem by checking that the closed strata in complementary dimension have nonsingular intersection matrix.

2.2. **Functoriality.** Let $f : X \to Y$ be a proper map of schemes. Then for any subvariety $A \subset X$, $f(A) \subset Y$ will be a subvariety.

Define the pushforward on cycles $f_* : Z(X) \to Z(Y)$ sending $[A]$ to 0 if $\dim(A) > \dim(f(A))$, and $[A]$ to $d[f(A)]$ otherwise,

$$d = [R(A) : R(f(A))].$$

Proposition. The homomorphism f_* descends to $f_* : A(X) \to A(Y)$.
Proof. Use the norm map on fields of rational functions, to be introduced later. □

Exercise. Find a non-proper map such that f_* is not well-defined (hint: try an open immersion).

If X is proper over a point, then point classes are nonzero, which gives the notion of *degree* for a 0-cycle.

Let $f : X \to Y$ be an arbitrary map of smooth quasi-projective varieties. We would like to define a notion of pullback.

Definition. A subvariety $A \subset Y$ is *generically transverse* to f if the pre-image $f^{-1}(A)$ is generically reduced and has the same codimension as A.

Theorem. There is a unique map of groups $f^* : A^k(Y) \to A^k(X)$ such that for any $A \subset Y$ generically transverse to f,

$$f^*[A] = [f^{-1}(A)].$$

Remark. This theorem actually gives the intersection product, by applying $i_* i^*$ for i the inclusion of B. For this reason, f^* actually gives a homomorphism of graded rings $A^*(Y) \to A^*(X)$. If f is proper too, we have $\alpha \cdot f_*(\beta) = f_*(f^*\alpha \cdot \beta)$. This implies that $f_* : A_*(X) \to A_*(Y)$ is a homomorphism of $A^*(Y)$-modules.

NB: In the case when f is flat, you can just define $f^* : Z(Y) \to Z(X)$ and it descends to Chow; no Moving Lemma is required. Fulton calls this the flat pullback, and the more general one the Gysin pullback denoted $f^!$.

2.3. **Refinements.** At this stage we have defined $[A] \cdot [B] \in A^*(X)$ when $A, B \in X$ are generically transverse. Fulton defined a refined intersection $[A] \cdot [B] \in A^*(A \cap B)$. A partial refinement is given by the (Gysin) pullback. More on this later.

Suppose that $A \cap B$ is a union of components C of the correct codimension. Then

$$[A] \cdot [B] = \sum m_C(A,B)[C] \in A^*(X).$$

Here are some features of the intersection multiplicities $m_C(A,B)$:

- $m_C(A,B) \in \mathbb{Z}_+$.
- $m_C(A,B) = 1$ iff A and B intersect generically transversely along C.
- If A and B are Cohen-Macaulay at a general point of C, then $m_C(A,B)$ is the scheme-theoretic multiplicity of C.
- If not, then you need to use the Serre formula to find $m_C(A,B)$.

2.4. **Grassmannians.** Recall that $\mathbb{G}(1,3) = G(2,4)$ is a quadric hypersurface in $\mathbb{P}^5 = \mathbb{P}(\wedge^2\mathbb{C}^4)$ given by the single equation $\alpha \wedge \alpha = 0$.

We will describe the Chow group $A(\mathbb{G}(1,3))$ by giving an affine stratification. Fix a complete flag $p \in L \subset H \subset \mathbb{P}^3$.

$$\begin{aligned}
\Sigma_{0,0} &= \mathbb{G}(1,3)\\
\Sigma_{1,0} &= \{\Lambda : \Lambda \cap L \neq 0\}\\
\Sigma_{2,0} &= \{\Lambda : \Lambda \cap p \neq 0\}\\
\Sigma_{1,1} &= \{\Lambda : \Lambda \subset H\}\\
\Sigma_{2,1} &= \{\Lambda : p \in \Lambda \subset H\}\\
\Sigma_{2,2} &= \{\Lambda : \Lambda = L\}
\end{aligned}$$

1. The dense open stratum $\Sigma^\circ_{0,0}$ can be thought of as the locus of $V \subset \mathbb{C}^4$ complementary to a fixed $L \subset \mathbb{C}^4$. Fix one such Ω (as the origin). All others will be graphs of linear maps from $\Omega \to L$, so we get an $\mathbb{A}^4$.

2. The next open $\Sigma^\circ_{1,0}$ is the set $\{\Lambda : \Lambda \cap L \neq 0, p \notin \Lambda, \Lambda \not\subset H\}$. Let H' be a plane containing p but not L. Any Λ above meets H' away from $H' \cap H$. We get maps $\Sigma^\circ_{1,0} \to L \smallsetminus p$ and $\Sigma^\circ_{1,0} \to H' \smallsetminus (H' \cap H)$, so $\mathbb{A}^1 \times \mathbb{A}^2 \simeq \mathbb{A}^3$.

3. The middle two can be thought of as $\mathbb{P}^2$ and $\mathbb{P}^{2*}$ respectively, and you are removing a $\mathbb{P}^1 = \Sigma_{2,1}$ in both cases. 4. $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$.

We now know the Chow group completely. What is the ring structure? It's easy to see that

$$\sigma_{1,1}^2 = 1, \sigma_{2,0}^2 = 1, \sigma_{1,1}\sigma_{2,0} = 0, \sigma_1\sigma_{2,1} = 1.$$

Next up, we see that $\sigma_1\sigma_2 = \sigma_{2,1}$ and $\sigma_1\sigma_{1,1} = \sigma_{2,1}$ also.

Lemma. $\sigma_1^2 = \sigma_{1,1} + \sigma_2$.

Proof. If Λ meets L_1 and L_2, and L_1 meets L_2, then either it passes through $L_1 \cap L_2$ or it lies in the plane $\overline{L_1L_2}$. Alternatively, you can just use the relations we already know. $\sigma_1^2 = a\sigma_2 + b\sigma_{1,1}$.

$$\begin{aligned}
a &= \sigma_1^2\sigma_2 = \sigma_1\sigma_{2,1} = 1\\
b &= \sigma_1^2\sigma_{1,1} = \sigma_1\sigma_{2,1} = 1. \quad \square
\end{aligned}$$

3. 2/5/20 - Moving Cycles

3.1. **Loose Ends.** Today we will always assume that $f : X \to Y$ is a morphism of smooth projective varieties (so automatically proper). Recall that $f^* : A^*(Y) \to A^*(X)$ is homomorphism of graded rings, and $f_* : A_*(X) \to A_*(Y)$ is a homomorphism of graded $A^*(Y)$-modules.

Definition. Let L/K be a finite extension of fields. The *norm* $N_{L/K} : L \to K$ is defined as follows. For $x \in L$, the multiplication $m_x : L \to L$ is K-linear.

$$N_{L/K}(x) := \det(m_x) \in K.$$

It can be also be computed in a normal closure of L as the product of conjugates:

$$N_{L/K}(x) = \prod_{\sigma} x^{\sigma}.$$

If $f : X \to Y$ is generically finite and dominant, then we have a finite extension of rational function fields $R(X)/R(Y)$. Given a rational function φ, its associated divisor is given by

$$\operatorname{div}(\varphi) = \varphi^{-1}(0) - \varphi^{-1}(\infty).$$

Lemma. If $\varphi \in R(X)$, then:

$$f_*(\operatorname{div}(\varphi)) = \operatorname{div}(N_{R(X)/R(Y)}(\varphi)).$$

A nice consequence of this is a more economical definition of rational equivalence $\mathrm{Rat}_k(X)$. Instead of taking all subvarieties $W \subset X \times \mathbb{P}^1$, it suffices to consider those which are graphs of rational functions on $W \subset X$ of dimension $k+1$. With this definition, it is easy to see that the proper pushforward descends to Chow groups.

Last time, we saw that $A^0(X) = \mathbb{Z}[X]$. The next case to consider is $A^1(X)$. Here, rational equivalence of divisors coincides with linear equivalence. By the correspondence between divisors and line bundles, we have an isomorphism

$$c_1 : \mathrm{Pic}(X) \to A^1(X).$$

Remark. If X is singular, then c_1 is no longer surjective; in that case $A^1(X) \simeq \mathrm{Cl}(X)$, the Weil divisor class group.

The pullback $f^* : A^1(Y) \to A^1(X)$ can be defined by pulling back line bundles. To get a sense of the size of $A^1(X)$, recall that for X a curve of genus g,

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathbb{Z} \to 0.$$

Over $\mathbb{C}$, we have $\mathrm{Pic}^0(X) \simeq \mathbb{C}^g/\mathbb{Z}^{2g}$ a complex torus (Abel's Theorem). More generally, for X smooth projective we have a cycle class map

$$cc^k : A^k(X) \to H^{2k}(X, \mathbb{Z}).$$

Its kernel can be quite large in general. In dimension 0, $cc^n : A^n(X) \to \mathbb{Z}$ is an isomorphism if X is rationally connected (any two points can be linked by a rational curve). But there are actually varieties of general type such that $A^n(X) \simeq \mathbb{Z}$.

Conjecture. (Bloch) If S is a surface with $H^{1,0}(S) = H^{2,0}(S) = 0$, then

$$A^2(X) \simeq \mathbb{Z}.$$

3.2. **Moving Lemma.** Let $A, B \subset X$ be subvarieties with $\dim(X) = n$. There exists $\alpha \in Z(X)$ such that $[\alpha] = [A] \in A^*(X)$ but each component of α intersects B generically transversely.

Proof. As a warm-up, let us prove the case where A is a divisor. Let L be an ample line bundle on X. For $m \gg 0$, we have that $L^{\otimes m}$ and $L^{\otimes m} \otimes \mathcal{O}_X(D)$ are very ample. General sections D_1 and D_2 of them meet B generically transversely by Bertini's Theorem. In that case, we can take $\alpha = (s_2) - (s_1)$.

To prove the lemma in general, embed X into some projective space $\mathbb{P}^N$, and let $\Lambda \subset \mathbb{P}^N$ be a general linear subspace of dimension $N - n - 1$, so $\Lambda \cap X = \emptyset$. Let $\pi_\Lambda : X \to \mathbb{P}^n$ be the linear projection, a finite map in this case since Λ is general. Let $\widetilde{A} = \pi_\Lambda^{-1}\pi_\Lambda(A) = X \cap \overline{\Lambda, A}$. The bar denotes *linear join*: the union of all lines linking Λ with A. Since π_Λ is finite, $\widetilde{A} = A + A'$, where A' is generically reduced with the same dimension as A (this follows from Bertini's Theorem). From here, the strategy consists of three steps:

(1) Show that $\widetilde{A}$ can be made transverse to B.
(2) Show that no component $C \subset A' \cap B$ is also a component of $A \cap B$.
(3) Show that A is generically transverse to $B^* = B \setminus (A \cap B)$.

The Moving Lemma will follow by induction: write $A = \widetilde{A} - A'$, and then apply the same construction to A'.

(1) Recall that $\widetilde{A}$ is equal to $X \cap \overline{\Lambda, A}$. By Kleiman's Theorem, a general PGL_{N+1}-translate of $\overline{\Lambda, A}$ in $\mathbb{P}^N$ will be generically transverse to X and to B.

(2) Every component $C \subset A \cap B$ contains points p such that Λ does not meet $\mathbb{T}_p X$. Indeed, this is a Zariski open condition, and we can choose Λ so that it's true for one point on each component. The projection $\pi_\Lambda : X \to \mathbb{P}^n$ is nonsingular at such points p. Hence $p \notin A \cap A'$ because $\widetilde{A} = A \cup A'$ is singular along $A \cap A'$.

(3) Here we punt a little and just show that A is *dimensionally transverse* to B^*. With another page of work, we could show that they are generically transverse, but with the theory of intersection multiplicities the weaker statement suffices. Consider

$$\Psi = \{(\Lambda, p, q) \in \mathbb{G}(N - n - 1, N) \times A \times B^* : \Lambda \cap \overline{pq} \neq \emptyset\}$$

The fiber over a point $(a, b) \in A \times B^*$ is the set Λ meeting a fixed line, which is codimension n inside $\mathbb{G}$. Thus Ψ is irreducible, and

$$\dim \Psi = \dim A + \dim B + \dim \mathbb{G} - n.$$

Now, a general fiber of $\Psi \to \mathbb{G}$ surjects onto $\widetilde{A} \cap B^* = A' \cap B^*$. That fiber has dimension $\dim A + \dim B - n$, so

$$\dim(\widetilde{A} \cap B^*) \leq \dim A + \dim B - n.$$

On the other hand, $\dim(\widetilde{A} \cap B^*) \geq \dim A + \dim B - n$ because codimension is subadditive, so we are done. $\square$

To get an idea of what is happening, consider the example where $X \subset \mathbb{P}^3$ is a smooth quadric, so isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and $A = B = \mathbb{P}^1$ is one of the rulings. In this case, Λ is a point and $\widetilde{A} = \mathbb{P}^1 \cup \mathbb{P}^1$. Now $\widetilde{A}$ can be moved to a general linear section of X (a smooth conic C), which meets B at a single point. On the other hand, A' already meets B in a single point. We see that $A \cdot B = 0$ using

$$A \sim C - A'.$$

3.3. **Applications.** Now we return to the Grassmannian $\mathbb{G}(1,3)$ to see what the intersection product can do for us.

Example. Let $C \subset \mathbb{P}^3$ be a smooth, nondegenerate curve. Consider the locus

$$\Sigma(C) = \{\text{lines } \Lambda \subset \mathbb{P}^3 \text{ meeting } C\}.$$

It is not hard to see that $\Sigma(C) \subset \mathbb{G}(1,3)$ is codimension 1. Since $A^1(\mathbb{G}(1,3)) = \mathbb{Z}\sigma_1$, we know that $[\Sigma(C)] = d\sigma_1$. To find d, we intersect with the complementary $\sigma_{2,1}$:

$$d = [\Sigma(C)] \cdot \sigma_{2,1}$$

Recall that $\Sigma_{2,1}$ a pencil of lines through a point p lying in a plane $H \subset \mathbb{P}^3$. Since H intersects C in $\deg(C)$ points, we find that $d = deg(C)$.

Example. Let $C \subset \mathbb{P}^3$ be as before. Consider the locus

$$\mathrm{Sec}(C) = \{\text{lines } \Lambda \subset \mathbb{P}^3 \text{ meeting } C \text{ twice}\}.$$

It is not hard to see that $\mathrm{Sec}(C) \subset \mathbb{G}(1,3)$ is codimension 2, so we know that

$$[\mathrm{Sec}(C)] = a\sigma_{1,1} + b\sigma_2$$

To find a and b, we intersect with the complementary cycles $\sigma_{1,1}$ and σ_2:

$$a = [\mathrm{Sec}(C)] \cdot \sigma_{1,1}$$
$$b = [\mathrm{Sec}(C)] \cdot \sigma_2.$$

Recall that $\Sigma_{1,1}$ is the locus of lines contained in H. Since H intersects C in $d = deg(C)$ points, we find that $a = \binom{d}{2}$.

Recall that Σ_2 is the locus of lines through p. The linear projection $\pi_p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ sends C to a degree d plane curve with b nodes. A smooth plane curve always has genus $\binom{d-1}{2}$ by the Riemann-Hurwitz formula. The number of nodes is the difference between the arithmetic and geometric genera of $\pi_p(C) \subset \mathbb{P}^2$:

$$b = \binom{d-1}{2} - g(C).$$

The genus $g(C)$ must be part of the problem; it can take any value from 0 to the Castelnuovo bound, which is quadratic in d.

4. 2/10/20 - Schubert Calculus

4.1. **Grassmannians.** Today we will discuss the Chow ring of a general $G(k, n)$. As in the case $G(2, 4)$ from last week, $G(k, n)$ admits an affine stratification where the strata are indexed by sequences $\underline{a}$ of integers of length k such that:

$$n - k \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 0.$$

This can be viewed as a Young tableau contained in a box of height $n-k$ and width k, with the bars decreasing in height from left to right. We will suppress trailing zeroes from the notation.

Remark. There are $\binom{n}{k}$ such tableaux, by counting possibilities for the jagged boundary of the tableau.

To define the stratification, fix a complete flag in $\mathbb{C}^n$:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n.$$

Given a sequence $\underline{a}$ as above, we set

$$\Sigma_{\underline{a}} := \{\Lambda \in G(k, n) : \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i\}.$$

To parse this, note that for Λ general, $\dim \Lambda \cap V_{n-k+i} = i$, so the non-generic strata correspond to earlier intersections than expected. If $\underline{a} \leq \underline{a}'$, then $\Sigma_{\underline{a}} \supset \Sigma_{\underline{a}'}$. The following are special cases:

Example. The locus of Λ contained inside some V_r corresponds to $\underline{a} = (n-r)^k$.

Example. The locus of Λ containing some V_r corresponds to $\underline{a} = (n-k)^r$.

Example. The locus of Λ meeting some V_l non-trivially corresponds to

$$\underline{a} = n - k - (l - 1).$$

For simplicity, from now on we assume that the flag is the standard flag in $\mathbb{C}^n$.

Proposition. The open stratum $\Sigma_{\underline{a}}^\circ$ is an affine space of codimension $|a| = \sum a_i$.

Proof. We can choose a basis for Λ_0 inductively which consists of vectors in $V_{n-k+i-a_i}$ as soon as they become available. If $\Lambda_0 \in G(k, n)$ is generic then this basis looks like:

$$n \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \to \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The highest column of 0's has height $k-1$. If Λ_0 is inside $\Sigma_{\underline{a}}$, then we can make the ith column of 0's higher by a_i, after performing the column operations, we can read off the dimension of the affine stratum. $\square$

Given a partition $\underline{a}$, there is a complementary partition $\underline{a}^*$ given by

$$a_i^* = n - k - a_{k+1-i}$$

Proposition. Schubert classes in complementary dimension have product 0, except for the case:

$$\sigma_{\underline{a}} \cdot \sigma_{\underline{a}^*} = 1.$$

Proof. The action of GL_n simply changes the flag, so by Kleiman's Theorem any intersection product can be computed using two generic flags, $V_\bullet$ and $W_\bullet$. To intersect $\Sigma_{\underline{a}}(V_\bullet)$ with $\Sigma_{\underline{b}}(W_\bullet)$ consider the conditions in pairs $(i, k+1-i)$:

$$\dim(\Lambda \cap V_{n-k+i-a_i}) \geq i$$
$$\dim(\Lambda \cap W_{n+1-i-b_{k+1-i}}) \geq k+1-i.$$

If you view both of these intersections inside Λ, their dimensions sum to $k+1$, so they must have non-trivial intersection. Since $V_\bullet$ and $W_\bullet$ are generic, their dimensions in $\mathbb{C}^n$ must also add up to $\geq n+1$. This forces $\underline{b} = \underline{a}^*$. There is a unique Λ by taking the span of the k vectors you get by intersection each complementary pair $V_{n-k+i-a_i} \cap W_{n+1-i-a^*_{k+1-i}=k+1-i+a_i}$. □

With this in hand, we can encode the full ring structure in terms of structure constants:

$$\gamma^{c}_{\underline{a},\underline{b}} = \sigma_{\underline{a}} \cdot \sigma_{\underline{b}} \cdot \sigma_{\underline{c}^*}.$$

Remark. The $\gamma^{c}_{\underline{ab}}$ are always positive because they are actually the same as Littlewood-Richardson coefficients. Irreducible representations of GL_k are classified by partitions of length k (Schur functors applied to the standard representation). If you tensor together two irreps, you can decompose the result into irreps. This gives a ring called the *representation ring*, $Rep(GL_k)$ which can also be expressed in terms of characters as polynomial functions in the entries of a matrix, invariant under conjugation. We have a sequence of isomorphisms

$$Rep(GL_k) \to Sym^*(\mathfrak{gl}_k^\vee)^{GL_k} \to H^{2*}(BGL_k)$$

(closure followed by Chern-Weil). The latter has a natural map to $\to H^{2*}(G(k,n))$ via the tautological bundle on $G(k,n)$. A particular Schur irrep maps to the corresponding Schubert class via the composition.

Consider the example $\mathbb{G}(1,n) = G(2,n+1)$ of lines in $\mathbb{P}^n$, which has dimension $2n-2$. A natural questions is what is its Plücker degree, that is σ_1^{2n-2}?

$$\Sigma_{a,b} = \{\Lambda \in \mathbb{G}(1,n) : \Lambda \cap \mathbb{P}^{2-a} \neq \emptyset,\, \Lambda \subset \mathbb{P}^{3-b}\}.$$

Proposition. $\sigma_1 \cdot \sigma_{a,b} = \sigma_{a+1,b} + \sigma_{a,b+1}$, where a summand may be zero if that sequence is not allowed.

Proof. Similar to our proof of the fact that $\sigma_1^2 = \sigma_{1,1} + \sigma_2$. Move the $\mathbb{P}^{n-2}$ defining Σ_1 so that it intersects $\mathbb{P}^{2-a}$ in codimension 1, and $\mathbb{P}^{3-b}$ in codimension 2. There are two ways for Λ to lie in $\Sigma_1 \cap \Sigma_{a,b}$: either it meets $\mathbb{P}^{n-2} \cap \mathbb{P}^{2-a}$, or it lies inside the span of $\mathbb{P}^{2-a}$ and $\mathbb{P}^{n-2} \cap \mathbb{P}^{3-b}$, which is a $\mathbb{P}^{3-b-1}$. □

Corollary. In $\mathbb{G}(1,n)$, the top intersection σ_1^{2n-2} has degree Catalan(n+1).

Proof. Stacking unit boxes inside the $(n+1)\times 2$ box such that each step is a Young tableau is the same as placing n+1 pairs of parentheses. $\square$

Theorem. (Pieri) In $G(k,n)$, $\sigma_c \cdot \sigma_{\underline{a}}$ is equal to the sum over all tableaux obtained by adding c blocks to $\underline{a}$, at most one block in each row.

These σ_c are called *special Schubert classes*. Actually, any class can be expressed as a polynomial in these special classes:

Theorem. (Giambelli) In $G(k,n)$, any $\sigma_{\underline{a}}$ can be expressed as the determinant of a matrix with special Schubert classes as the entries:

$$\sigma_{a_1,a_2,\ldots,a_l} = \begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \cdots & \sigma_{a_1+l-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \cdots & \sigma_{a_2+l-2} \\ \vdots & & \ddots & \\ \sigma_{a_l-l+1} & \sigma_{a_l-l+2} & \cdots & \sigma_{a_l} \end{vmatrix}.$$

Proof. Use the Pieri formula with induction by expanding along the last column. For example:

$$\sigma_{a_1,a_2} = \begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} \\ \sigma_{a_2-1} & \sigma_{a_2} \end{vmatrix} = \sigma_{a_1}\sigma_{a_2} - \sigma_{a_1+1}\sigma_{a_2-1}. \quad \square$$

Pieri and Giambelli together give an algorithm to compute any intersection product. Vakil gave a more efficient algorithm for computing LR coefficients in terms of checkers puzzles. Positivity is clear from Vakil's approach.

Lemma. $(1-\sigma_1+\sigma_{1,1}-\cdots\pm\sigma_{1^k})(1+\sigma_1+\sigma_2+\cdots+\sigma_{n-k}) = 1.$

Proof. For $d>0$, using Pieri we have

$$\sum_{i=0}^{d} = \sigma_d - (\sigma_d + \sigma_{d-1,1}) + (\sigma_{d-1,1} + \sigma_{d-2,1,1}) - \cdots \pm (\sigma_{2,1^{d-2}} + \sigma_{1^d}) \mp \sigma_{1^d}$$

4.2. **Chern classes.** The cohomology of $BGL_k = G(k,\infty)$ is a free polynomial ring on special Schubert classes called *Chern classes* c_i $(1 \le i \le k)$:

$$c_i = (-1)^i \sigma_{1^i}.$$

These give a set of characteristic classes. They are uniquely determined by $c_1(\mathcal{O}\mathbb{P}^n(1)) = H$ and the Whitney sum formula: for any short exact sequence

$$0 \to E \to F \to G \to 0,$$

the total Chern classes $c = \sum c_i$ satisfy $c(F) = c(E)c(G)$. Below we will give a geometric characterization of c_i for globally generated bundles:

Proposition. Let $\mathcal{E}$ be a globally generated vector bundle on X, and let $\sigma_0, \sigma_1, \sigma_{r-i}$ be $r-i+1$ general global sections. Then $Z(\sigma_0 \wedge \sigma_1 \wedge \ldots \sigma_{r-i})$ is generically reduced of codimension i, and it represents $c_i(E)$.

Proof. Suppose that $H^0(\mathcal{E})$ has dimension m. Let $\phi : X \to G(m-r, m)$ be the morphism sending $p \in X$ to the kernel of the evaluation map.

$$0 \to K \to \mathcal{O}^{\oplus m} \to \mathcal{E} \to 0.$$

If $U \subset H^0(E)$ is the span of the chosen sections, then $Z(\sigma_0 \wedge \sigma_1 \wedge \dots \sigma_{r-i})$ is $\phi^{-1}(\Sigma_k(U))$, the locus where K_p meets U non-trivially. $\square$

On any Grassmannian $G(k, n)$, we have a tautological sequence

$$0 \to S \to \mathcal{O}^{\oplus n} \to Q \to 0.$$

The proposition above implies that $c_i(Q) = \sigma_i$. The Whitney sum formula then implies that $c_i(S) = (-1)^i \sigma_{1^i}$. The tangent bundle to the Grassmannian is

$$\mathrm{Hom}(S, Q) = S^* \otimes Q.$$

In the case where $k = 1$, we recover the Euler sequence for $\mathbb{P}^n$:

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{n+1} \to Q \to 0$$
$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T_{\mathbb{P}^n} \to 0$$

Theorem. (Poincaré-Hopf) Let X be a compact complex manifold.

$$c_{top}(X) = \chi_{top}(X)$$

Example. We can now compute the Euler characteristic of a smooth hypersurface of degree d in $\mathbb{P}^n$.

$$0 \to T_X \to T_{\mathbb{P}^n}|_X \to N_{X/\mathbb{P}^n} \to 0$$

Now as a sheaf, $N_{X/\mathbb{P}^n}$ is $\mathcal{O}_X(X)$, so we have

$$\begin{aligned}
c(T_X) &= \frac{(1+H)^{n+1}}{(1+dH)} \\
&= \left(1 + (n+1)H + \binom{n+1}{2} H^2 + \dots\right)(1 - dH + d^2H^2 - \dots) \\
c_{n-1}(T_X) &= \sum_{i=0}^{n-1} \binom{n+1}{i} (-d)^{n-1-i} H_X^{n-1} \\
&= \sum_{i=0}^{n-1} (-1)^{n+1+i} \binom{n+1}{i} d^{n-i}
\end{aligned}$$

For example, when $n = 3, d = 4$ you get $4^3(1) - 4^2(4) + 4(6) = 24$ (K3 surface).

5. 2/12/20 - Chern Classes in Geometry

Theorem. (Splitting Principle) Any identity involving the Chern classes of vector bundles on X holds if and only if it is true for vector bundles which are direct sums of line bundles on X.

Proof. Suppose that a vector bundle $\mathcal{E}$ on X admits a filtration

$$0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \mathcal{F}_r = \mathcal{E},$$

such that the successive quotients $\mathcal{F}_{i+1}/\mathcal{F}_i$ are line bundles. By the Whitney sum formula, the Chern classes of $\mathcal{E}$ can be computed from a direct sum of line bundles. Not every $\mathcal{E}$ admits such a filtration. To prove the theorem in general, we will produce a projective morphism $\pi : Y \to X$ such that (1) $\pi^*\mathcal{E}$ admits a filtration as above, and (2) the pull-back map $\pi^* : A^*(X) \to A^*(Y)$ is injective. For (1) we induct on the rank r of $\mathcal{E}$. Consider the projectivization of $\mathcal{E}^\vee$:

$$\pi_1 : \mathbb{P}\mathcal{E}^\vee = \operatorname{Proj}_X(\operatorname{Sym}^*\mathcal{E}) \to X.$$

The pull-back $\pi_1^*\mathcal{E}^\vee$ fits into the relative tautological sequence

$$0 \to S \to \pi_1^*\mathcal{E}^\vee \to Q \to 0$$

$$0 \to Q^\vee \to \pi_1^*\mathcal{E} \to S^\vee \to 0$$

By induction, $Q^\vee$ admits a filtration to we are done. For (2), if $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}^\vee}(1))$ is the relative hyperplane class, then for any $\alpha \in A^*(X)$,

$$\alpha = \pi_{1*}\left(\pi_1^*(\alpha) \cdot \zeta^{r-1}\right). \quad \square$$

As a first application of this splitting principle, we can count lines on hypersurfaces. Let $X_d \subset \mathbb{P}^n$ be a hypersurface of degree d.

$$F(X) = \{\text{lines on } X\} \subset \mathbb{G}(1,n).$$

Before we find the class of $F(X)$, we compute its dimension using an incidence correspondence:

$$\Psi = \{(X,L) : L \subset X\} \subset \mathbb{P}^{N=\binom{n+d}{d}-1} \times \mathbb{G}(1,n).$$

The second projection pr_2 has fiber a linear subspace of $\mathbb{P}^N$ of codimension $d+1$, that is the kernel of the restriction

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(\mathbb{P}^1, \mathcal{O}(d)).$$

Hence, Ψ is smooth of dimension $2n-2+N-(d+1)$. If the first projection is dominant, then by properness it will be surjective. Then for a general hypersurface X its Fano variety of lines will have dimension

$$\dim F(X) = 2n-d-3.$$

Harris and Eisenbud show dominance of pr_1 (when this number is positive) by producing an explicit hypersurface X containing a line L whose normal bundle $N_{L/X}$ has $2n-d-3$ sections.

Conjecture. (Debarre-de Jong) If $d \leq n$, then for *every* smooth hypersurface,

$$\dim F(X) = 2n-d-3.$$

In other words, the word *general* encompasses all smooth hypersurfaces for small d.

Remark. The term "Fano variety" has two meanings. In this context it means the variety of lines contained in X. More broadly, a Fano variety is a variety whose anticanonical line bundle is ample. The meanings do not match; $F(X)$ is often not a Fano variety in the second sense. To add to the confusion, the hypersurfaces appearing in the Debarre-de Jong Conjecture are Fano in the second sense.

We can compute the class of $F(X) \in A^{d+1}(\mathbb{G}(1,n))$ as the top Chern class of a vector bundle $\mathcal{E}$ on $\mathbb{G}(1,n) = G(2,n+1)$. There is a tautological sub-bundle $S \subset \mathcal{O}^{\oplus n+1}$, and we set $\mathcal{E} = \mathrm{Sym}^d(S^\vee)$. The fiber of $\mathcal{E}$ at a line $L \simeq \mathbb{P}^1 \subset \mathbb{P}^n$ is canonically identified with $H^0(L, \mathcal{O}(d))$. The equation defining $X \subset \mathbb{P}^n$ is a degree d polynomial, an element of $H^0(\mathbb{P}^n, \mathcal{O}(d))$. For any line L, we can restrict the polynomial to $H^0(L, \mathcal{O}(d))$, so there is a section $\sigma_X \in H^0(\mathcal{E})$. The zero locus of σ_X consists of lines contained in X, which is precisely the locus $F(X)$.

To be precise, dualizing and taking the symmetric power of the tautological inclusion $S \hookrightarrow \mathcal{O}^{\oplus n+1}$, we have

$$\mathcal{O}_{G(2,n+1)} \otimes H^0(\mathbb{P}^n, \mathcal{O}(d)) \simeq \mathrm{Sym}^d(\mathcal{O}_{G(2,n+1)}^{\vee \oplus n+1}) \to \mathrm{Sym}^d(S^\vee) \to 0$$

Taking global sections, we have a map $H^0(\mathbb{P}^n, \mathcal{O}(d)) \to H^0(\mathcal{E})$ sending the equation defining X to σ_X.

To compute $c_{d+1}(\mathcal{E})$, we use the Splitting Principle. If $S^\vee$ were isomorphic to a direct sum of line bundles $L \oplus M$, then

$$\mathrm{Sym}^d(S^\vee) \simeq (L^{\otimes d}) \oplus (L^{\otimes d-1} \otimes M) \oplus \cdots \oplus (L^{d-1} \otimes M) \oplus (M^{\otimes d}).$$

Using the Whitney sum formula, we know that if $c_1(L) = \alpha$ and $c_1(M) = \beta$,

$$c(L \oplus M) = (1+\alpha)(1+\beta) = 1 + (\alpha+\beta) + (\alpha\beta).$$

We can rewrite the symmetric expression

$$c(\mathrm{Sym}^d(S^\vee)) = \prod_{i=0}^{d} (1 + i\alpha + (d-i)\beta)$$

as a polynomial in $(\alpha+\beta)$ and $\alpha\beta$, and the result gives an identity relating the Chern classes of $\mathrm{Sym}^d(S^\vee)$ with those of $S^\vee$.

As a first example, we carry this for $X \subset \mathbb{P}^3$ a cubic surface. We want to compute

$$c_4(\mathrm{Sym}^3(S^\vee)) \in A_0(\mathbb{G}(1,3)).$$

The degree 4 term of the product formula above (for the pretend split case) reads

$$\begin{aligned}(3\alpha)(2\alpha+\beta)(\alpha+2\beta)(3\beta) &= 9(\alpha\beta)(2\alpha^2 + 2\beta^2 + 5\alpha\beta) \\ &= 9(\alpha\beta)(2(\alpha+\beta)^2 + \alpha\beta) \\ &= 9c_2(S^\vee)(2c_1(S^\vee)^2 + c_2(S^\vee)) \\ &= 9\sigma_{11}(2\sigma_1^2 + \sigma_{11}) \\ &= 9\sigma_{11}(2\sigma_2 + 3\sigma_{11}) = 27\sigma_{22}.\end{aligned}$$

This proves that there are 27 lines on a generic cubic surface $X_3 \subset \mathbb{P}^3$. In fact, *all* smooth cubic surfaces contain 27 lines. A similar calculation allows us to count 2875 lines on a generic quintic threefold $X_5 \subset \mathbb{P}^4$. The reader can readily count lines on any hypersurface with $2n-d-3=0$. What happens if $2n-d-3=1$? There, $F(X)$ will be a curve (smooth for generic choice of X), and the Chern class calculation tells us its degree as a multiple of the generator $\sigma_{n-1,n-2} \in A_1(\mathbb{G}(1,n))$.

Question. What is the genus of $F(X)$ when $2n-d-3=1$?

To answer this, we need a general version of the *adjunction formula*. Recall that if Y is a smooth projective variety, and $D \subset Y$ is a divisor, then

$$K_D = (K_Y + D)|_D.$$

This can be proved by taking c_1 of the terms in the normal bundle short exact sequence:

$$0 \to T_D \to T_Y|_D \to N_{D/Y} \to 0,$$

together with the observation that $N_{D/Y} \simeq \mathcal{O}(D)|_D$. The latter is true because

$$N^\vee_{D/Y} \simeq I_D/I_D^2 \simeq I_D \otimes \mathcal{O}_Y/I_D \simeq \mathcal{O}_Y(-D) \otimes \mathcal{O}_D = \mathcal{O}(-D)|_D.$$

If instead $D \subset Y$ is subvariety of codimension r cut out by a section of a rank r vector bundle $\mathcal{E}$, then we have $N_{D/Y} \simeq \mathcal{E}|_D$. To prove this, take the Koszul complex for $\mathcal{O}_Y \to \mathcal{O}_D$, and restrict it to D. Putting all this together, we deduce the following formula for the genus g of $F(X)$ in the case where $2n-d-3=1$:

$$\begin{aligned} 2g-2 = \deg K_{F(Y)} &= \deg \Big(K_{\mathbb{G}(1,n)} + c_1(\mathrm{Sym}^d(S^\vee)) \Big) \Big|_{F(X)} \\ &= \deg \Big(K_{\mathbb{G}(1,n)} + c_1(\mathrm{Sym}^d(S^\vee)) \Big) \cdot c_{top}(\mathrm{Sym}^d(S^\vee)). \end{aligned}$$

The splitting principle can be used to prove several identities, which we leave as exercises. For vector bundles E and F or ranks e and f, respectively:

$$\begin{aligned} c_k(E^\vee) &= (-1)^k c_k(E); \\ c_1(E \otimes F) &= f c_1(E) + e c_1(F); \\ c_k(E \otimes L) &= \sum_{i=0}^{k} \binom{r-k+i}{i} c_1(L)^i c_{k-i}(E); \\ c_{ef}(E \otimes F) &= \mathrm{Resultant}(c_t(E), c_t(F)). \end{aligned}$$

Given any sequence of symmetric analytic functions

$$f_1(x_1), f_2(x_1,x_2), f_3(x_1,x_2,x_3), \ldots$$

one can define a characteristic class of algebraic vector bundles. If $\mathcal{E}$ is a bundle of rank r, rewrite f_r as a power series in the elementary symmetric polynomials p_k, and then replace each p_k with c_k. For particular choices of f_r the resulting characteristic class may have nice geometric properties. For example:

$$f_r = \prod_{i=1}^{r} (1+x_i) \qquad \text{(Total Chern class)}$$

$$f_r = \sum_{i=1}^{r} e^{x_i} \qquad \text{(Chern character)}$$

$$f_r = \prod_{i=1}^{r} \frac{x_i}{1-e^{-x_i}} \qquad \text{(Todd class).}$$

6. 2/19/20 - The Riemann-Roch Theorem

The Chern character (defined last time) has the advantage that it is additive for short exact squences:

$$0 \to E \to F \to G \to 0$$

$$\mathrm{ch}(F) = \mathrm{ch}(E) + \mathrm{ch}(G)$$

and also multiplicative for tensor products:

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\mathrm{ch}(F).$$

In other words, we have a ring homomorphism

$$\mathrm{ch} : K_0(X) \to A^*(X) \otimes \mathbb{Q}.$$

Here $K_0(X)$ is the Grothendieck group of X, generated by vector bundles E on X modulo the relation that $[F] = [E] + [G]$ is there is a short exact sequence of vector bundles $0 \to E \to F \to G \to 0$. The product structure on $K_0(X)$ is given by the tensor product. In is natural to ask for a functoriality property for morphisms $f : X \to Y$ of smooth quasi-projective varieties. From basic properties of Chern classes, the pullback functors fit into a commutative square:

$$\begin{tikzcd}
K_0(Y) \arrow[r, "f^*"] \arrow[d, "\mathrm{ch}"'] & K_0(X) \arrow[d, "\mathrm{ch}"] \\
A^*(Y)_{\mathbb{Q}} \arrow[r, "f^*"] & A^*(X)_{\mathbb{Q}}.
\end{tikzcd}$$

For covariant (pushfoward) functoriality, we must define a candidate for the proper pushforward of a vector bundle E. The pushforward of the locally sheaf $\mathcal{E}$ is not necessary locally free. For example, the pushforward $\mathcal{O}_Z$ via a closed immersion $Z \hookrightarrow X$ will be a torsion sheaf. The correct setting to define the proper pushforward is on the (larger) abelian category of coherent sheaves on X.

Theorem. For X smooth projective, $K_0(X) \simeq K_0(\mathrm{Coh}(X))$.

Proof. It suffices to show that every coherent sheaf admits a resolution by locally free sheaves. This follows from the Hilbert Syzygy Theorem. $\square$

For f_* to be well-defined on $K_0(\mathrm{Coh}(X))$, we need to use the *derived pushforward.*

$$f_*[\mathcal{E}] = \sum_{i \geq 0} [R^i f_*(\mathcal{E})].$$

The higher pushforwards $R^i f_*(\mathcal{E})$ are explicitly given by sheafifying the presheaf

$$(U \subset Y) \mapsto H^i(f^{-1}(U), \mathcal{E}|_{f^{-1}(U)}).$$

The fact that f_* is well-defined on K_0 follows from the long exact sequence. The fact that it is a covariant functor follows from the spectral sequence for compositions.

Now we may hope for a commutative square for all proper $f : X \to Y$:

$$\begin{tikzcd} K_0(X) \arrow[r, "f_*"] \arrow[d, "\mathrm{ch}"'] & K_0(Y) \arrow[d, "\mathrm{ch}"] \\ A^*(X)_{\mathbb{Q}} \arrow[r, "f_*"] & A^*(Y)_{\mathbb{Q}}. \end{tikzcd}$$

But alas, this is still not so. Consider the case when X is a curve and Y is a point.

$$h^0(L) - h^1(L) \neq \deg(L).$$

The Riemann-Roch Theorem (for curves) says that

$$h^0(L) - h^1(L) = \deg(L) + 1 - g = \deg(L).$$

The Riemann-Roch Theorem (for surfaces) says that

$$h^0(L) - h^1(L) + h^2(L) = \frac{c_1(L)^2 + c_1(L)c_1(T_X)}{2} + \frac{c_1(T_X)^2 + c_2(T_X)}{12}.$$

Notice that this is the codimension 2 part of

$$\left(1 + c_1(L) + \frac{c_1(L)^2}{2}\right)\left(1 + \frac{c_1(T_S)}{2} + \frac{c_1(T_X)^2 + c_2(T_X)}{12}\right)$$

The right hand factor is the 2-truncated part of the Todd class. To see a bit more:

$$1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1c_2}{24} + \frac{-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4}{720} + \dots$$

Lemma. If E has rank r, then

$$\sum_{i=0}^{r}(-1)^i \mathrm{ch}(\wedge^i E^\vee) = c_r(E) \cdot \mathrm{Td}(E)^{-1}.$$

Proof. This is a straightforward application of the splitting principle:

$$\sum_{i=0}^{r}(-1)^i \mathrm{ch}(\wedge^i E^\vee) = \prod_{i=1}^{r}(1 - e^{-\alpha_i}) = (\alpha_1\alpha_2 \dots \alpha_r)\prod_{i=1}^{r}\frac{1 - e^{-\alpha_i}}{\alpha_i}. \quad \square$$

Theorem. (Grothendieck) Let $f : X \to Y$ be a morphism of smooth projective varieties and $\mathcal{E}$ a coherent sheaf on X. Then in $A_*(Y)$ we have the equality:

$$\mathrm{ch}(f_*\mathcal{E}) \cdot \mathrm{Td}(Y) = f_*\left(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{Td}(X)\right).$$

$$\begin{tikzcd} K_0(X) \arrow[r, "f_*"] \arrow[d, "\mathrm{ch}(-)\cdot\mathrm{Td}(X)"'] & K_0(Y) \arrow[d, "\mathrm{ch}(-)\cdot\mathrm{Td}(Y)"] \\ A^*(X)_{\mathbb{Q}} \arrow[r, "f_*"] & A^*(Y)_{\mathbb{Q}}. \end{tikzcd}$$

Proof. It is more convenient to write $\mathrm{ch}(f_*\mathcal{E}) = f_*\left(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{Td}(X) \cdot f^*\mathrm{Td}(Y)^{-1}\right)$.
Step 1: Check that if the identity holds for $f : X \to Y$ and $g : Y \to Z$ separately, then it holds for the composite $g \circ f : X \to Z$.

$$\begin{aligned} \mathrm{ch}((g \circ f)_*\mathcal{E}) &= \mathrm{ch}(g_*f_*\mathcal{E}) \\ &= g_*(\mathrm{ch}(f_*\mathcal{E}) \cdot \mathrm{Td}(Y/Z)) \\ &= g_*(f_*(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{Td}(X/Y)) \cdot \mathrm{Td}(Y/Z)) \\ &= g_*f_*(\mathrm{ch}(E) \cdot \mathrm{Td}(X/Z)) \end{aligned}$$

Step 2: Check that the identity holds for $X = \mathbb{P}^m \times Y \to Y$. This can be reduced to checking it for $Y = pt$: the box product $K_0(\mathbb{P}^m) \otimes K_0(Y) \to K_0(\mathbb{P}^m \times Y)$ is surjective, by induction on m with the localization sequence for K_0. If the outer and upper squares commute, then lower square commutes:

$$\begin{tikzcd}
K_0(\mathbb{P}^m) \otimes K_0(Y) \arrow[r] \arrow[d, "\boxtimes"'] & K_0(pt) \otimes K_0(Y) \arrow[d, "\sim"] \\
K_0(\mathbb{P}^m \times Y) \arrow[r, "f_*"] \arrow[d, "\mathrm{ch}(-)\cdot\mathrm{Td}(\mathbb{P}^m)\cdot\mathrm{Td}(Y)"'] & K_0(Y) \arrow[d, "\mathrm{ch}(-)\cdot\mathrm{Td}(Y)"] \\
A^*(\mathbb{P}^m \times Y) \arrow[r, "f_*"] & A^*(Y).
\end{tikzcd}$$

On the generating set $[\mathcal{O}(n)]$, $0 \le n \le m$, for $K_0(\mathbb{P}^m)$, we have:

$$\chi(\mathbb{P}^m, \mathcal{O}(n)) = H^0(\mathbb{P}^m, \mathcal{O}(n)) = \binom{m+n}{n}.$$

On the other hand,

$$\begin{aligned}
\left[e^{nH} \cdot \left(\frac{H}{1-e^{-H}} \right)^{m+1} \right]_m &= \mathrm{Res}_{x=0} \frac{e^{nx}}{(1-e^{-x})^{m+1}} \\
&= \mathrm{Res}_{y=0} \frac{(1-y)^{-n}}{y^{m+1}} \\
&= \binom{m+n}{n}.
\end{aligned}$$

Step 3: Check that the identity holds for a closed immersion $f : X \hookrightarrow Y$. This can be reduced to checking it for $Y = \mathbb{P}(N \oplus \mathcal{O})$ an arbitrary completed bundle on X, with f the zero section $x \mapsto [0:1]$, and $p : Y \to X$ the structure map. Start by computing $\mathrm{ch}(f_*\mathcal{O}_X)$ using a Koszul resolution. The image of X in $\mathbb{P}(N \oplus \mathcal{O})$ is cut out by a section of Q, the tautological quotient bundle on Y:

$$\mathcal{O}_Y \to p^*N \oplus \mathcal{O}_Y \to Q.$$

This gives us a resolution of $f_*\mathcal{O}_X$ by vector bundles:

$$0 \to \wedge^r Q^\vee \to \cdots \to \wedge^2 Q^\vee \to Q^\vee \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to 0.$$

By the additivity of the Chern character and the earlier Lemma,

$$\mathrm{ch}(f_*\mathcal{O}_X) = \sum_{i=0}^r (-1)^i \mathrm{ch}(\wedge^i Q^\vee) = c_r(Q) \cdot \mathrm{Td}(Q)^{-1}.$$

Then using the fact that $f^*Q = N_{X/Y} = N$ we obtain

$$c_r(Q) \cdot \mathrm{Td}(Q)^{-1} = f_*\left(f^*\mathrm{Td}(Q)^{-1}\right) = f_*(\mathrm{Td}(N)^{-1}) = f_*(\mathrm{Td}(X/Y)).$$

The proof for $\mathrm{ch}(f_*\mathcal{E})$ is similar; the Koszul complex above tensored with $p^*\mathcal{E}$ gives a resolution of $f_*\mathcal{E}$ and the rest follows.

Putting everything together, we use the fact that any morphism $f : X \to Y$ can be factored $X \to \Gamma_f \subset \mathbb{P}^m \times Y$ as a closed immersion followed by a simple projection with fiber $\mathbb{P}^m$. The deformation to the normal cone trick (next lecture) allows us to reduce the statement for closed embeddings to the setting of Step 3. $\square$

Corollary. The Chern character gives an isomorphism of rings:

$$\mathrm{ch} : K_0(X) \otimes \mathbb{Q} \to A^*(X) \otimes \mathbb{Q}.$$

Proof. There is a natural filtration F^i on $K_0(X)$ given by the codimension of the support of a sheaf being $\geq i$, and $A^*(X)$ has a similar filtration (it is already graded). By the GRR formula, for a k-codimensional subvariety $Z \subset X$ we have

$$\mathrm{ch}[\mathcal{O}_Z] = i_{Z*}(1 + \alpha) \in [Z] + F^{k+1}A^*(X).$$

Hence, the assignment $[Z] \mapsto [\mathcal{O}_Z]$ defines an isomorphism

$$A^*(X) \to \mathrm{gr}^* K_0(X).$$

Since it is an isomorphism at the level of graded $\mathbb{Q}$-algebras, it is an isomorphism at the level of filtered $\mathbb{Q}$-algebras too. $\square$

Example. When $X = \mathbb{P}^n$ the Chern character isomorphism sends $x = [\mathcal{O}_{\mathbb{P}^n}(1)]$ to $e^h \in A^*(\mathbb{P}^n)$:

$$K_0(\mathbb{P}^n) \otimes \mathbb{Q} \simeq \mathbb{Q}[x]/(x-1)^{n+1} \to \mathbb{Q}[h]/(h^{n+1}) \simeq A^*(\mathbb{P}^n) \otimes \mathbb{Q}$$

Notice that x is invertible in the quotient because the geometric series is finite.

7. 2/24/20 - Deformation to the Normal Cone

Today we will take a step back and go through some foundational material, culminating in two key statements (starred) that were used in the proof of GRR last time.

Definition. Given a vector bundle $E \to X$, its projectivization is given by

$$\mathbb{P}E = \operatorname{Proj}(\operatorname{Sym}^* E^\vee) \xrightarrow{p} X.$$

As such, it has a line bundle $\mathcal{O}_{\mathbb{P}E}(1)$ which restricts to $\mathcal{O}(1)$ on each fiber. The relative tautological sequence for $\mathbb{P}E \to X$ reads

$$0 \to S \to p^*E \to Q \to 0,$$

and $S \simeq \mathcal{O}_{\mathbb{P}E}(-1)$, which can be checked locally by trivializing the bundle.

Universal Property. Given a scheme Y with a morphism $\pi : Y \to X$, there is bijection between commutative triangles

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \mathbb{P}E \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

and sub-line bundles $L \subset \pi^*E$ over Y. To go from a commutative triangle to a line bundle, simply pull back the tautological line bundle S via f:

$$f^*S \subset f^*p^*E = \pi^*E$$

To go the other direction, cover X by open sets trivializing both E and L, and define $f|_U$ using the universal property for projective space.

Proposition. If E (resp. L) is a vector (resp. line) bundle on X, then setting $E' = E \otimes L$, we have an isomorphism

$$\mathbb{P}E \simeq \mathbb{P}E'.$$

Proof. Define a morphism $f : \mathbb{P}E' \to \mathbb{P}E$ via the universal property: tensoring $\mathcal{O}_{\mathbb{P}E'}(-1) \subset p'^*E' \simeq p'^*E \otimes p'^*L$ with p'^*L^{-1}, we get

$$\mathcal{O}_{\mathbb{P}E'}(-1) \otimes p'^*L^{-1} \subset p'^*E.$$

We can define a similar morphism $g : \mathbb{P}E \to \mathbb{P}E'$. Their composition is the identity, as can be seen by pulling back the line bundles through the composition. $\square$

Remark for experts. As a consequence of the construction above,

$$f^*\mathcal{O}_{\mathbb{P}E}(1) \simeq \mathcal{O}_{\mathbb{P}E'}(1) \otimes p'^*L.$$

So if L was ample, then $\mathcal{O}_{\mathbb{P}E'}(1)$ is less ample than $\mathcal{O}_{\mathbb{P}E}(1)$ via the isomorphism f. This may seem backwards, and indeed it reflects the different convention between Fulton and Grothendieck in the definition of projectivization. To make things consistent, an ample vector bundle is one for which $\mathcal{O}_{\mathbb{P}E^\vee}(1)$ is ample.

In what follows, $\zeta = c_1(\mathcal{O}_{\mathbb{P}E}(1))$ is the relative hyperplane class, which depends on the choice of E, not just on the projective bundle $\mathbb{P}E$. We will assume that E has rank $r+1$ for now so that $\mathbb{P}E$ has fiber $\mathbb{P}^r$.

Theorem. The Chow group of $\mathbb{P}E$ is given by

$$A(\mathbb{P}E) \simeq \bigoplus_{i=0}^{r} p^*A(X) \cdot \zeta^i.$$

NB: We will sometimes suppress p^* in the notation. This is not so egregious because since p is flat, $p^*[A] = [p^{-1}A]$ always.

Proof. Define $\psi : A(\mathbb{P}E) \to A(X)^{\oplus r+1}$ by

$$\psi(\beta) = \bigoplus_i p_*(\zeta^{r-i} \cdot \beta).$$

Define $\varphi : A(X)^{\oplus r+1} \to A(\mathbb{P}E)$ by

$$\varphi((\alpha_i)_{i=0}^r) = \sum_i \zeta^i \cdot p^*\alpha_i.$$

Now using the fact that (for dimensional reasons)

$$p_*(\zeta^i p^*\alpha) = p_*(\zeta^i)\alpha = \begin{cases} \alpha \text{ if } i = r \\ 0 \text{ if } i < r \end{cases},$$

we see that $\psi \circ \varphi$ is upper triangular with 1's along the diagonal, so ψ is injective. To show surjectivity, we use induction to write a general cycle as a sum of cycles of the form $\zeta^i \cdot p^*\alpha$:

Lemma. Given a subvariety $Z \subset \mathbb{P}E$ of dimension k, let $W = p(Z) \subset X$ have dimension $l \le k$, so the general fiber of $p|_Z : Z \to W$ has dimension $k - l$. We can always write

$$Z \sim Z' + \sum B_j,$$

where $[Z'] = \zeta^{r-k+l}[W]$ and $\dim(p(B_j)) < l$.

Proof. We start with the case of projective space $\mathbb{P}^r$ (so $X = pt$) and $\dim(Z) = n$. It's possible to find coordinates x_i on $\mathbb{P}^r$ such that $(x_0 = x_1 = \cdots = x_n = 0)$ is disjoint from Z. Define

$$g_t = \begin{pmatrix} I_{n+1} & 0 \\ 0 & tI_{r-n} \end{pmatrix} \in PGL(r+1).$$

The flat limit of $g_t(Z)$ as $t = 0$ will be the nonreduced cycle $\deg(Z) \cdot \mathbb{P}^n$, so

$$Z \sim \zeta^{r-n} \deg(Z).$$

The case of $\mathbb{P}E$ is similar. If L is ample on X then for $N \gg 0$, $E^\vee \otimes L^{\otimes N}$ is globally generated. Replacing E with $E' = E \otimes L^{-\otimes N}$ does not affect the projectivization. Fix a point $x \in W \subset X$. For a general choice of global sections $\tau_0, \tau_1, \ldots, \tau_r$,

(1) $(\tau_0)_x, (\tau_1)_x, \ldots, (\tau_n)_x$ forms a basis for the fiber $E_x^\vee$.
(2) $(\tau_0 = \tau_1 = \cdots = \tau_{k-l} = 0)$ is disjoint from Z_x.

In fact, both conditions are Zariski open in X, so they are true on some $U \subset X$. Using condition (1),

$$\mathbb{P}E|_U = U \times \mathbb{P}^r,$$

so we can perform the same construction as in the point case over U.

$$g_t = \begin{pmatrix} I_{k-l+1} & 0 \\ 0 & tI_{r-k+l} \end{pmatrix} \in PGL(r+1).$$

Set $Z_t = \overline{g_t(Z_U)}$. The flat limit Z_0 will have class $d \cdot [\mathbb{P}^{k-l} \times U]$ over U, plus some components B_i over $W \smallsetminus W \cap U$. $\square$

Theorem. As a ring,

$$A^*(\mathbb{P}E) \simeq A^*(X)[\zeta]/(\zeta^{r+1} + c_1(E)\zeta^r + \cdots + c_{r+1}(E))$$

Proof. It suffices to prove that the monic polynomial relation above is satisfied. There can be no other relations by our description of the group structure. From

$$0 \to S \to p^*E \to Q \to 0,$$

we have $c(S)c(Q) = c(p^*E)$, so $c(Q) = c(p^*E)(1 + \zeta + \zeta^2 + \dots)$. Since $\mathrm{rk}(Q) = r$,

$$0 = c_{r+1}(Q) = \zeta^{r+1} + c_1(E)\zeta^r + \cdots + c_{r+1}(E). \quad \square$$

Remark. Fulton uses this result as the definition of (algebraic) Chern classes.

Remark. Applying the theorem to the trivial bundle $\mathcal{O}_X^{\oplus r+1}$, we obtain

$$A^*(X \times \mathbb{P}^r) \simeq A^*(X) \otimes A^*(\mathbb{P}^r).$$

This is called a Chow-Künneth formula. No such formula holds for general products.

For the rest of the lecture, we will assume that E has rank r because the main player will be $\mathbb{P}(E \oplus \mathcal{O}_X)$, which has fiber $\mathbb{P}^r$.

Proposition. If $F \subset E$ is a sub-bundle of rank s, then $\mathbb{P}F \subset \mathbb{P}E$. The class of $\mathbb{P}F$ in $CH^{r-s}(\mathbb{P}E)$ is given by

$$[\mathbb{P}F] = \zeta^{r-s} + \gamma_1\zeta^{r-s-1} + \cdots + \gamma_{r-s},$$

where $\gamma_i = c_i(E/F)$.

Proof. Consider the composition $S \to p^*E \to p^*(E/F)$. The composition vanishes identically over points of $\mathbb{P}F$. This means that we can compute $[\mathbb{P}F]$ as the top Chern class of the bundle

$$\mathcal{H}om(S, p^*(E/F)) \simeq \mathcal{O}_{\mathbb{P}E}(1) \otimes p^*(E/F).$$

The formula now follows easily from the Splitting Principle. $\square$

We will refer to $\mathbb{P}(E \oplus \mathcal{O}_X)$ as the *completion* of E. It has a *zero section*

$$j : X \to \mathbb{P}(E \oplus \mathcal{O}_X)$$

whose image is $\mathbb{P}(0 \oplus \mathcal{O}_X)$, and it contains a *hyperplane at* ∞, which is $\mathbb{P}(E \oplus 0)$. These two cycles are clearly disjoint, and by the previous proposition their classes in $A^*(\mathbb{P}(E \oplus \mathcal{O}_X))$ are given by

$$[\mathbb{P}\mathcal{O}_X] = \zeta^r + c_1(E)\zeta^{r-1} + \cdots + c_r(E)$$
$$[\mathbb{P}E] = \zeta$$

Proposition. The zero section $\mathbb{P}\mathcal{O}_X \subset \mathbb{P}(E \oplus \mathcal{O}_X)$ has self-intersection $j_* c_r(E)$.

Proof. To compute, $[\mathbb{P}\mathcal{O}_X] \cdot [\mathbb{P}\mathcal{O}_X]$ we write one of them in terms of ζ and leave the other alone. Since zero and ∞ are disjoint, $[\mathbb{P}\mathcal{O}_X] \cdot \zeta = 0$.

$$\begin{aligned}[\mathbb{P}\mathcal{O}_X]^2 &= [\mathbb{P}\mathcal{O}_X] \cdot (\zeta^r + c_1(E)\zeta^{r-1} + \cdots + c_r(E)\\ &= [\mathbb{P}\mathcal{O}_X] \cdot p^* c_r(E)\\ &= j_*(c_r(E)). \quad \square\end{aligned}$$

Proposition. ($\star$) More generally, for any cycle $\alpha \in A^*(X)$, $j^* j_*(\alpha) = \alpha \cdot c_r(E)$.

Proof. For any $\beta \in A^*(\mathbb{P}(E \oplus \mathcal{O}_X))$, by the Moving Lemma $j^*\beta = p_*(\beta \cdot [\mathbb{P}\mathcal{O}_X])$.

$$\begin{aligned}j^* j_*(\alpha) &= p_*(j_*(\alpha) \cdot [\mathbb{P}\mathcal{O}_X])\\ &= p_*(p^*\alpha \cdot [\mathbb{P}\mathcal{O}_X] \cdot [\mathbb{P}\mathcal{O}_X])\\ &= p_*(p^*\alpha \cdot j_* c_r(E))\\ &= \alpha \cdot p_* j_* c_r(E)\\ &= \alpha \cdot c_r(E). \quad \square\end{aligned}$$

Definition. Let $Z \subset X$ a closed subscheme defined by an ideal sheaf $\mathcal{I}$. The *blow-up* of X along Z is a new scheme given by Proj of the Rees algebra:

$$\mathrm{Bl}_Z X := \mathrm{Proj}\left(\mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \dots\right) \xrightarrow{\epsilon} X.$$

If we restrict ϵ to $Z \subset X$, the fibered product is obtained by tensoring the Rees algebra with $\mathcal{O}_X/\mathcal{I}$:

$$E_Z X := \mathrm{Proj}\left(\mathcal{O}_X/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2 \oplus \mathcal{I}^2/\mathcal{I}^3 \oplus \dots\right)$$

If Z is smooth (or more generally l.c.i.), then $\mathcal{I}^k/\mathcal{I}^{k+1} \simeq \mathrm{Sym}^k N_{Z/X}^\vee$. In that case,

$$E_Z X \simeq \mathrm{Proj}\left(\mathrm{Sym}^* N_{Z/X}^\vee\right) = \mathbb{P}N_{Z/X}.$$

By a dimension count, we find that $E_Z X \subset \mathrm{Bl}_Z X$ is a divisor. In fact, the blow up is the *minimal* way to replace Z with a Cartier divisor.

Lemma. If $Y \subset X$ is a subvariety, its proper transform in $\mathrm{Bl}_Z X$ is:

$$\overline{\epsilon^{-1}\left(Y \smallsetminus (Y \cap Z)\right)} \simeq \mathrm{Bl}_{Y \cap Z} Y.$$

The proof is technical, so we omit it. Instead, we will finish by introducing the "deformation to the normal cone" ($\star$). Consider $X \times \mathbb{P}^1$, viewed as a family over $\mathbb{P}^1$, and blow up $Z \times \{0\} \subset X \times \mathbb{P}^1$ to get $M \to \mathbb{P}^1$. The exceptional divisor E is

$$\mathbb{P}(N_{Z \times \{0\}/X \times \mathbb{P}^1}) \simeq \mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z),$$

the completed normal bundle. The proper transform of $X \times \{0\}$ is $\mathrm{Bl}_Z X$, and it meets E along $E_Z X$, which coincides with the hyperplane at ∞ in $\mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z)$. The proper transform of $Z \times \mathbb{P}^1$ is simply $Z \times \mathbb{P}^1$ because blowing up a Cartier divisor does nothing, but it meets E along the zero section $\mathbb{P}\mathcal{O}_Z$.

We have a flat family $M \to \mathbb{P}^1$. Over $t \neq 0$, $Z \subset X$ is embedded somehow. Over $t = 0$, $Z \subset \mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z)$ is embedded as the zero section in its completed normal bundle. Over $t = 0$, there is a *Zariski* neighborhood of Z isomorphic $N_{Z/X}$. This is analogous to the tubular neighborhood theorem in differential geometry.

8. 2/26/20 - Gysin Pullback and Excess Intersection

The deformation to the normal cone construction produces a family of varieties

$$M = \mathrm{Bl}_{Z\times\{0\}} X \times \mathbb{P}^1 \to \mathbb{P}^1$$

with general fiber X, and with special fiber a reducible, normal crossing variety

$$M_0 = \mathrm{Bl}_Z X \underset{E_Z X}{\cup} \mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z)$$

Here $E_Z X \simeq \mathbb{P}N_{Z/X}$ is the hyperplane at ∞ in $\mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z)$. Recall that

$$A^*(\mathbb{P}(N \oplus \mathcal{O})) \simeq A^*(Z)[\zeta]/(\zeta^{r+1} + c_1(N)\zeta + \cdots + c_r(N)\zeta).$$

The relation factors as $\zeta(\zeta^r + \zeta^{r-1}c_1(N) + \cdots + c_r(N)) = 0$ which corresponds to

$$[\mathbb{P}\mathcal{O}] \cdot [\mathbb{P}N] = 0.$$

The localization sequence for Chow groups gives

$$A_*(\mathbb{P}N) \to A_*(\mathbb{P}(N \oplus \mathcal{O})) \to A_*(N) \to 0,$$

where the first map sends $\alpha \mapsto \zeta \cdot \alpha$. This implies that the ideal (ζ) vanishes in $A^*(N)$, so we have

$$A^*(N) \simeq A^*(Z).$$

To be more explicit, the pull-back $\pi^* : A^*(Z) \to A^*(N)$ is an isomorphism. We can give the inverse map on a cycle $\beta \in A^*(N)$ by taking its closure $\overline{\beta} \in A^*(\mathbb{P}(N \oplus \mathcal{O}))$, and then pushing forward the intersection with the zero section: $p_* \left(\overline{\beta} \cdot [\mathbb{P}\mathcal{O}]\right)$.

Fulton and MacPherson had the idea to specialize cycles in X to cycles in $N_{Z/X}$ in

$$M^\circ := M \setminus \mathrm{Bl}_Z X \to \mathbb{P}^1.$$

At the level of sets, a subvariety $B \subset X$ specializes to the normal cone

$$C_{B\cap Z/B} \subset N_{Z/X},$$

and this descends to a map on Chow groups:

Theorem. The assignment $[B] \mapsto [C_{B\cap Z/B}]$ gives a well defined map

$$\sigma : A^*(X) \to A^*(N_{Z/X})$$

Proof. Consider the localization sequence again:

$$\begin{tikzcd}
A_{*+1}(N_{Z/X}) \arrow[r, "i_*"] & A_{*+1}(M^\circ) \arrow[r] \arrow[d, "i^*"'] & A_{*+1}(X \times \mathbb{A}^1) \arrow[r] \arrow[dl, dashed] & 0 \\
& A_*(N_{Z/X}) & A_*(X). \arrow[u, "\sim"'] &
\end{tikzcd}$$

The dotted arrow exists because for $i : N_{Z/X} \hookrightarrow M^\circ$ the inclusion, $i^* i_* = 0$. □

This allows us to give a definition for Gysin pullback without appealing to the Moving Lemma. If $j : Z \to X$ is the inclusion, then

$$j^* := (\pi^*)^{-1} \circ \sigma.$$

Remark. Fulton uses this construction to define the intersection product.

With a few formal observations, we can get a lot of mileage out of this. Recall that the zero section $\mathbb{P}\mathcal{O} \subset \mathbb{P}(N \oplus \mathcal{O})$ is cut out by a section of the tautological quotient bundle Q. If $\beta \in A_k(N)$, then we have

$$\begin{aligned}
(\pi^*)^{-1}\beta &= p_* \left(\overline{\beta} \cdot [\mathbb{P}\mathcal{O}]\right) \\
&= p_* \left(\overline{\beta} \cdot c_r(Q)\right) \\
&= p_* \left(\overline{\beta} \cdot c(Q)\right)_{k-r} \\
&= p_* \left(\overline{\beta} \cdot c(p^*(N \oplus \mathcal{O})) \cdot (1 + \zeta + \zeta^2 + \dots)\right)_{k-r} \\
&= \left[c(N) \cdot p_* \left(\overline{\beta} \cdot (1 + \zeta + \zeta^2 + \dots)\right)\right]_{k-r}
\end{aligned}$$

The factor $p_* \left(\overline{\beta} \cdot (1 + \zeta + \zeta^2 + \dots)\right)$ is an example of a *total Segre class.*

Definition. Let $E \to X$ be a vector bundle of rank r. The total Segre class

$$s(E) := p_*(1 + \zeta + \zeta^2 + \dots) \in A^*(X),$$

where $p : \mathbb{P}(E \oplus \mathcal{O}_X) \to X$. The graded parts of $s(E)$ are recovered by

$$s_i(E) = p_*(\zeta^{r+i})$$

for dimensional reasons. NB: we could have used $\mathbb{P}(E)$ instead of $\mathbb{P}(E \oplus \mathcal{O})$.

Proposition. $s(E) \cdot c(E) = 1$.

Proof. Apply p_* to the identity $c(Q) = c(p^*E)(1 + \zeta + \zeta^2 + \dots)$ and use the push-pull formula. $\square$

The total Segre class contains the same information as the total Chern class, but it has the advantage of being definable using only Chow groups. For this reason, the definition can be generalized to *cones*. Any graded $\mathcal{O}_X$-algebra $S = \bigoplus_{i \geq 0} S^i$ such that $\mathcal{O}_X \twoheadrightarrow S^0$ and S generated by S^1 gives rise to a cone scheme $C = \operatorname{Spec}(S) \to X$. The Segre classes of C are defined in the analogous way, using $p : \mathbb{P}(C \oplus \mathcal{O}_X) \to X$ and $\zeta = c_1(\mathcal{O}_{\mathbb{P}(C \oplus \mathcal{O})}(1))$.

$$s(C) = p_* \left(1 + \zeta + \zeta^2 + \dots\right) \in A^*(X).$$

Recall that in the specialization construction above, $\beta = [C_{B \cap Z/B}] \in A^*(N_{Z/X})$ and $\overline{\beta} = [\mathbb{P}(C_{B \cap Z/B} \oplus \mathcal{O}_Z)] \in A^*(\mathbb{P}(N_{Z/X} \oplus \mathcal{O}_Z))$. The hyperplane at ∞ (class ζ) restricts well, so the formula above becomes

$$j^*[B] = (\pi^*)^{-1}[C_{B \cap Z/B}] = \left[c(N_{Z/X}) \cdot g_* s(C_{B \cap Z/B})\right]_{k-r} \in A^*(Z),$$

where $g : B \cap Z \to Z$ is the inclusion. By the push-pull formula, this can be refined:

$$[B \cap Z] = \left[g^* c(N_{Z/X}) \cdot s(C_{B \cap Z/B})\right]_{k-r} \in A^*(B \cap Z).$$

This is referred to as the *excess intersection formula*. It allows us to find an intersection product without moving the cycles, and the answer is supported on their set-theoretic intersection. If Z and B are generically transverse, then the $(k - r)$ piece will be 1, and their is no need to consider normal cones.

Remark. There is an asymmetry between Z and B. One of them must have a normal *bundle* (in our case Z) which is equivalent to it being *regularly embedded*, that is cut out by a regular sequence of length r. In that case

$$\mathrm{Sym}^k(\mathcal{I}_Z/\mathcal{I}_Z^2) \simeq \mathcal{I}_Z^k/\mathcal{I}_Z^{k+1}.$$

If Z, B, and $B \cap Z$ are all smooth, then using the normal bundle sequence

$$0 \to N_{B\cap Z/B} \to N_{B\cap Z/X} \to N_{B/X}|_{B\cap Z} \to 0$$

we can rewrite the excess intersection formula more symmetrically:

$$[B\cap Z] = \left[\frac{c(N_{Z/X}|_{B\cap Z}) \cdot c(N_{B/X}|_{B\cap Z})}{c(N_{B\cap Z/X})}\right]_{k-r}.$$

On a smooth variety X, the diagonal $\Delta : X \to X \times X$ is a regular embedding. This is enough to define an intersection product for arbitrary cycles A, B; apply the Gysin pullback Δ^* to $A \times B \subset X \times X$. Note: X is not necessarily quasi-projective!

The most general form of the excess intersection formula is stated for $Z \to X$ a regular embedding and $B \to X$ an arbitrary morphism. The fibered product $W = B \times_X Z$ fits into a Cartesian square

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "g"'] & B \arrow[d, "f"] \\ Z \arrow[r, "j"] & X. \end{tikzcd}$$

Theorem. For any class $\alpha \in A_k(Z)$, we have

$$f^* j_*(\alpha) = i_* \left(g^*(\alpha \cdot c(N_{Z/X})) \cdot s(C_{W/B})\right)_{k+\dim(B)-\dim(X)}.$$

Corollary. Suppose that $\phi : X \to Y$ is generically finite. If $y \in Y$ is a special point for which $F = \phi^{-1}(y)$ is not finite, then

$$\deg(\varphi) = [s(N_{F/X})]_0.$$

For example, for $\mathrm{Bl}_0\mathbb{P}^2 \to \mathbb{P}^2$ the normal bundle to the exceptional fiber is $\mathcal{O}_{\mathbb{P}^1}(-1)$. As a more exotic example, Donagi-Smith computed the degree of the Prym map $R_6 \to A_5$ by considering a special fiber, and the answer was 27!

Lastly, let us return to the reduction step in the proof of the Grothendieck-Riemann-Roch theorem. To relate the case of an arbitrary closed embedding to the zero section of the completed normal bundle, we cited the deformation to the normal cone trick. The key fact is that $M \to \mathbb{P}^1$ is flat, by the following "Miracle Flatness:"

Theorem. If $q : M \to N$ is a surjective morphism of schemes, with M Cohen-Macaulay and N regular, such that for all $n \in N$, $\dim(M) = \dim(N) + \dim(q^{-1}(n))$, then q is flat.

Let $\mathcal{E}$ be a vector bundle on Z, and pull it back to $\tilde{\mathcal{E}}$ on $Z \times \mathbb{P}^1$. Via the proper transform, we have an embedding $\iota : Z \times \mathbb{P}^1 \to M$. Take a resolution $G_\bullet$ of $\iota_* M$. Since $M \to \mathbb{P}^1$ is flat, the G_i are flat over $\mathbb{P}^1$, so the restrictions of $G_\bullet$ to M_0 and to M_∞ remain exact. This observation combined with repeated application of the projection formula completes the desired reduction.

9. 3/2/20 - The Five Conics Problem

Question. How many conics are tangent to five general conics?

Let $C_1, \ldots, C_5$ be general conics in $\mathbb{P}^2$. Let $Z_i \subset \mathbb{P}^5$ be the locus of conics tangent to C_i. Consider the incidence correspondence

$$\Omega_i = \{(C, p) : C \text{ tangent to } C_i \text{ at } p\} \subset \mathbb{P}^5 \times C_i.$$

The fiber of pr_2 at $p \in C_i$ is a codimension 2 linear subspace of $\mathbb{P}^5$ - defined by the vanishing of the first two coefficients in the Taylor expansion of the degree 4 polynomial $C|_{C_i}$ at p. Since pr_1 is finite, we see that Z_i is a hypersurface. To find its degree, intersect it with a line in $\mathbb{P}^5$, that is a pencil of conics. Restricting the pencil to $C_i \simeq \mathbb{P}^1$, we get a basepoint free linear series of degree 4 divisors on $\mathbb{P}^1$. That is the same a degree 4 map $\mathbb{P}^1 \to \mathbb{P}^1$, which by Riemann-Hurwitz must have 6 branch points.

Since each Z_i is a sextic hypersurface, and we intersecting them inside $\mathbb{P}^5$, it is tempting to conclude that the answer is

$$\deg[Z_i]^5 = 6^5 = 7776.$$

But that is incorrect. To see why, observe that any double line is "tangent" to C_i, so each of the Z_i contains the Veronese surface

$$S = v_2(\mathbb{P}^2) \subset \mathbb{P}^5.$$

This means that set-theoretically

$$\bigcap_{i=1}^{5} Z_i = S \cup \Gamma,$$

where Γ is the finite set that we want to count. The correct answer is 6^5 minus the excess contribution supported on $S \simeq \mathbb{P}^2$. We will use $h = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$.

$$\text{excess } = \deg \prod_{i=1}^{5} c(N_{Z_i/\mathbb{P}^5}|_S) \cdot s(C_{T/\mathbb{P}^5}).$$

Here T is the component of the scheme-theoretic intersection supported on S. First,

$$\prod_{i=1}^{5} c(N_{Z_i/\mathbb{P}^5}|_S) = (1 + 12h)^5 = 1 + 60h + 1440h^2.$$

On the other hand,

$$s(N_{S/\mathbb{P}^5}) = \frac{c(T_S)}{c(T_{\mathbb{P}^5}|_S)} = \frac{1 + 3h + 3h^2}{1 + 12h + 30h^2} = 1 - 9h + 51h^2$$

The ideal of T is $(\mathcal{I}_S)^2$ essentially because each Z_i contains S with multiplicity 2. This means you can just replace ζ with 2ζ in the definition of Segre class. This has the effect of multiplying s_k by 2^{3+k} in this case:

$$s(N_{T/\mathbb{P}^5}) = 8 - 144h + 1632h^2.$$

Hence, the excess contribution is $1440 \cdot 8 - 60 \cdot 144 + 1632 = 4512$, so we have

$$|\Gamma| = 7776 - 4512 = 3264.$$

Question. How many conics are tangent to five general lines?

By the Riemann-Hurwitz argument, Z_i is a quadric, so the computation becomes

$$\begin{aligned}2^5 - \deg(1+4h)^5(1-9h+51h^2) &= 2^5 - \deg(1+20h+160h^2)(1-9h+51h^2)\\ &= 32-31\\ &= 1\end{aligned}$$

But this is not a surprise because the dual of a conic is a conic, and 5 general points lie on a unique conic. In general, the dual of a smooth hypersurface $X \subset \mathbb{P}^n$ has degree $d(d-1)^{n-1}$, by pulling back hyperplanes via the Gauss map $\mathcal{G} : X \to \mathbb{P}^{n*}$.

A second approach to the five conics problem involves blowing up the Veronese surface S to tease apart the scheme-theoretic behavior there. Luckily for us,

$$M \simeq \mathrm{Bl}_S\mathbb{P}^5$$

has a nice moduli theoretic description. It happens to be the space of *complete conics.* This is the closure in $\mathbb{P}^5 \times \mathbb{P}^{5*}$ of the locus U consisting of (C, C^*) where C is a smooth conic and C^* its dual. It also happens to be isomorphic to the coarse moduli scheme of the *Kontsevich stack of stable maps*:

$$\overline{\mathcal{M}}_0(\mathbb{P}^2, 2h) = \{f : C \to \mathbb{P}^2 |\ C \text{ is conn. nodal genus } 0,\ f_*[C] = 2h,\ |\mathrm{Aut}(f)| < \infty\}.$$

The fibers of the exceptional $E \to S$ are $\mathbb{P}^2 \simeq \mathrm{Sym}^2\mathbb{P}^1$, encoding the branch points of the double cover $C \to \mathbb{P}^1$. From these models, it is clear that the proper transforms $\widetilde{Z}_i$ intersect only in a finite set away from the exceptional divisor:

$$\bigcap_{i=1}^{5} \widetilde{Z}_i = \Gamma.$$

So it suffices to compute the intersection product inside the blow-up $\mathrm{Bl}_S\mathbb{P}^5$.

Proposition. Let $\epsilon : \mathrm{Bl}_Z X \to X$ be the blow up of a regularly embedded subvariety Z. Then we have

$$A_*(\mathrm{Bl}_Z X) \simeq \tilde{A}_*(E_{Z/X}) \oplus A_*(X),$$

where $\tilde{A}_*(E) \subset A_*(E \simeq \mathbb{P}N)$ consists of cycles $\zeta^i p^*\alpha$ where $i < \mathrm{codim}(Z/X) - 1$.

Proof. The proper pushforward $\epsilon_* : A_*(\mathrm{Bl}_Z X) \to A_*(X)$ is surjective, and the kernel is precisely $\tilde{A}_*(E_{Z/X})$. The pullback ϵ^* provides a splitting. $\square$

Proposition. The ring structure on $A^*(\mathrm{Bl}_Z X)$ is given in terms of generators in $\epsilon^* A^*(X)$ and $j_* A^*(E)$ by the following formulas:

$$\begin{aligned}\epsilon^*(\alpha) \cdot \epsilon^*(\alpha) &= \epsilon^*(\alpha \cdot \alpha');\\ \epsilon^*(\alpha) \cdot j_*(\beta) &= j_*(\beta \cdot p^* i_Z^*(\alpha));\\ j_*(\beta) \cdot j_*(\beta') &= -j_*(\beta \cdot \beta' \cdot \zeta).\end{aligned}$$

Proof. The first follows from the fact that ϵ^* is a ring homomorphism. The second follows from the push-pull formula. The third follows from the fact that

$$N_{E/\mathrm{Bl}_Z X} \simeq \mathcal{O}_{\mathbb{P}N}(-1). \quad \square$$

The class of $\widetilde{Z}_i$ in the blow up is $6H - 2E = 2(3H - E)$, because Z_i vanishes with multiplicity 2 along S. To prove this, note that a general pencil of conics containing a double line meets Z_i at only 4 other points, by the Riemann-Hurwitz argument.

$$\begin{aligned}
(3H-E)^5 &= 243H^5 - 5\cdot 81H^4E + 10\cdot 27H^3E^2 - 10\cdot 9H^2E^3 + 5\cdot 3HE^4 - E^5 \\
&= 243[pt] - 90\cdot j_*\left((2h)^2\zeta^2\right) + 15\cdot j_*\left((2h)(-\zeta^3)\right) - j_*(\zeta^4) \\
&= 243[pt] - 360[pt] + 30j_*\left(h(\zeta^2 9h + \zeta 30h^2)\right) + j_*\left((\zeta^2 9h + \zeta 30h^2)\zeta\right) \\
&= (243 - 360 + 270 - 81 + 30)[pt] \\
&= 102[pt].
\end{aligned}$$

Recall that in $A^*(\mathbb{P}N)$, we have $\zeta^2h^2 = [pt]$, $\zeta^3 = -\zeta^2c_1(N) - \zeta c_2(N)$, and we are repeatedly using the fact that $c(N_{S/\mathbb{P}^5}) = 1 + 9h + 30h^2$. To conclude,

$$\deg[\widetilde{Z}]^5 = 2^5 \cdot 102 = 3264.$$

Two Generalizations. A quadric φ on $\mathbb{P}V \simeq \mathbb{P}^n$ can be viewed as an element of $\mathrm{Sym}^2(V^\vee)$ or as a symmetric transformation $V \to V^\vee$. One can associated to φ a sequence of symmetric transformations

$$\varphi_i : \wedge^i V \to \wedge^i V^\vee \simeq (\wedge^i V)^\vee.$$

The space M of complete quadrics on $\mathbb{P}^n$ is defined to be the closure of $\varphi \mapsto (\varphi_1, \ldots, \varphi_n)$ for φ non-degenerate.

$$M \subset \prod_{i=1}^{n} \mathbb{P}\left(\mathrm{Sym}^2(\wedge^i V^\vee)\right).$$

M can alternatively be obtained by successively blowing up the strata of degenerate quadrics, starting with the most degenerate one. The result is a Mori Dream Space.

On the other hand, a degree d rational curve in $\mathbb{P}^2$ can be viewed as a morphism $f : \mathbb{P}^1 \to \mathbb{P}^2$, where two morphisms f and g are *equivalent* if they differ by a reparametrization. We can define a pre-stack which associates to a scheme T, the groupoid of diagrams:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d] & \mathbb{P}^2 \\
T &
\end{tikzcd}$$

such that $F_*[C_t] = dh$. This pre-stack turns out to be a smooth algebraic stack. To make it proper, we allow the curves $\mathcal{C}_t$ to be nodal, and require that any contracted component have ≥ 3 nodes. This makes the isotropy groups finite. A coarse space for a stack $\mathfrak{X}$ is a morphism $m : \mathfrak{X} \to X$ to an algebraic space such that

$$m\left(\overline{k}\right) : \mathfrak{X}\left(\overline{k}\right) \to X\left(\overline{k}\right)$$

is a bijection, and any morphism to an algebraic space factors through m.

Theorem. (Keel-Mori) Any stack with finite inertia has a coarse moduli space.

The map m looks like $[T_x\mathfrak{X}/G] \to T_x\mathfrak{X}/G$ formally locally around a smooth point $x \in \mathfrak{X}$ with isotropy group G. By a theorem of Chevalley-Shephard-Todd, if G is generated by pseudo-reflections, i.e. elements fixing a hyperplane, then the quotient is smooth (and conversely). This is true when $d = 2$, but fails when $d \geq 3$.

10. 3/4/20 - Intersection Theory on Singular Spaces

Question. Can we do intersection theory on singular spaces? What about smooth algebraic stacks?

On a singular quadric surface, $Q \subset \mathbb{P}^3$, a line L through the vertex p fails the Moving Lemma. Indeed, any curve rationally equivalent to L must pass through p because curves which do not have even degree in $\mathbb{P}^3$. Note that $H \cap C = 2\deg(\pi)$, where π is the linear projection from p. To have a well-defined intersection product, $(2L)^2 = 2$ forces us to set

$$L^2 = \frac{1}{2}.$$

Mumford defined an intersection product on $A^1(S) \otimes \mathbb{Q}$ for S a normal surface. Take a resolution $\epsilon : \tilde{S} \to S$. For a curve $A \subset S$, we use $\tilde{A}$ to denote its proper transform in $\tilde{S}$.

Lemma. There is a unique $\mathbb{Q}$-curve class A' in $\tilde{S}$ supported on the exceptional locus such that $\tilde{A} \cdot E + A' \cdot E = 0$ for each component E of the exceptional locus.

Proof. Let $E_1, \ldots, E_r$ be the components of the exceptional. We want to solve

$$\sum_{j=1}^{r} \lambda_j E_j \cdot E_i = -\tilde{A} \cdot E_i$$

for all i. This follows from the Hodge Index Theorem, which says that the intersection matrix of the E_i is negative definite. $\square$

Mumford's intersection product is defined by

$$A \cdot B = (\tilde{A} + A') \cdot (\tilde{B} + B') \in \mathbb{Q}.$$

In the case of the line on the singular quadric S, we get $A' = \frac{1}{2}E$, since $E^2 = -2$:

$$L^2 = (\tilde{A} + A')^2 = \tilde{A}^2 + \tilde{A}E + \frac{1}{4}E^2 = 0 + 1 - \frac{1}{2} = \frac{1}{2}.$$

Things are a bit worse for the singular quadric threefold X. All lines in X are rationally equivalent to a line through the vertex, but on the other hand we have $L_i \cdot H_j = \delta_{ij} pt$. This implies that $pt = 0$ which is problematic.

Intersections with $\mathbb{Q}$-Cartier divisors in $A^*(X) \otimes \mathbb{Q}$ can be done in general.

Definition. A divisor D is $\mathbb{Q}$-Cartier if for some $m \in \mathbb{Z}$, mD is Cartier.

For such D, we can define $D \cdot Z$ for any cycle $[Z] \in A_k(X)$ by pushing forward

$$D|_Z := \frac{1}{m} c_1\left(\mathcal{O}(mD)|_Z\right).$$

This recovers Mumford's construction, since a Cartier divisor D satisfies

$$\epsilon^* D \cdot E = 0.$$

More generally, Chern classes of vector bundles can be defined and pulled back via morphisms of schemes. The correct setting for a ring structure on the full Chow group is that of smooth algebraic stacks.

A category fibered in groupoids (CFG) $p : \mathfrak{X} \to Sch/k$ is the same as a 2-functor from $(Sch/k)^{op} \to Grpd$, sending T to $p^{-1}(T)$, the subcategory of $\mathfrak{X}$ whose objects lie over T and whose morphisms lie over id_T. Any scheme T has an associated CFG given by its functor of points, which is actually valued in Set, and we will abuse notation slightly by referring to this CFG as T also.

Lemma. (2-Yoneda) For $T \in Sch/k$, there is an equivalence of groupoids between

$$\mathrm{Hom}(T, \mathfrak{X}) \simeq p^{-1}(T)$$

For this reason, we often write $\mathfrak{X}(T)$ for $p^{-1}(T)$.

Definition. An algebraic (or Artin) stack $\mathfrak{X}$ over k is a CFG $p : \mathfrak{X} \to Sch/k$, and a topos on Sch/k (usually fppf), such that

1. For $x, y \in \mathfrak{X}(U)$, the presheaf $\mathrm{Isom}(x, y)(V) := \mathrm{Isom}(x|_V, y|_V)$ is a sheaf.
2. Every descent datum is effective, that is for every collection of objects $x_i \in \mathfrak{X}(U_i)$ with isomorphisms $f_{ij} : x_i|_{U_{ij}} \to x_j|_{U_{ij}}$ satisfying the cocycle condition, there is an object $x \in \mathfrak{X}(U)$ and isomorphisms $f_i : x|_{U_i} \to x_i$.
3. The diagonal $\Delta : \mathfrak{X} \to \mathfrak{X}\times\mathfrak{X}$ is representable, separated, and quasi-compact.
4. There exists a smooth epimorphism $Y \to \mathfrak{X}$ from a scheme (called an atlas).

The typical examples one should have in mind for Artin stacks are coherent sheaves on a scheme - $\mathrm{Coh}(X)$, curves of arithmetic genus g - $\mathfrak{M}_g$, and $BG = [pt/G]$ for G an algebraic group. A motivating example is $B\mathbb{G}_m$ which classifies line bundles. It is easy to find two line bundles which are isomorphic (both trivial) on every subset in a covering U_i, but non-isomorphic on U. This highlights the subtle necessity of the descent axiom.

Axioms (1) and (2) assert that $\mathfrak{X}$ is a 2-sheaf on the site Sch/k; isomorphisms glue, and objects 2-glue. Axiom (3) implies that any morphism from a scheme to $\mathfrak{X}$ is representable, via the Magic Square (below), which is Cartesian in any category with fibered products. This allows us to describe maps from schemes to $\mathfrak{X}$.

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \times Y \arrow[d] \\ Z \arrow[r, "\Delta"] & Z \times Z. \end{tikzcd}$$

Representable morphisms between stacks can be modified by any adjective that modifies morphisms of schemes, is local on both the domain and target, and is preserved under base change (e.g. flat, smooth, finite, étale). Hence, Axiom (4) only makes sense given Axiom (3).

Definition. An algebraic stack is Deligne-Mumford (DM) if it satisfies the sheaf conditions (1) and (2) above with respect to the *étale* topos, and furthermore the following equivalent conditions are satisfied:

1. The diagonal $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is unramified (and quasi-finite).
2. There exists an étale epimorphism $Y \to \mathfrak{X}$ from a scheme (called an atlas).

The typical examples one should have in mind for Deligne-Mumford stacks are smooth curves - $\mathcal{M}_g$, stable curves - $\overline{\mathcal{M}}_g$, and $BG = [pt/G]$ for G a finite group. Any scheme S is a stack (via its functor of points).

NB: The hypothesis on Δ for a DM stack is slightly weaker than the hypothesis of the Keel-Mori theorem (existence of coarse spaces) which was finiteness of inertia stack $\mathcal{I}_{\mathfrak{X}} \to \mathfrak{X}$, defined as $\mathcal{I}_{\mathfrak{X}} = \mathfrak{X} \underset{\mathfrak{X}\times\mathfrak{X}}{\times} \mathfrak{X}$, but this distinction rarely matters.
Objects of $\mathcal{I}_{\mathfrak{X}}$ are pairs of isomorphisms from $x \to y$. This category is equivalent to one whose objects are pairs (x, ϕ) where ϕ is an automorphism of x.

Absolute properties of $\mathfrak{X}$ like being reduced or finite type are defined from the atlas scheme Y. Smoothness and properness can also be defined via lifting criteria.

Definition. A stack $\mathfrak{X}$ is *proper* if for every discrete valuation ring R with fraction field K, any morphism $\operatorname{Spec} K \to \mathfrak{X}$ extends to $\operatorname{Spec} R \to \mathfrak{X}$.

Definition. A stack $\mathfrak{X}$ is *smooth* if for every Artinian ring A with square-zero ideal I, any morphism $\operatorname{Spec} A/I \to \mathfrak{X}$ extends to $\operatorname{Spec} A \to \mathfrak{X}$.

Theorem. (Vistoli) Deligne-Mumford stacks have Chow groups and, if smooth, an intersection product defined over $\mathbb{Q}$. The product can be defined directly on the coarse space. This gives a $\mathbb{Q}$-intersection theory for all Q-varieties, i.e. varieties which are étale locally the quotient of a smooth variety by a finite group.

Theorem. (Kresch) Artin stacks have Chow groups. If they are smooth and (1) Deligne-Mumford, or (2) stratified by global quotients stacks, then there is an intersection product defined over $\mathbb{Z}$.

Vistoli uses the naive definition of Chow groups:

$$Z(\mathfrak{X}) := \mathbb{Z}\{\text{integral closed substacks of } \mathfrak{X}\}$$

A closed substack is simply a representable morphism $\mathfrak{Y} \to \mathfrak{X}$ which is a closed embedding. A rational function on an integral stack $\mathfrak{W}$ is a morphism from an open substack to $\mathbb{A}^1_k$. This leads to the notion of rational equivalence:

$$\operatorname{Rat}(\mathfrak{X}) := \bigoplus_{\mathfrak{W}} k^*(\mathfrak{W});$$
$$A_*(\mathfrak{X}) := Z(\mathfrak{X})/\partial \operatorname{Rat}(X).$$

A dominant morphism of integral stacks has a well-defined *degree* valued in $\mathbb{Q}$, defined in terms of extensions of rational function fields on étale atlases. For example,

$$\deg([pt/G] \to pt) = \frac{1}{|G|}$$

Representable morphisms have integral degree. This leads to a definition of proper pushforward. The main technical challenge is to define the Gysin pullback without using deformation to the normal cone. Vistoli defines pull-backs through finite type, unramified maps (called local embeddings). The diagonal morphism $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is a local embedding.

Kresch uses an idea of Graham-Eddidin (next lecture), who defined equivariant Chow groups $A_*^G(X) = A_*[X/G]$ for any G-variety X by considering $(X \times V)/G$ for sufficiently large G-representations V. The G-representations are replaced with arbitrary vector bundles on the Artin stack $\mathfrak{X}$.

Definition. A quasi-coherent sheaf on a stack $\mathfrak{X}$ (with structure functor $p : \mathfrak{X} \to Sch/k$) consists of the following data. For every object $x \in \mathfrak{X}$, a sheaf $\xi(x)$ on the scheme $p(x)$, and for every morphism $f : x \to y$ over $F : p(x) \to p(y)$, an isomorphism $F^*\xi(x) \to \xi(y)$, satisfying the obvious compatibility for compositions.

Example. A vector bundle on BG is the same as a G-representation. Given a vector bundle $\mathcal{V}$ on BG, consider its value on the trivial G-bundle over a point. This is a vector space V, with an action of G coming from automorphisms of the trivial G-bundle. Conversely, given a G-representation V, we can construct a vector bundle on B for every principal G-bundle $P \to B$ by forming

$$(V \times P)/G.$$

Let $B_{\mathfrak{X}}$ denote the category of vector bundles E on $\mathfrak{X}$ with the partially ordering $E \leq F$ if there exists $F \twoheadrightarrow E$. Kresch defines:

$$\hat{A}_k(\mathfrak{X}) := \lim_{\to B_{\mathfrak{X}}} A_{k+\mathrm{rk}(E)}(E).$$

The natural morphism $A_k(\mathfrak{X}) \to \hat{A}_k(\mathfrak{X})$ is an isomorphism when $\mathfrak{X}$ is a scheme. Next, we take a further enlargement to obtain $A_k^{\mathrm{Kr}}(\mathfrak{X})$, which satisfies all the axioms. The issue is that while $\mathfrak{X}$ may have no non-trivial vector bundles, it may contain a global quotient stack, which has many. Let $U_{\mathfrak{X}}$ be the set of projective morphisms $f : \mathfrak{Y} \to \mathfrak{X}$, partially ordered by inclusions of components.

$$A_k^{\mathrm{Kr}}(\mathfrak{X}) = \lim_{\to U_{\mathfrak{X}}} \hat{A}_k(\mathfrak{Y})/\hat{B}_k(\mathfrak{Y}),$$

where $\hat{B}_k(\mathfrak{Y})$ is a union over all stacks $\mathfrak{T}$ with pairs of projective morphisms $p_1, p_2 : \mathfrak{T} \to \mathfrak{Y}$ such that $f \circ p_1 \simeq f \circ p_2$ of the following abelian group:

$$\{p_{1*}\beta_1 - p_{2*}\beta_2 | \, (\beta_1, \beta_2) \in \hat{A}_k^{p_1}(\mathfrak{T}) \oplus \hat{A}_k^{p_2}(\mathfrak{T}) \text{ satisfies } \iota_{p_1}(\beta_1) = \iota_{p_2}(\beta_2)\}.$$

To make sense of this, we need to define

$$\hat{A}_k^p(\mathfrak{T}) := \lim_{\to B_{\mathfrak{Y}}} A_{k+\mathrm{rk}(E)}(p^*E); \qquad \iota_p : \hat{A}_k^p(\mathfrak{T}) \to \hat{A}_k(\mathfrak{T}).$$

There is a natural sequence of morphisms

$$A_k(\mathfrak{X}) \to \hat{A}_k(\mathfrak{X}) \to A_k^{\mathrm{Kr}}(\mathfrak{X}).$$

The last one is an isomorphism when $\mathfrak{X}$ a global quotient. The composite is an isomorphism over $\mathbb{Q}$ when $\mathfrak{X}$ is smooth and Deligne-Mumford.

11. 3/9/20 - Quotient Stacks

Let G be an algebraic group. Recall that a coherent sheaf on $BG = [pt/G]$ is the same as a G-representation. We can define $K_0(BG)$ as the Grothendieck group of the abelian category of G-representations, with multiplication given by tensor product. This ring is sometimes called the *representation ring*, $\mathrm{Rep}(G)$.

If $H \subset G$ is a subgroup, there is a morphism of stacks $BH \to BG$. The pullback on K_0 is the restriction of representations, and the pushforward on K_0 is the induction of representations.

More generally, if X is a smooth G-variety, we have the stack quotient $[X/G]$. A coherent sheaf on $[X/G]$ is the same as a G-equivariant sheaf on X. We define $K_0[X/G]$ to be the Grothendieck group of G-equivariant vector bundles on X. Our goal is to formulate an equivariant version of Grothendieck-Riemann-Roch.

Theorem-Definition. (Totaro) Let V be a G-rep such that the action is free outside a closed subset S of codimension $s > i$. Define the Chow groups of BG as:

$$A^i(BG) := A^i((V - S)/G).$$

The definition is independent of the choice of S because if $S' \supset S$ is a larger subset, then we have the localization sequence

$$A_*((S' - S)/G) \to A_*((V - S)/G) \to A_*((V - S')/G) \to 0.$$

For $* > \dim(S') - \dim(G) = n - s - \dim(G)$, the first group vanishes, so the second two groups are isomorphic. Matching the dimension notation with codimension notation in the statement above,

$$* = n - \dim(G) - i > n - s - \dim(G).$$

This is equivalent to $i < s$. The definition is independent of the representation V as well; suppose we have chosen (V, S_V) and (W, S_W) which both satisfy the conditions. Then we have two vector bundles.

$$((V - S_V) \times W)/G \to (V - S_V)/G, \qquad (V \times (W - S_W))/G \to (W - S_W)/G$$

By S-independence applied to $V \oplus W$ with $S_V \times W$ versus $V \times S_W$, the total spaces of the bundles have isomorphic Chow groups A^i, and this implies that the base spaces also have isomorphic Chow groups too. To show that G-representations V with S of large codimension always exist, take any faithful representation V_0 of G of dimension m, and set

$$V = \mathrm{Hom}(k^{m+N}, V_0),$$

for $N \gg 0$, and S the locus of non-surjective linear maps. Note that in this case, $(V - S)/G$ is actually quasi-projective! $\square$

Definition. (Edidin-Graham) For a smooth G-variety X, choose V and S as before, and define the G-equivariant Chow groups as:

$$A^i[X/G] := A^i((X \times (V - S))/G).$$

Theorem. (Edidin-Graham) The intersection product on $A^*(X/G)$ is well-defined.

If $E \to X$ is a G-equivariant vector bundle, we can form

$$(E \times (V - S))/G \to (X \times (V - S))/G$$

which is an ordinary vector bundle, so we have G-equivariant chern classes

$$c_i^G(E) \in A^i[X/G].$$

We now have all the necessary pieces to formulate the equivariant GRR theorem. The wrinkle is that we must take the completions with respect to augmentation ideals for the theorem to work. This is essentially because $A^i(BG)$ can be nonzero for i arbitrarily large.

Theorem. (Edidin-Graham) The map $\tau(V) = \mathrm{ch}(V) \cdot \mathrm{Td}(T_X - \mathfrak{g})$, where $\mathfrak{g}$ is the trivial bundle with the adjoint action of G, defines a ring homomorphism

$$\tau : K_0[X/G] \to \hat{A}^*[X/G] \otimes \mathbb{Q}$$

which factors through an isomorphism

$$\hat{\tau} : \hat{K}_0[X/G] \otimes \mathbb{Q} \to \hat{A}^*[X/G] \otimes \mathbb{Q}.$$

The completion of $K_0[X/G]$ is taken with respect to the kernel of $\mathrm{rk} : K_0[X/G] \to \mathbb{Z}$. The completion of $A^*[X/G]$ is taken with respect to the ideal of positive degree elements, so

$$\hat{A}^*[X/G] = \prod_{i \geq 0} A^i[X/G].$$

Theorem. (Equivariant GRR) The map τ is covariant for proper, representable morphisms of stacks which are global quotients.

12. 3/11/20 - Applications of Equivariant GRR

Example. Consider the case of $B\mathbb{C}^\times$. The K_0 ring is

$$\mathrm{Rep}(\mathbb{C}^\times) \simeq \mathbb{Z}[t, t^{-1}]$$

The Chow ring is $\mathbb{Z}[h]$ because $B\mathbb{C}^\times$ is approximated by projective spaces $\mathbb{P}^l$, $l \gg 0$. The tangent bundle is trivial, and $\mathfrak{g}$ is a trivial representation so $\tau : \mathbb{Z}[t, t^{-1}] \to \mathbb{Q}[\![h]\!]$ is simply the Chern character: $\tau(t) = \exp(h)$. This does NOT descend to a ring isomorphism $\mathbb{Q}((t)) \to \mathbb{Q}[\![h]\!]$. If instead we complete $\mathbb{Q}[t, t^{-1}]$ at the ideal $(t-1)$, then we do get an isomorphism:

$$\hat{\tau} : \mathbb{Q}[\![u]\!] \to \mathbb{Q}[\![h]\!]$$
$$u \mapsto \exp(h) - 1 = h + \frac{h^2}{2} + \frac{h^3}{3!} + \dots$$

Example. Consider the case of $[\mathbb{P}^n/\mathbb{C}^\times]$, where the action of $\mathbb{C}^\times$ has weights $(a_0, \dots, a_n)$. The approximations to Chow are given by $(\mathbb{P}^n \times (\mathbb{C}^l - 0))/\mathbb{C}^\times \to \mathbb{P}^{l-1}$, which is isomorphic to:

$$\mathbb{P}(\mathcal{O}(a_0) \oplus \cdots \oplus \mathcal{O}(a_n)) \to \mathbb{P}^{l-1}$$

The Chow ring of this variety is given by:

$$\mathbb{Z}[h, \zeta]/(h^l, \zeta^{n+1} + \sigma_1 \zeta^n + \cdots + \sigma_{n+1}),$$

where σ_i is the ith symmetric polynomial in the variables $a_0 h, a_1 h, \dots, a_n h$. As we send l to infinity, we are left with

$$\mathbb{Z}[h, \zeta]/(\zeta^{n+1} + \sigma_1(a) h \zeta^n + \cdots + h^{n+1} \sigma_{n+1}(a)).$$

In particular, if $\mathbb{C}^\times$ acts on $\mathbb{P}^1$ with weights ± 1, we get $\mathbb{Z}[h, \zeta]/(\zeta^2 - h^2)$.

Theorem. (Weyl) Let G be a semisimple Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra. The irreducible representations Π of G are classified by their highest weight, λ. For such an irrep, the character of $e^H \in T$ is given by

$$\mathrm{ch}_\Pi(e^H) = \frac{\sum_{w \in W} \epsilon(w)\, e^{w(\lambda+\rho)(H)}}{\prod_{\alpha \in \Delta^+} \left(e^{\alpha(H)/2} - e^{-\alpha(H)/2}\right)}.$$

The character formula is enough to determine ch_Π on all of G because any semisimple element can be conjugated into T, and semisimple elements are Zariski dense. Setting $H = 0$, we recover the dimension formula for irreps.

Proof. In the case of SL_2, the torus is $\mathbb{C}^\times$. The highest weight is a nonnegative integer n, and the representation can be realized as $H^0(\mathbb{P}^1, \mathcal{O}(n))$. Weyl's formula in this case reads

$$\operatorname{tr} \pi_n \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin((n+1)\theta)}{\sin(\theta)}$$

This will be a consequence of EGRR for the action of $\mathbb{C}^\times$ on $\mathbb{P}^1$ with weights ± 1.

$$\begin{tikzcd}
K_0[\mathbb{P}^1/\mathbb{C}^\times] \arrow[r, "\chi^{\mathbb{C}^\times}"] \arrow[d, "\tau"'] & K_0[pt/\mathbb{C}^\times] \arrow[d, "\tau"] \\
\hat{A}^*_{\mathbb{Q}}[\mathbb{P}^1/\mathbb{C}^\times] \arrow[r, "\pi_*"] & \hat{A}^*_{\mathbb{Q}}[pt/\mathbb{C}^\times]
\end{tikzcd}$$

The equivariant Euler characteristic of $\mathcal{O}_{\mathbb{P}^1}(n)$ is an element of $K_0(B\mathbb{C}^\times) = \mathbb{Z}[t, t^{-1}]$. Since $\mathcal{O}_{\mathbb{P}^1}(n)$ has no higher cohomology, we get the representation $\Pi_n = H^0(\mathbb{P}^1, \mathcal{O}(n))$.

$$\tau(\Pi_n) = \pi_*(\mathrm{ch}(\mathcal{O}_{\mathbb{P}^1}(n)) \cdot \mathrm{Td}(\mathbb{P}^1)) = \pi_*\left(e^{n\zeta}\frac{2\zeta}{1-e^{-2\zeta}}\right)$$

Lemma. For $p(\zeta) \in A^*_{\mathbb{Q}}[\mathbb{P}^1/\mathbb{C}^\times] = \mathbb{Q}[h,\zeta]/(\zeta^2 - h^2)$,

$$\pi_*(p(\zeta)) = \frac{p(h) - p(-h)}{2h}$$

and the same is true after completion.

Proof. From the $\mathbb{P}^1$-bundle model, any polynomial in h only gets killed by π_*. Since $\zeta^2 = h^2$, all even degree terms in $p(\zeta)$ go to 0, whereas an odd degree term $a_{2k+1}\zeta^{2k+1} = a_{2k+1}\zeta h^{2k}$ goes to $a_{2k+1}h^{2k}$. $\square$

Now we compute using the lemma:

$$\tau(\Pi_n) = \pi_*\left(2\zeta\frac{e^{(n+1)\zeta}}{e^{\zeta} - e^{-\zeta}}\right) = \frac{e^{(n+1)h} - e^{-(n+1)h}}{e^h - e^{-h}}.$$

Note that τ is the sum of exponentials of the weights, under the diagonalization isomorphism (which holds for any torus T):

$$\mathrm{Rep}(T) \otimes \mathbb{C} \simeq \mathbb{C}[T].$$

The Weyl character formula for general G can be proven in a similar way, using an action of a maximal torus $T \subset B \subset G$ on the flag variety G/B (exercise). $\square$

The morphism $[X/G] \to pt$ is not representable, so GRR does not apply directly. However, there is a generalization of Hirzebruch-Riemann-Roch in the case when $[X/G]$ is proper and Deligne-Mumford. We would like a formula for the Euler characteristic of a bundle V on $[X/G]$ in terms of intersection numbers (degrees of 0-cycles) coming from Chern characters and Todd classes. A 0-dimensional substack of $[X/G]$ is a *gerbe* for a finite group G_x, and we set:

$$\deg(BG_x) = 1/|G_x|.$$

Occasionally we write deg as an integral over $[X/G]$, in analogy with cohomology.

Definition. Let V be a vector bundle on $[X/G]$ a proper DM stack. Define

$$\chi([X/G], V) := \sum_i (-1)^i \dim H^i(X,V)^G.$$

This is a finite sum because if $m : [X/G] \to M$ is the coarse space map, then

$$H^i(X,V)^G = H^i(M, p_*V).$$

Remark. This definition of χ is the correct definition of pushforward to pt because applying K_0 to $pt \to BG \to pt$, we get

$$\mathbb{Z} \to Rep(G) \to \mathbb{Z}$$
$$1 \mapsto \mathbb{C}[G] \to 1.$$

13. 3/30/20 - Localization Theorem in Equivariant K-theory

Our goal is to prove a version of Hirzebruch-Riemann-Roch for smooth proper DM stacks which are global quotients $[X/G]$. This will allow us e.g. to compute e.g. the dimensions of spaces of classical modular forms. Along the way, we will gain a better understanding of equivariant K-theory in general.

Recall that $K_0[X/G]$ is a ring, but it also a module over $K_0[pt/G]$ via the pullback through $[X/G] \to [pt/G]$. Geometrically, $K_0[X/G]_{\mathbb{C}}$ is a coherent sheaf over $\operatorname{Spec} K_0[pt/G]_{\mathbb{C}}$. If G is a diagonalizable group (subgroup of a torus), then

$$\operatorname{Spec} K_0[pt/G]_{\mathbb{C}} = \operatorname{Spec} \operatorname{Rep}(G)_{\mathbb{C}} \simeq G.$$

For a more general linear algebraic group,

$$\operatorname{Spec} K_0[pt/G]_{\mathbb{C}} = \operatorname{Spec} \operatorname{Rep}(G)_{\mathbb{C}} \simeq G /\!\!/_{\mathrm{ad}} G,$$

whose points are conjugacy classes of semisimple elements of G. Both of these isomorphisms send a representation to its character. The augmentation ideal in $\operatorname{Rep}(G)_{\mathbb{C}}$ corresponds to the identity element $1 \in G$, since the trace 1 recovers the dimension of a representation.

Lemma. (Thomason) If $h \in G$ does not fix any points in X, then the localization

$$(K_0[X/G]_{\mathbb{Q}})_h = 0.$$

If $[X/G]$ is Deligne-Mumford, then only finite order elements $h \in H$ can have fixed points in X. This implies that $K_0[X/G]_{\mathbb{C}}$ is supported over *finitely many* points h_i:

$$K_0[X/G]_{\mathbb{Q}} \simeq \prod_{i=1}^{k} (K_0[X/G]_{\mathbb{Q}})_{h_i}.$$

Theorem. (Edidin-Graham) If $[X/G]$ is proper DM and $\alpha \in K_0[X/G]_{\mathbb{Q}}$, then

$$\deg \operatorname{ch}(\alpha)\operatorname{Td}(T_X - \mathfrak{g}) = \chi([X/G], \alpha_1),$$

where α_1 is the part of α over the augmentation ideal (called 1).

Proof. EGRR says that τ is an isomorphism after taking the completion $\hat{K}_0$ at 1, but τ it is only covariant for representable morphisms. There exists a smooth G-variety X' with a finite G-equivariant surjection $f : X' \to X$ such that $[X'/G]$ is represented by a smooth variety. Now use the covariance of τ and the surjectivity of f_* on Chow groups, which implies surjectivity of f_* on $\hat{K}_0$:

$$\begin{tikzcd}
K_0(X'/G)_{\mathbb{Q}} \arrow[r, "f_*"] \arrow[d, "\sim"'] & \hat{K}_0[X/G]_{\mathbb{Q}} \arrow[r] \arrow[d, "\tau"', "\sim"] & K_0(pt)_{\mathbb{Q}} = \mathbb{Q} \arrow[d, "\sim"'] \\
A^*(X'/G)_{\mathbb{Q}} \arrow[r, "f_*"] & \hat{A}^*[X/G]_{\mathbb{Q}} \arrow[r] & A^*(pt)_{\mathbb{Q}} = \mathbb{Q}.
\end{tikzcd}$$

The left square commutes by EGRR, and large square commutes by HRR. □

To compute the full $\chi([X/G], \alpha)$, we must add up contributions from the remaining *sectors*, closed points in the support of $K_0[X/G]_{\mathbb{Q}}$ over $\operatorname{Rep}(G)_{\mathbb{Q}}$. Each contribution will be the degree of a Hirzebruch-Riemann-Roch type expression. Our main tools will be localization in K-theory, followed by twist operators.

Theorem. (Thomason) Let G be a diagonalizable group acting on a scheme X.

$$(K_0[X/G])_h \simeq \left(K_0[X^h/G]\right)_h$$

as modules over $K_0[pt/G] \simeq \mathrm{Rep}(G)$. This is referred to as the *localization theorem.*

Proof. We start by proving it in the case of $\mathbb{C}^\times$ acting on $\mathbb{P}^n$ with weights $(a_0, a_1, \ldots, a_n)$. The general case can be reduced to this one. The equivariant K-theory of projective space is given in terms of $x = [\mathcal{O}(1)]$ by

$$K_0[\mathbb{P}^n/\mathbb{C}^\times] = \mathbb{Z}[t, t^{-1}][x]/\prod_{i=0}^{n}(x - t^{-a_i}).$$

To see this, a version of the Hilbert syzygy theorem for multi-graded modules implies that $\mathbb{Z}[t, t^{-1}, x, x^{-1}]$ generate K_0. To check the relation, consider the equivariant tautological sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{n+1} \to Q \to 0$$

which implies that $x \cdot \sum t^{a_i} = 1 + qx$, where $q = [Q]$. Now apply the λ operation on K-theory (which sends sums to products):

$$\lambda(E) := \sum_{i=0}^{\mathrm{rk} E}(-1)^i[\wedge^i E].$$

The equation above becomes

$$\prod_{i=0}^{n} \lambda(t^{a_i}x) = 0 \cdot \lambda(qx)$$

$$\prod_{i=0}^{n}(1 - t^{a_i}x) = 0$$

as desired (up to units). To check that $1, x, x^2, \ldots, x^n$ are independent, simply apply χ (the ordinary Euler characteristic) to a linear relation, multiplied by x^{-m} to make all but one of the exponents negative.

Exercise. Check that the equivariant Chern character

$$\mathbb{Z}[t, t^{-1}][x]/\prod_{i=0}^{n}(x - t^{-a_i}) \to \mathbb{Q}[\![h, \zeta]\!]/\prod_{i=0}^{n}(\zeta + a_i h)$$

is well-defined (send $t \mapsto e^h$ and $x \mapsto e^\zeta$).

Now, the localization theorem follows from geometry! If the weights are distinct and h is general, then there are $n + 1$ fixed points, so

$$K_0[(\mathbb{P}^n)^h/\mathbb{C}^\times] = \prod_{i=0}^{n} K_0(B\mathbb{C}^\times) = \prod_{i=0}^{n} \mathbb{Z}[t, t^{-1}]$$

If $m > 1$ weights coincide, then there is a $\mathbb{P}^{m-1}$ in the fixed locus which contributes

$$K_0(B\mathbb{C}^\times) \otimes \mathbb{Z}[x]/(x-1)^m = \mathbb{Z}[t, t^{-1}][x]/(x-1)^m.$$

When you localize at h, the two Artinian rings match geometrically (graph $x = t^{-a}$), and this correspondence can be refined for h a root of unity.

To go from the case of projective space to a general smooth projective X, first embed $X \hookrightarrow \mathbb{P}^n$ equivariantly (by choosing a G-linearized very ample line bundle). Pulling back give ring homomorphisms

$$K_0[(\mathbb{P}^n \smallsetminus (\mathbb{P}^n)^h)/G] \to K_0[(X \smallsetminus X^h)/G]$$
$$K_0[(\mathbb{P}^n \smallsetminus (\mathbb{P}^n)^h)/G]_h \to K_0[(X \smallsetminus X^h)/G]_h.$$

Since $K_0[(\mathbb{P}^n \smallsetminus (\mathbb{P}^n)^h)/G]_h = 0$ by localization for $\mathbb{P}^n$, $K_0[(X \smallsetminus X^h)/G]_h = 0$ as well - this is Thomason's Lemma above for $\mathbb{P}^n$. Next we use the excision sequence for K-theory, and the fact that localization is an exact functor:

$$\cdots \to K_1[(X \smallsetminus X^h)/G] \to K_0[X^h/G] \to K_0[X/G] \to K_0[(X \smallsetminus X^h)/G] \to 0$$
$$\cdots \to K_1[(X \smallsetminus X^h)/G]_h \to K_0[X^h/G]_h \to K_0[X/G]_h \to K_0[(X \smallsetminus X^h)/G]_h \to 0.$$

We have not defined K_1 yet, but it is a module over K_0. Since the latter vanishes in this case, the former also vanishes, and we get the full localization theorem. $\square$

Explicitly, the localization theorem isomorphism is induced by the pullback i^* for

$$i : [X^h/G] \to [X/G]$$

the inclusion. Using the flat deformation to the normal cone, we know that

$$i^* i_*(\beta) = \lambda(N^\vee) \cdot \beta,$$

where $N = N_{X^h/X}$ is the normal bundle, so the inverse of the isomorphism in the theorem is given by

$$\beta \mapsto i_* \left(\frac{\beta}{\lambda(N^\vee)} \right).$$

Remark. The class $\lambda(N^\vee)$ is invertible in $\left(K_0[X^h/G]\right)_h$ because it is a product over weights complementary to the fixed locus weights.

Using the inverse formula, along with functoriality of $[X^h/G] \to [X/G] \to pt$, we find that for $\alpha_h \in (K_0[X/G])_h$,

$$\chi([X/G], \alpha_h) = \chi \left([X^h/G], \frac{i^* \alpha_h}{\lambda(N^\vee)} \right).$$

14. 4/1/20 - Twisted Sectors

Let G be a diagonalizable group. Consider a G-variety Y such that $h \in G$ acts trivially, and assume that h has finite order, so we have $H = \langle h \rangle \simeq \mathbb{Z}/m$. Given a G-equivariant vector bundle V on Y, it has a direct sum decomposition

$$V \simeq \bigoplus_{\xi \in H^\vee} V_\xi$$

over characters of H. In other words, $K_0[Y/H] \simeq \mathrm{Rep}(H) \otimes K_0(Y)$.

Definition. The twist operator $\mathrm{tw}_h : K_0[Y/G] \to K_0[Y/G]$ is defined by

$$[V] \mapsto \sum_{\xi \in H^\vee} \xi(h)[V_\xi].$$

Proposition. The twist operator tw_h sends

$$(K_0[Y/G])_h \to (K_0[Y/G])_1 .$$

Proof. The pull-back morphisms $K_0[pt/G] \to K_0[Y/G] \to K_0[Y/H]$ are both tw_h-equivariant. Geometrically, the twist can be viewed as the multiplication by h^{-1} from $H \to H$. Indeed, if $f \in \mathbb{C}[H]$, then $f = \sum a_i \xi_i$, then we have

$$\mathrm{tw}_h f(g) = \sum c_i \mathrm{tw}_h \xi_i(g) = \sum c_i \xi_i(h)\xi_i(g) = \sum c_i \xi_i(hg) = f(hg).$$

If f vanishes at a point g, then $\mathrm{tw}_h f$ vanishes at $h^{-1}g$. $\square$

The other key property of tw_h is that it preserves G-invariants, so it preserves the Euler characteristic that we seek. Let Y be the fixed locus X^h from last time. The operator tw_h moves the sector at h to the sector at 1 so that we can apply the Hirzebruch-Riemann-Roch theorem. Recall that $\alpha \in K_0[X/G]$ decomposes into components $\alpha_h \in (K_0[X/G])_h \simeq \left(K_0[X^h/G]\right)_h$ and χ is additive, so we have:

Theorem. If $[X/G]$ is proper DM and $\alpha \in K_0[X/G]_\mathbb{Q}$, then

$$\chi([X/G], \alpha) = \sum_{h \in \mathrm{supp}\, K_0[X/G]} \int_{[X^h/G]} \mathrm{ch}\left(\mathrm{tw}_h\left(\frac{i^*\alpha_h}{\lambda(N^\vee_{X^h/X})}\right)\right) \mathrm{Td}(T_{X_h} - \mathfrak{g})$$

There is a slicker way to write this formula as a single integral over the inertia stack $\mathcal{I}_{[X/G]}$ if you care to unwind the statement. Let

$$I(X, G) = \{(x, h) : hx = x\} \subset X \times G,$$

with the G-action given by $g \cdot (x, h) = (gx, ghg^{-1})$. The inertia stack is a global quotient in this case: $\mathcal{I}_{[X/G]} = [I(X, G)/G]$. Note that $I(X, G)$ admits a finite G-equivariant decomposition into connected components indexed by $h \in \mathrm{supp}\, K_0[X/G]$, so one can define a global twist operator tw.

Theorem. (Edidin) If $\mathfrak{X} = [X/G]$ is proper DM, $\alpha \in K_0[X/G]_\mathbb{Q}$, and $f : \mathcal{I}_\mathfrak{X} \to \mathfrak{X}$ is the natural morphism from the inertia stack, then

$$\chi(\mathfrak{X}, \alpha) = \int_{\mathcal{I}_\mathfrak{X}} \mathrm{ch}\left(\mathrm{tw}\left(\frac{f^*\alpha}{\lambda(N_f^\vee)}\right)\right) \mathrm{Td}(\mathcal{I}_\mathfrak{X}).$$

Example. Consider the stack $\mathbb{P}(4,6) = (\mathbb{C}^2 \smallsetminus \{0\})/\mathbb{C}^\times$, acting with weights $(4,6)$. To compute K_0, we use

$$K_0[\{0\}/\mathbb{C}^\times] \to K_0[\mathbb{C}^2/\mathbb{C}^\times] \to K_0(\mathbb{P}(4,6)) \to 0.$$

The first two terms are both isomorphic to $\mathbb{Z}[t,t^{-1}]$, since all vector bundles on $\mathbb{C}^2$ are trivial. The pushforward is multiplication by

$$\lambda(T_0^\vee) = (1-t^{-4})(1-t^{-6}).$$

Hence, $K_0(\mathbb{P}(4,6)) \simeq \mathbb{Z}[t,t^{-1}]/(t^4-1)(t^6-1)$. As a sheaf over $\operatorname{Spec}\mathbb{Z}[t,t^{-1}]$ it is supported at $\pm 1, \pm i, \eta = e^{\pm \pi i/3}, \omega = e^{\pm 2\pi i/3}$. Using the formula above,

$$\chi(\mathbb{P}(4,6), t^k) = \frac{1}{24}(k+5) + \frac{(-1)^k}{24}(k+5) + \frac{i^k + (-i)^k}{8} + \frac{\omega^k + \omega^{-k} + \eta^k + \eta^{-k}}{12}.$$

This recovers the dimension formula for spaces of classical modular forms because $\mathbb{P}(4,6)$ is isomorphic to the moduli space of elliptic curves, $\overline{M}_{1,1}$. In this case, $\chi(t^k) = h^0(t^k)$ because the higher cohomologies vanish.

$$\dim M_k(SL_2(\mathbb{Z})) = \begin{cases} 0 & k \text{ odd} \\ \lfloor k/12 \rfloor + 1 & k \not\equiv 2(12) \\ \lfloor k/12 \rfloor & k \equiv 2(12). \end{cases}$$

15. 4/6/20 - Bundles of Principal Parts

Let X be a smooth projective variety, and $\mathcal{E}$ locally free sheaf on X. For $k \geq 0$, it is natural to seek a vector bundle $P^k(\mathcal{E})$ whose fiber at a point $x \in X$ is given by

$$\begin{aligned} P^k(\mathcal{E})_x &= H^0(\mathcal{E} \otimes \mathcal{O}_{X,x}/\mathfrak{m}_x^{k+1}) \\ &= \{\text{germs of sections of } \mathcal{E} \text{ at } x\}/\{\text{those vanishing to order } k+1 \text{ at } x\}. \end{aligned}$$

This can be done globally by thickening the diagonal $\Delta_X \subset X \times X$:

Definition. $P^k(\mathcal{E}) = pr_{2*}\left(pr_1^*\mathcal{E} \otimes \mathcal{O}_{X\times X}/\mathcal{I}_\Delta^{k+1}\right)$.

Note that $P^0(\mathcal{E}) = \mathcal{E}$ itself. Consider the short exact sequence

$$0 \to \mathcal{I}_\Delta^k/\mathcal{I}_\Delta^{k+1} \to \mathcal{O}_{X\times X}/\mathcal{I}_\Delta^{k+1} \to \mathcal{O}_{X\times X}/\mathcal{I}_\Delta^k \to 0,$$

tensor with $pr_1^*\mathcal{E}$, and then apply pr_{2*} to get

$$0 \to \mathcal{E} \otimes \mathrm{Sym}^k(\Omega_X) \to P^k(\mathcal{E}) \to P^{k-1}(\mathcal{E}) \to 0.$$

We have used the fact that $\mathcal{I}_\Delta^k/\mathcal{I}_\Delta^{k+1} \simeq \mathrm{Sym}^k(\Omega_X)$. This allows us to compute the total Chern class inductively, using the Whitney sum formula:

$$c(P^k(\mathcal{E})) = \prod_{i=0}^{k} c\left(\mathcal{E} \otimes \mathrm{Sym}^i(\Omega_X)\right).$$

Remark. The short exact sequence for $k = 1$,

$$0 \to \mathcal{E} \otimes \Omega_X \to P^1(\mathcal{E}) \to \mathcal{E} \to 0,$$

gives us a naturally defined class $a(\mathcal{E}) \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_X) = H^1(\mathrm{End}(\mathcal{E}) \otimes \Omega_X)$. This is called the *Atiyah class* of $\mathcal{E}$, and its vanishing is equivalent to the existence of a holomorphic connection on $\mathcal{E}$ - the correct setting is for X a compact Kähler manifold. If $\mathcal{E} = \mathcal{L}$ is a line bundle, then $a(\mathcal{L}) \in H^1(\Omega_X) \simeq H^{1,1}(X, \mathbb{C})$, and $a(\mathcal{L}) = -2\pi i \cdot c_1(\mathcal{L})$. More generally, by the splitting principle

$$\mathrm{tr}(\wedge^k a(\mathcal{E})) = (-2\pi i)^k c_k(\mathcal{E}) \in H^k(\Omega_X^k).$$

If $a(\mathcal{E}) = 0$, then all the rational Chern classes of $\mathcal{E}$ vanish, but the converse is false; consider $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. When $\dim(X) = 1$, any holomorphic connection is integrable (flat). The Atiyah class is also important in deformation theory: given a first order deformation $\delta \in H^1(T_X)$ of X and a coherent sheaf $\mathcal{E}$ on X, the obstruction to deforming $\mathcal{E}$ with X is $\delta \cdot a(\mathcal{E}) \in H^2(\mathrm{End}(\mathcal{E}))$. Illusie generalized this to the case of singular varieties X, replacing Ω_X with $\mathbb{L}_X$.

Our first application of principal parts will be counting singular elements in linear series. A pencil of plane curves is given by a line $\mathbb{P}^1 \simeq \langle F, G\rangle \subset \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}(d)) \simeq \mathbb{P}^N$. A general pencil will contain curves with at worst ordinary double points, so we consider $P^1(\mathcal{O}_{\mathbb{P}^2}(d))$, a bundle of rank 3. The global sections F, G of $\mathcal{O}_{\mathbb{P}^2}(d)$ induce sections τ_F, τ_G of $P^1(\mathcal{O}_{\mathbb{P}^1}(d))$, and we want to count the points where the latter two sections become linearly dependent. This is given by c_2:

$$\deg c_2\left(P^1(\mathcal{O}_{\mathbb{P}^2}(d)\right) = 3(d-1)^2.$$

A similar computation with $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$ counts singular elements in a general pencil of hypersurfaces:

$$\deg c_n\left(P^1(\mathcal{O}_{\mathbb{P}^n}(d)\right) = (n+1)(d-1)^n.$$

This formula can alternatively be obtained by looking at the universal singularity $\Sigma \subset \mathbb{P}^n \times \mathbb{P}^N$, which is a complete intersection of $n+1$ hypersurfaces of bidegree $(d-1,1)$, and pushing it forward to $\mathbb{P}^N$.

Proposition. The total Chern class for the bundle $P^k(\mathcal{O}_{\mathbb{P}^n}(d))$ is given by

$$c\left(P^k(\mathcal{O}_{\mathbb{P}^n}(d))\right) = (1-(d-k)h)^{\binom{n+k}{n}}$$

Proof. The Euler sequence for $\mathbb{P}^n$ tensored with $\mathcal{O}_{\mathbb{P}^n}(d)$ reads:

$$0 \to \Omega_{\mathbb{P}^n}(d) \to \mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0.$$

The middle term is not isomorphic to $P^1(\mathcal{O}_{\mathbb{P}^n}(d))$, but they have the same Chern classes! This handles the case $k = 1$. We leave the general case as an exercise. $\square$

Proposition. Let $\mathcal{L}$ be a line bundle on X with $\dim(X) = n$. For a pencil in $|\mathcal{L}|$ such that each fiber has at most one singularity (a double point) lying outside the base locus, the number of singular fibers is given by:

$$\deg c_n\left(P^1(\mathcal{L})\right) = \deg \sum_{i=0}^{n}(i+1)c_1(\mathcal{L})^i c_{n-i}(\Omega_X).$$

Proof. Using the Whitney sum formula, $c(P^1(\mathcal{L})) = c(\mathcal{L})c(\mathcal{L}\otimes\Omega_X)$. The degree n part of the product becomes

$$\begin{aligned} c_n\left(P^1(\mathcal{L})\right) &= c_n(\mathcal{L}\otimes\Omega_X) + c_1(\mathcal{L})c_{n-1}(\mathcal{L}\otimes\Omega_X) \\ &= \sum_{i=0}^{n} c_1(\mathcal{L})^i c_{n-i}(\Omega_X) + c_1(\mathcal{L})\sum_{i=0}^{n-1}(i+1)c_1(\mathcal{L})^i c_{n-1-i}(\Omega_X) \\ &= \sum_{i=0}^{n}(i+1)c_1(\mathcal{L})^i c_{n-i}(\Omega_X). \quad \square \end{aligned}$$

Next, we can ask for the degree of the locus of plane curves with triple points, which has codimension 4 in $\mathbb{P}^N$. Consider $P^2(\mathcal{O}_{\mathbb{P}^2}(d))$, a bundle of rank 6. Take a general $\langle F_0, F_1, F_2, F_3, F_4\rangle \simeq \mathbb{P}^4 \subset \mathbb{P}^N$, and consider the associated sections τ_{F_i} of $P^2(\mathcal{O}_{\mathbb{P}^2}(d))$. The locus where they become linearly dependent has size:

$$\deg c_2\left(P^2(\mathcal{O}_{\mathbb{P}^2}(d))\right) = 15(d^2-4d+4).$$

For example, the locus of asterisks in the space of plane cubics ($\mathbb{P}^9$) has degree 15. Taking this argument to the extreme, the locus of cones inside the space $\mathbb{P}^N$ of degree d hypersurfaces in $\mathbb{P}^n$ has degree:

$$\deg c_n\left(P^{d-1}(\mathcal{O}_{\mathbb{P}^n}(d))\right) = \binom{\binom{n+d-1}{n}}{n}.$$

A second application of principal parts is counting lines meeting a hypersurface with high multiplicity. For example, how many lines meet a quintic surface $S_5 \subset \mathbb{P}^3$ at a single point? Let $U \to \mathbb{G}(1,3)$ be the universal line, that is the projectivization

$\mathbb{P}S$ of the tautological sub-bundle on $G(2,4)$, and let $\mathcal{L}$ be the pull-back of $\mathcal{O}_{\mathbb{P}^3}(d)$ via $U \to \mathbb{P}^3$. We form the bundle of relative principal parts using $\Delta_U \subset U \times_{\mathbb{G}} U$:

$$P^k_{U/\mathbb{G}}(\mathcal{L}) = pr_{2*}\left(pr_1^*(\mathcal{L}) \otimes \mathcal{O}_{U\times_{\mathbb{G}} U}/\mathcal{I}_\Delta^{k+1}\right).$$

Its fiber at a point $(M,p) \in U$ is given by

$P^k_{U/\mathbb{G}}(\mathcal{L})_{(M,p)} = \{$germs of sections of $\mathcal{L}|_M$ at $p\}/\{$those vanishing to order $k+1\}$.

These bundles fit into the same short exact sequences as before, only with the cotangent sheaf Ω_U replaced by its relative version, $\Omega_{U/\mathbb{G}}$. The same relative construction works for any smooth proper morphism $X \to Y$.

Proposition. The number of lines meeting a general surface $S_d \subset \mathbb{P}^3$ ($d \geq 5$) at a point with multiplicity ≥ 5 is given by:

$$\deg c_5\left(P^4_{U/\mathbb{G}}(\mathcal{L})\right) = 35d^3 - 200d^2 + 240d,$$

Proof. We compute using the Whitney sum formula applied to the 5 pieces

$$\mathcal{L}, \mathcal{L}\otimes\Omega_{\mathbb{P}S/\mathbb{G}}, \mathcal{L}\otimes\mathrm{Sym}^2(\Omega_{\mathbb{P}S/\mathbb{G}}), \mathcal{L}\otimes\mathrm{Sym}^3(\Omega_{\mathbb{P}S/\mathbb{G}}), \mathcal{L}\otimes\mathrm{Sym}^4(\Omega_{\mathbb{P}S/G}),$$

aided by the fact that the relative cotangent bundle $\Omega_{\mathbb{P}S/\mathbb{G}}$ is a line bundle. Its first Chern class is $-2\zeta + \sigma_1$ by the relative Euler sequence, so

$$c_5\left(P^4_{U/\mathbb{G}}(\mathcal{L})\right) = \prod_{i=0}^{4}((d-2i)\zeta + i\sigma_1) \in A^*(\mathbb{P}S). \quad \square$$

Note: the d^4 and d^5 terms vanish because $\zeta^4 = \zeta^5 = 0$ (ζ is pulled back from $\mathbb{P}^3$). The formula gives 575 lines for the quintic surface.

Question. What is the maximum number $M(d)$ of lines that can appear on a smooth surface $S_d \subset \mathbb{P}^3$ with $d \geq 4$?

For reasons of Kodaira dimension, the number of lines must be finite, and indeed for general S_d there are no lines at all. However, the Fermat surface

$$V(w^d + x^d + y^d + z^d = 0) \subset \mathbb{P}^3$$

has $3d^2$ lines on it (exercise). This gives a uniform lower bound on $M(d)$, although it can be improved for certain d. To produce an upper bound, look at the curve Γ in U of lines meeting S_d at a point with multiplicity ≥ 4. It has class

$$c_4\left(P^3_{U/\mathbb{G}}(\mathcal{L})\right) = \prod_{i=0}^{3}((d-2i)\zeta + i\sigma_1) \in A^4(\mathbb{P}S).$$

The image of Γ in $\mathbb{P}^3$ has degree given by

$$\zeta \cdot c_4\left(P^3_{U/\mathbb{G}}(\mathcal{L})\right) = 11d^2 - 24d.$$

If S_d contains a line, then that line must be a *component* of this curve, so $11d^2 - 24d$ is an upper bound for $M(d)$. The true value is not known past $M(4) = 64$.

Remark. The degree of the codimension $d-3$ locus in $\mathbb{P}^N$ consisting of surfaces which contain a line also provides an upper bound for $M(d)$, but it is higher. Exercise: the answer is $\deg c_4\left(\mathrm{Sym}^d S^\vee\right)$ on $\mathbb{G}(1,3)$, which is $O(d^4)$.

16. 4/8/20 - Classical Moduli Spaces

Today we will survey some results on the Chow groups of some standard moduli spaces. The most studied examples are:

- $\mathcal{M}_g$ - moduli of smooth curves of genus g,
- $\mathcal{A}_g$ - moduli of principally polarized abelian varieties of dimension g, and
- $\mathcal{K}_g$ - moduli of polarized K3 surfaces of degree $2g-2$.

Note that the g denotes something different for each row, but they are related. The Jacobian of a genus g curve is an abelian variety of dimension g, and a general element of the linear system on a polarized K3 surface is a curve of genus g. The dimension of each moduli space is $\dim M_g = 3g-3$, $\dim \mathcal{A}_g = g(g+1)/2$, $\dim(\mathcal{K}_g) = 19$. Smooth compactifications with divisorial boundary are best for doing intersection theory (this excludes the Baily-Borel compactifications). The following arrangement of the smooth examples highlights certain analogies.

$$\begin{array}{lll} \mathcal{M}_g \subset \mathcal{M}_g^{ct} \subset \overline{\mathcal{M}}_g & & \\ & \mathcal{A}_g \subset \overline{\mathcal{A}}_g^{\mathrm{vor}} & \\ & & \mathcal{K}_g \subset \widetilde{\mathcal{K}}_g \subset \overline{\mathcal{K}}_g^{\Sigma}? \end{array}$$

Except for $\mathcal{M}_0$ and $\mathcal{M}_1$, all of these moduli spaces are Deligne-Mumford stacks. Recall that the coarse space morphism induces an isomorphism on Chow groups $A^*(\mathcal{M})_{\mathbb{Q}} \simeq A^*(M)_{\mathbb{Q}}$. The integral Chow groups of a moduli stack are typically hard to compute; here is a sample of what is known.

Theorem. (Mumford) $A^1(\mathcal{M}_{1,1}) \simeq \mathbb{Z}/12$ and $A^1(\overline{\mathcal{M}}_{1,1}) \simeq \mathbb{Z}$.

Theorem. (Vistoli) $A^*(\mathcal{M}_2) \simeq \mathbb{Z}[\lambda_1, \lambda_2]/(10\lambda_1, 2\lambda_1^2 - 24\lambda_2)$.

Theorem. (Larson)

$$A^*(\overline{\mathcal{M}}_2) \simeq \mathbb{Z}[\lambda_1, \lambda_2, \delta_1]/(24\lambda_1^2 - 48\lambda_2, 20\lambda_1\lambda_2 - 4\delta_1\lambda_2, \delta_1^3 + \delta_1^2\lambda_1, 2\delta_1^2 + 2\delta_1\lambda_1).$$

The first result that we will prove is Mumford's relation in $A^1(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$:

Theorem. Let $\pi : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ be the universal family of curves with its relative dualizing sheaf $\omega = \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}$, and let $E = \pi_*(\omega)$ be the Hodge bundle. For $\lambda = c_1(E)$, $\kappa = \pi_*(c_1(\omega)^2)$, and δ the boundary divisor class, we have:

$$\lambda = \frac{\kappa + \delta}{12} \in A^1(\overline{\mathcal{M}}_g)_{\mathbb{Q}}.$$

Proof. It suffices to check the relation on any family of curves $\pi : S \to B$ over a base curve B. We apply the GRR formula to ω:

$$\mathrm{ch}(E) - 1 = \pi_*\left(e^{c_1(\omega)} \cdot \mathrm{Td}(S/B)\right)$$

The LHS is given by $(g-1)[B] + \lambda$. The inside of the expression on the RHS has codimension 2 part given by:

$$\frac{c_1(\omega)^2}{2} + c_1(\omega) \cdot \mathrm{Td}_1(S/B) + \mathrm{Td}_2(S/B) \tag{1}$$

We can compute the total Chern class

$$\begin{aligned}\frac{c(T_S)}{\pi^* c(T_B)} &= (1 + c_1(T_S) + c_2(T_S))(1 - \pi^* c_1(T_B)) \\ &= 1 - c_1(\omega) + c_2(T_S) - c_1(T_S) \cdot \pi^* c_1(T_B)\end{aligned}$$

The codimension 2 part has degree equal to δ via topological Euler characteristics:

$$\begin{aligned}\chi_{top}(S) &= (2 - 2g)\chi_{top}(B) + \delta; \\ c_2(T_S) &= -c_1(\omega) \cdot \pi^* c_1(T_B) + \delta \\ &= c_1(T_S) \cdot \pi^* c_1(T_B) + \delta.\end{aligned}$$

Using the expression for the Todd class,

$$\operatorname{Td}(S/B) = 1 - \frac{c_1(\omega)}{2} + \frac{c_1(\omega)^2 + \delta}{12}. \tag{2}$$

Mumford's relation now follows from substituting (2) into (1). $\square$

Let us describe $A^1_{\mathbb{Q}} = \operatorname{Pic}_{\mathbb{Q}}$ for some moduli spaces. First, I claim that the Picard group is discrete in each of the standard examples. By the localization sequence for Chow groups, we have

$$A^1(\Delta) \to A^1(\overline{\mathcal{M}}) \to A^1(\mathcal{M}) \to 0,$$

so it suffices to show that $\operatorname{Pic}(\overline{\mathcal{M}}) \simeq H^1(\overline{\mathcal{M}}, \mathcal{O}^\times)$ is discrete. By the exponential long exact sequence

$$\cdots \to H^1(\overline{\mathcal{M}}, \mathcal{O}) \to H^1(\overline{\mathcal{M}}, \mathcal{O}^\times) \xrightarrow{c_1} H^2(\overline{\mathcal{M}}, \mathbb{Z}) \to \dots,$$

it suffices to prove that the irregularity $H^1(\overline{\mathcal{M}}, \mathcal{O}) = 0$. By Hodge theory, it suffices to show that $H^1(\overline{\mathcal{M}}, \mathbb{C}) = 0$. To see this, note that over $\mathbb{C}$ each of the examples above is the quotient of a contractible space by a discrete group.

- $\mathcal{M}_{1,1}(\mathbb{C}) \simeq BG$ where G is $SL_2(\mathbb{Z})$.
- $\mathcal{M}_g(\mathbb{C}) \simeq BG$ where G is the mapping class group MCG_g.
- $\mathcal{A}_g(\mathbb{C}) \simeq BG$ where G is the arithmetic group $Sp_{2g}(\mathbb{Z})$.
- $\mathcal{K}_g(\mathbb{C}) \simeq BG$ where G is the arithmetic group $O'(\Lambda_{2g-2})$.

Each of these groups G has finite abelianization (MCG_g is perfect for $g \geq 3$, and $\mathrm{MCG}_2^{ab} = \mathbb{Z}/10$). Passing from $\mathcal{M}$ to $\overline{\mathcal{M}}$ only makes π_1 smaller, so $H_1(\overline{\mathcal{M}}, \mathbb{C}) = 0$.

Theorem. (Harer) The cohomology groups of $\mathcal{M}_g$ stabilize: for $3k - 1 \leq g$,

$$H^k(\mathcal{M}_g) \simeq H^k(\mathcal{M}_{g+1}).$$

Mumford conjectured that the stable cohomology ring $H^*(\mathcal{M}, \mathbb{Q})$ is a polynomial ring in the κ classes $\kappa_i = \pi_*(c_1(\omega)^{i+1})$.

Theorem. (Madsen-Weiss) Mumford's Conjecture is true:

$$H^*(\mathcal{M}_\infty, \mathbb{Q}) \simeq \mathbb{Q}[\kappa_1, \kappa_2, \dots].$$

In particular, this means that $\operatorname{Pic}(\mathcal{M}_g)_{\mathbb{Q}} = \mathbb{Q}\kappa = \mathbb{Q}\lambda$ for all $g \geq 5$. We will see that the same is true for $g \geq 3$, while $\operatorname{Pic}(\mathcal{M}_2)_{\mathbb{Q}} = 0$. Borel showed that $H^2(BSp_{2g}(\mathbb{Z}))_{\mathbb{Q}} \simeq \mathbb{Q}$ for $g \geq 4$, so in fact $\operatorname{Pic}(\mathcal{A}_g)_{\mathbb{Q}} = \mathbb{Q}\lambda$ for $g \geq 4$ as well.

In what follows, we compute $\operatorname{Pic}_{\mathbb{Q}}(\mathcal{M}_g)$ for $2 \leq g \leq 5$, using GIT models:

To build a moduli space, we need to embed all the varieties X into the same ambient space P, consider the appropriate Hilbert scheme, and then take the quotient by the automorphism group of P. If every aut of X is induced by an aut of the ambient space, then we have successfully constructed a moduli space. Canonical embeddings are nice because any automorphism of X preserves the canonical bundle. Since not every canonical map is an embedding, pluricanonical maps are used to prove existence, but the canonical map is easier to study (see below).

$g = 2$. Every smooth curve C of genus 2 is hyperelliptic, and the canonical morphism $\phi_K : C \to \mathbb{P}^1$ has 6 distinct branch points. To build M_2, we take the GIT quotient of $\mathrm{Sym}^6(\mathbb{P}^1) \simeq \mathbb{P}^6$ by the action of $PGL(2)$. An orbit is GIT stable (resp. semi-stable) iff < 3 (resp. ≤ 3) points collide, so all nodal curves are stable. $\dim(\mathcal{M}_2) = 6 - 3 = 3$. $\dim \mathrm{Pic}(\mathcal{M}_2)_{\mathbb{Q}} = 1 - 1 = 0$.

$g = 3$. Let C be a smooth curve of genus 3. The canonical bundle gives a map $\phi_K : C \to \mathbb{P}^2$. There are two cases: either C is hyperelliptic and ϕ_K is the double cover of a conic in $\mathbb{P}^2$, or C is non-hyperelliptic and ϕ_K is an embedding (this is the generic case). To build M_3, consider space $\mathbb{P}^{14}$ of plane quartics, and take the GIT quotient by the action of $PGL(3)$. Every smooth or nodal quartic is GIT stable. The double conics are strictly semi-stable. Blowing up the double conic locus gives a model for $\mathcal{M}_3$. A degenerating family of smooth quartics gives 8 points on the conic, which are precisely the branch points of the hyperelliptic double cover. $\dim(\mathcal{M}_3) = 14 - 8 = 6$. $\dim \mathrm{Pic}(\mathcal{M}_3)_{\mathbb{Q}} = 1 + 1 - 1 = 1$.

$g = 4$. Let C be a smooth curve of genus 4. The canonical bundle gives a map $\phi_K : C \to \mathbb{P}^3$. The hyperelliptic locus is no longer a divisor, so we can ignore it. All remaining curves are complete intersections of a quadric surface and a cubic surface in $\mathbb{P}^3$. To build M_4, consider the $\mathbb{P}^{15}$-bundle over $\mathbb{P}^9 = \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}(2))$, and take the GIT quotient by the action of $PGL(4)$.

$$0 \to H^0(\mathbb{P}^3, \mathcal{O}(1)) \to H^0(\mathbb{P}^3, \mathcal{O}(3)) \to H^0(\mathcal{O}_Q(3)) \to 0$$

Every smooth or nodal curve is GIT stable. Those curves lying on a singular quadric surface are called *theta-null*. $\dim(\mathcal{M}_4) = 24 - 15 = 9$. $\dim \mathrm{Pic}(\mathcal{M}_4)_{\mathbb{Q}} = 2 - 1 = 1$.

$g = 5$. Let C be a smooth curve of genus 5. The canonical bundle gives a map $\phi_K : C \to \mathbb{P}^4$, where the general curve is a complete intersection of three quadrics. To build M_4, consider the Grassmannian $G(3, 15)$ and take the GIT quotient by the action of $PGL(5)$. $\dim(\mathcal{M}_5) = 36 - 24 = 12$. $\dim \mathrm{Pic}(\mathcal{M}_5)_{\mathbb{Q}} = 1 - 1 = 0$? There is actually a divisor of trigonal curves, for which the ideal $I_C(2)$ cuts out an $\mathbb{F}_1 \subset \mathbb{P}^4$ containing C as a trisection, so $\dim \mathrm{Pic}(\mathcal{M}_5)_{\mathbb{Q}} = 1$.

The hyperelliptic, theta-null, and trigonal loci can be used to produce an affine stratication of M_g for $2 \leq g \leq 5$. This gives a $\mathbb{Q}$-basis for all the Chow groups.

To complete the description of $\mathrm{Pic}(\mathcal{A}_g)_{\mathbb{Q}}$ for $g \leq 3$, note that $A_1 \simeq M_{1,1}$, $A_2 \simeq M_2^{ct}$ and $A_3 \simeq M_3^{ct}$, so they have Picard ranks 0, 1, and 2, respectively.

Theorem. $\dim \mathrm{Pic}(\mathcal{K}_g)_{\mathbb{Q}} \to \infty$ as $g \to \infty$. (next time)

17. 4/13/20 - Test Curves on $\overline{\mathcal{M}}_g$ and $\mathcal{K}_g$

Using the affine stratifications described last time, the rational Chow ring is:

Theorem. (Loojienga-Fontanari) $A^*(\mathcal{M}_g)_{\mathbb{Q}} = \mathbb{Q}[\lambda]/\lambda^{g-1}$ for $g \leq 5$.

Looijenga proved (1995) that for all $\mathcal{M}_g$, a polynomial in the kappa classes κ_i of total degree $g-1$ vanishes in $A^*(\mathcal{M}_g)$. This is consistent with S. Diaz's theorem (1984) that any complete subvariety of $\mathcal{M}_g$ has dimension $\leq g-2$.

Conjecture. (Faber) The tautological subring $R^*(\mathcal{M}_g) \subset A^*(\mathcal{M}_g)_{\mathbb{Q}}$, which is generated by the κ_i classes, is an even Poincaré duality algebra of dimension $g-2$.

Is there a closed manifold with $R^*(\mathcal{M}_g)$ as its even cohomology? Wide open.

Next, we turn to $\overline{\mathcal{M}}_g$. The boundary $\Delta = \overline{\mathcal{M}}_g \smallsetminus \mathcal{M}_g$ is a union:

$$\Delta = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{\lfloor g/2 \rfloor}$$

The general element of Δ_0 is a smooth curve of genus $g-1$ with two points glued at a node. The general element of Δ_i $(i > 0)$ is a pair of smooth curves of genera i and $g-i$, glued at a node. To compute $\operatorname{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$, we use the localization sequence

$$\mathbb{Z}^{1+\lfloor g/2 \rfloor} \to \operatorname{Pic}(\overline{\mathcal{M}}_g) \to \operatorname{Pic}(\mathcal{M}_g) \to 0.$$

Proposition. The components Δ_i of the boundary of $\overline{\mathcal{M}}_g$ have linearly independent classes. In particular, $\dim \operatorname{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}} = 2 + \lfloor g/2 \rfloor$ for $g \geq 3$.

Proof. The approach is to produce $1 + \lfloor g/2 \rfloor$ complete curves $C_i \subset \overline{\mathcal{M}}_g$, such that the matrix $\deg(C_i \cdot \Delta_j)$ of intersection numbers is nonsingular. The test curve C_i will be inside Δ_i for each $i = 0, 1, \ldots, \lfloor g/2 \rfloor$, constructed by sliding the point of attachment. For C_0, fix a smooth curve Σ of geometric genus $g-1$, and glue marked points p and q. As the point p moves along Σ, we get a curve in Δ_0. For C_i $(i > 0)$, fix two marked curves (Σ_1, p) and (Σ_2, q) of genera i and $g-i$, respectively, and glue the marked points. As the point p moves along Σ_1, we get a curve in Δ_i. The intersection numbers are given by:

$$\deg(C_i \cdot \Delta_j) = \begin{pmatrix} 2-2g & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -2 & 0 & \cdots \\ 0 & 0 & 0 & -4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

To see this, we use the fact (from the deformation theory of nodes) that the fiber of the normal bundle $N_{\Delta_i/\overline{\mathcal{M}}_g}$ at a nodal curve Σ is naturally isomorphic to the tensor product of the tangent spaces to the two branches of Σ at the node point. By the excess intersection formula,

$$\deg(C_i \cdot \Delta_i) = \deg N_{\Delta_i/\overline{\mathcal{M}}_g}|_{C_i}.$$

For $i > 0$, C_i is isomorphic to the component Σ_1, and $N_{\Delta_i/\overline{\mathcal{M}}_g}|_{C_i} \simeq T_{\Sigma_1}$, so $\deg(C_i \cdot \Delta_i) = 2i - 2$. For C_0, observe that the stable limit as p collides with q is isomorphic to Σ glued to a rational nodal tail, a curve whose moduli point lies in $\Delta_0 \cap \Delta_1$. To construct the stable family explicitly, consider

$$S = \mathrm{Bl}_{(q,q)}(\Sigma \times \Sigma) \to \Sigma,$$

with the proper transforms of Δ_Σ and $\Sigma \times \{q\}$ identified, to produce a family of nodal curves over $\Sigma \simeq C_0$. The normal bundles of the two section curves have degrees $2-2(g-1)-1$ and -1, respectively, which gives $\deg C_0 \cdot \Delta_0 = 2-2g$. $\square$

Theorem. (O'Grady) For any N, there exists g such that $\dim \mathrm{Pic}(\mathcal{K}_g)_{\mathbb{Q}} > N$.

Proof. Recall that points of $\mathcal{K}_g$ are pairs (S, L) where S is a K3 surface, meaning that $H^1(\mathcal{O}_S) = 0$ and $K_S \simeq \mathcal{O}_S$, and $L \in \mathrm{Pic}(S)$ is primitive and ample, with $c_1(L)^2 = 2g-2$. To construct examples of polarized K3 surfaces, take the double cover of a surface T branched along a smooth curve $B \in |-2K_T|$. If $T = \mathbb{P}^1 \times \mathbb{P}^1$, then $\varphi : S \to T$ is branched along curve of bidegree $(4,4)$. For any integers $a, b > 0$,

$$L = \varphi^*(ah_1 + bh_2)$$

is ample with $c_1(L)^2 = 4ab$, and it is primitive as long as $(a,b) = 1$. Hence, for each pair a, b of coprime positive integer we have a divisor $D_{a,b} \subset \mathcal{K}_{1+2ab}$. Indeed, counting dimensions, $h^0(T, 4h_1 + 4h_2) = 25$ so we have $\dim \mathbb{P}^{24} - \dim \mathrm{Aut}(T) = 18$.

For each $D_{a,b}$, we will construct a test curve $C_{a,b} \subset D_{a,b}$ such that:

- $\deg C_{a,b} \cdot D_{a,b} < 0$,
- $\deg C_{a,b} \cdot D_{a,b} = 0$ if $\min\{c,d\} > 2\min\{a,b\}$ or $\min\{c,d\} < \frac{1}{2}\min\{a,b\}$.

The second property comes from the fact that $C_{a,b}$ and $D_{c,d}$ are disjoint under the given inequalities. For $g = 1 + 2p_1p_2p_3 \dots p_{2N}$ (with p_i distinct primes), we can produce a diagonal intersection matrix (of size N) with nonzero determinant.

Let $\mathbb{P}^1 \to \mathbb{P}^{24}$ be a general pencil of $(4,4)$ curves in $\mathbb{P}^1 \times \mathbb{P}^1$. There are 68 nodal members of the pencil; the corresponding K3 surface $S \to T$ will have a node. Let $C \to \mathbb{P}^1$ be the double cover branched at those 68 points. The family of curves

$$Y \subset C \times T \to C$$

has tridegree $(2,4,4)$, and the total space Y has 68 nodes. Let $X \to C \times T$ be the double cover branched along Y, with 68 ordinary double points. A small resolution $\hat{X} \to X \to C$ gives a family of smooth K3 surfaces. If we pullback the primitive ample line bundle $L = ah_1 + bh_2$ from T, we get a family of *polarized* K3 surfaces, which corresponds to a moduli map $C_{a,b} = C \to \mathcal{K}_{2ab+1}$.

By Noether-Lefschetz theory (next time), the normal bundle $N_{D_{a,b}/\mathcal{K}_g}$ is isomorphic to the restriction of the dual Hodge line bundle to $D_{a,b}$. The Hodge bundle is ample. For disjointness, assume WLOG that $a = \min\{a,b\}$, and suppose that we have found an automorphism f of a K3 surface S such that $f^*\varphi^*(ch_1 + dh_2) \simeq \varphi^*(ah_1 + bh_2)$. Replacing φ^* with $\sim$ for readability, we have

$$f_*(\tilde{h}_2) \cdot (c\tilde{h}_1 + d\tilde{h}_2) = \tilde{h}_2(a\tilde{h}_1 + b\tilde{h}_2) = 2a$$

On the other hand, using the fact that h_2 is nef, $f_*(\tilde{h}_2) \cdot (c\tilde{h}_1 + d\tilde{h}_2) \geq \min\{c,d\} > 2a$, contradiction. Swapping the roles of $\{a,b\}$ with $\{c,d\}$ gives a contradiction for the second inequality too. $\square$

18. 4/15/20 - Borcherds Modularity

To get more detailed information about the moduli spaces $\mathcal{K}_g$ of polarized K3 surfaces, we need to introduce some Hodge theory. Given (S, L), the primitive cohomology lattice $\langle c_1(L)\rangle^\perp \subset H^2(S,\mathbb{Z})$ is abstractly isomorphic to

$$\Lambda_g = \mathbb{Z}(2-2g) \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}(-1).$$

If we choose an isomorphism $H^2(S,\mathbb{Z}) \simeq \Lambda_g$, then the Hodge decomposition gives

$$\Lambda_g \otimes \mathbb{C} = H^{0,2} \oplus H^{1,1} \oplus H^{2,0} \simeq \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}^{19} \oplus \mathbb{C}.$$

The moduli space of Hodge structures on Λ_g is a Hermitian symmetric domain $\mathcal{D}$ of Type IV, meaning that it has a complex structure and an action of the real orthogonal group $O(\Lambda_g \otimes \mathbb{R}) \simeq O(2,19)$ such that

$$\mathcal{D} \simeq O(2,19)/O(2) \times O(19).$$

To see the complex structure, we may view $\mathcal{D}$ as an open subset of the quadric hypersurface $Q \subset \mathbb{P}(\Lambda_g \otimes \mathbb{C})$ defined by the lattice pairing $(v,v) = 0$, with the open condition on $\mathbb{C}v \in \mathbb{P}(\Lambda_g \otimes \mathbb{C})$ given by $(v, \overline{v}) > 0$. For a dual lattice vector $x \in \Lambda_g^\vee$, $x^\perp$ is a hyperplane section of the quadric Q, and it is non-empty iff $(x,x) < 0$.

The moduli space of polarized K3 surfaces of degree $2g-2$ has a period map:

$$\rho : \mathcal{K}_g \to \mathcal{D}/\Gamma,$$

where $\Gamma = O'(\Lambda_g) \subset O(\Lambda_g)$ is the subgroup of lattice automorphisms which act trivially on the discriminant $\Lambda_g^\vee/\Lambda_g$, or equivalently extend to an automorphism of the whole cohomology lattice Λ_{K3} fixing $c_1(L)$.

Theorem. (Piatetskii-Shapiro, Shafarevich) The period map ρ is an open embedding, and the complement of the image consists of a union of hyperplanes $r^\perp$, for $r \in \Lambda_g$ with $(r,r) = -2$. Since Γ acts transitively on the set of such r, we get an irreducible hypersurface in the quotient $\mathcal{D}/\Gamma$.

Remark. If we relax the ampleness condition on L and only require that it be nef, then we get an isomorphism $\widetilde{\mathcal{K}}_g \simeq \mathcal{D}/\Gamma$, the moduli space of quasi-polarized K3 surfaces. Geometrically this corresponds to adding mildly singular (ADE) K3 surfaces to the moduli space, using simultaneous resolution.

There is a natural Hodge line bundle λ on $\mathcal{D}$, whose fiber is $H^{2,0}$, and it descends to the quotient $\mathcal{D}/\Gamma$. Baily-Borel proved that some power of λ has enough sections to embed $\mathcal{D}/\Gamma$ into projective space (λ is ample). The closure of $\mathcal{D}/\Gamma$ in projective space is a singular projective variety, often denoted $(\mathcal{D}/\Gamma)^{BB}$.

For any $x \in \Lambda_g^\vee$, the hyperplane $x^\perp$ and its Γ-translates descend to a hypersurface in $\mathcal{D}/\Gamma$. These divisors are called Noether-Lefschetz divisors because they parametrize K3 surfaces with Picard number > 1. If $\mathrm{Pic}(S)$ contains a sublattice

$$\langle c_1(L), \beta\rangle = \begin{pmatrix} 2g-2 & k \\ k & 2m \end{pmatrix},$$

then the orthogonal projection of β is an element $x \in \Lambda_g^\vee$. If we index these hypersurfaces by $-n = (x,x) \in \mathbb{Q}_{>0}$, we can make a formal power series with classes in $A^1(\widetilde{\mathcal{K}}_g)$ as coefficients:

$$f(q) = -\lambda + \sum_{n \in \mathbb{Q}_{>0}} \sum_{\substack{x \in \Lambda_g^\vee/\Gamma \\ (x,x)=-2n}} [x^\perp] q^n.$$

Note: n can have a denominator of at most $2g-2$.

Theorem. (Borcherds) $f(q)$ is the Fourier series in $q = e^{2\pi i \tau}$ of a modular form of weight $21/2$ and level $\Gamma(2g-2)$, with values in $A^1(\widetilde{\mathcal{K}}_g)$.

Theorem. (Bergeron-Li-Millson-Moeglin) $A^1(\widetilde{\mathcal{K}}_g)_{\mathbb{Q}}$ is spanned by the classes $[x^\perp]$.

Although there are infinitely many divisors $[x^\perp]$, their span is finite-dimensional in $A^1(\widetilde{\mathcal{K}}_g)_{\mathbb{Q}}$. Indeed, for each element of $\mu \in A^1(\widetilde{\mathcal{K}}_g)_{\mathbb{Q}}^\vee$, we have

$$\mu \cdot f(q) \in \mathrm{Mod}(21/2, 2g-2),$$

so we have an injective morphism $A^1(\widetilde{\mathcal{K}}_g)_{\mathbb{Q}}^\vee \to \mathrm{Mod}(21/2, 2g-2)$. Bruinier was able to describe the image of this morphism with a simple vanishing criterion, which leads to a dimension formula for $\mathrm{Pic}(\mathcal{K}_g)_{\mathbb{Q}}$ using the Selberg trace formula[2] for spaces of modular forms $\dim \mathrm{Pic}(\mathcal{K}_g)_{\mathbb{Q}} \sim \frac{31}{24} g$.

Remark. The growth of $\dim \mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$ came from the boundary divisors, which were related to lower genus curves. The growth of $\dim \mathrm{Pic}(\mathcal{K}_g)_{\mathbb{Q}}$ comes from the NL-divisors, which are themselves locally symmetric spaces for the group $O(2,18)$.

Theorem. (Maulik) For any infinite sequence of elements $x_i \in \Lambda_g^\vee$, a complete curve $C \to \mathcal{K}_g$ must intersect one of the NL-divisors $x_i^\perp$.

Corollary. (BKPS) Any complete family with constant Picard number is isotrivial.

Proof. The crucial fact is ampleness of the Hodge line bundle λ (Baily-Borel). It suffices to express $a\lambda$ as a combination of the NL-divisors $x_i^\perp$. Let $\theta(q)$ be the Siegel theta function of weight $1/2$, and let $E_{10}(q)$ be the Eisenstein series of weight 10. If $f(q) \in \mathrm{Mod}(21/2, 2g-2)$ with constant term c_0, then

$$f(q) - c_0 \theta(q) E_{10}(q) \in \mathrm{Mod}(21/2, 2g-2)$$

is a cusp form. The coefficients of a cusp form grow like $O(n^{wt/2+\epsilon}) = O(n^{11/4+\epsilon})$. On the other hand, the coefficients of $\theta(q)E_{10}(q)$ are bounded from below by $O(n^9)$. These estimates imply that the coefficients of $f(q)$ are eventually nonzero. Next consider the composition:

$$\bigoplus_i \mathbb{Q} e_i \xrightarrow{\alpha} \mathrm{Mod}(21/2, 2g-2)^\vee \to \mathrm{Pic}(\mathcal{K}_g)_{\mathbb{Q}},$$

which sends e_i to $[x_i^\perp]$. If an element $f(q) \in \mathrm{Mod}(21/2, 2g-2)^\vee$ vanishes on the span of $\alpha(e_i)$, then it vanishes on e_0 as well. Hence $\alpha(e_0) \in \mathrm{span}_i(\alpha(e_i))$, so

$$\lambda \in \mathrm{span}_i [x_i^\perp]. \quad \square$$

[2]The Borcherds result is slightly more refined; you get a vector-valued modular form for the Weil representation on $\mathbb{C}[\Lambda_g^\vee/\Lambda_g]$. This is why we use the trace formula to compute the dimension.

19. 4/20/20 - Hirzebruch-Mumford Proportionality

Hirzebruch proved a statement relating certain top intersection products on $X = \Gamma\backslash\mathcal{D}$ (for $\mathcal{D} = G/K$ a symmetric space and $\Gamma \subset \mathrm{Aut}(\mathcal{D})$ torsion-free, cocompact) with those on Y, the flag variety containing $\mathcal{D}$ as an open subset. The relation is a universal constant of proportionality $c(\Gamma)$ between the two.

Remark. Every arithmetic group has a torsion-free subgroup of finite index. Torsion-free discrete groups Γ act freely on symmetric spaces[3].

To set this up, let G be a semi-simple $\mathbb{R}$-group, and let K be a maximal compact subgroup such that $\mathcal{D} = G/K$ has a complex structure. The standard examples are $SL(2,\mathbb{R})/SO(2)$, more generally $Sp(2n,\mathbb{R})/U(n)$, $O(2,n)/SO(2)\times SO(n)$, and $U(1,n)/U(1)\times U(n)$. The *compact dual* of $\mathcal{D}$ is a flag variety $Y = G_{\mathbb{C}}/P$, where $P = K_{\mathbb{C}}\cdot P_+$ is a parabolic subgroup with unipotent radical P_+ such that $P\cap G = K$.

To produce (complex) vector bundles on $\mathcal{D}$ take a representation $\rho : K \to GL_n(\mathbb{C})$, and form the bundle

$$V_\rho = G \times_K \mathbb{C}^n \to G/K = \mathcal{D},$$

which descends to $\Gamma\backslash\mathcal{D}$. These are called *automorphic bundles*. There is a natural extension of V to a vector bundle $\tilde{V}$ on the compact dual Y, by extending the representation linearly from K to $K_{\mathbb{C}}$ and trivially to $P = K_{\mathbb{C}}\cdot P_+$, and then performing the same quotient construction.

Theorem. (Hirzebruch) Any top intersection $c_\alpha(V_\alpha)$ on $X = \Gamma\backslash\mathcal{D}$ is proportional to the analogous top intersection $c_\alpha(\tilde{V}_\alpha)$ on Y, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a multi-index with $\sum \alpha_i = \dim(X) = \dim(Y)$. The constant depends only on Γ, and it is equal to $(-1)^{\dim(X)} \cdot \mathrm{vol}(\Gamma\backslash\mathcal{D})$, with respect to Haar measure.

Mumford gave a prescription for constructing *toroidal compactifications* $\overline{\Gamma\backslash\mathcal{D}}^{\Sigma}$, which depend on a choice of fan Σ, and how to extend automorphic bundles. By studying singular metrics on these bundles, he proved a proportionality statement for when $\Gamma\backslash\mathcal{D}$ is non-compact.

Theorem. (Mumford) The same statement as above, replacing $X = \Gamma\backslash\mathcal{D}$ with $\overline{\Gamma\backslash\mathcal{D}}^{\Sigma}$ a toroidal compactification that is smooth as a stack, so Γ can have torsion.

Our main application of the principle will be to $\mathcal{A}_g$, the moduli space of ppav's, and its toroidal compactifications. Here $\mathcal{D} = \mathbb{H}_g$ is the Siegel upper-half space, and the compact dual is $Y_g = Sp(2g,\mathbb{C})/P$ is the Lagrangian Grassmannian of $(\mathbb{C}^{2g},\omega)$. To compute the cohomology of Y_g, fix a flag of isotropic subspaces:

$$0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_g \subset \mathbb{C}^{2g},$$

and extend it by taking $Z_{g+i} = Z_{g-i}^{\perp}$. Given a partition μ satisfying $\mu_i - 1 \leq \mu_{i+1} \leq \mu_i$ fitting inside the $g\times g$ box (there are 2^g such partitions), we define

$$\Sigma_\mu(Z_\bullet) = \{L \in Y_g : \dim(L \cap Z_{g+i-\mu_i}) \geq i\} \subset Y_g.$$

[3]Note that discreteness of Γ implies that for $C \subset \mathcal{D}$ compact, $\{g \in \Gamma : gC \cap C \neq \emptyset\}$ is finite.

These are the only Schubert cycles in $G(g,2g)$ which lie in Y_g because of the fact that $\dim(L\cap Z_i^{\perp}) = \dim(L\cap Z_i) + g - i$.

Proposition. The Chow ring of Y_g the following (graded) presentation:

$$A^*(Y_g) \simeq \mathbb{Z}[u_1,u_2,\dots,u_g]/(1+u_1+u_2+\cdots+u_g)(1-u_1+u_2-\cdots\pm u_g)-1.$$

Proof. We have the tautological sequence on Y_g given by

$$0\to S\to \mathcal{O}_{Y_g}\otimes\mathbb{C}^{2n}\to Q\to 0,$$

and using the symplectic form ω, we have $Q\simeq S^\vee$. The Whitney sum formula then gives the desired relation, and one checks that there are no further relations by comparing the Betti numbers with the numbers of strict partitions. □

Remark. A more concise way of writing the relation is $\mathrm{ch}_{2k}(S)=0$ for $k\geq 1$.

The restriction of the bundle S from Y_g to the open subset $\mathbb{H}_g$ coincides with the Hodge bundle $\mathbb{E}=\pi_*(\Omega^1_{\mathcal{X}_g/\mathcal{A}_g})$ pulled back from $\mathcal{A}_g$. It is the automorphic bundle associated to the standard representation of $U(n)$. We define the tautological ring $R^*(\mathcal{A}_g)\subset A^*(\mathcal{A}_g)_{\mathbb{Q}}$ to be generated by the Chern classes of the Hodge bundle. Since $\mathbb{E}$ extends over any toroidal compactification $\overline{\mathcal{A}}_g^{\Sigma}$, we define the tautological ring of smooth compactifications in the same way. The Proportionality Principle implies (sending $u_i\mapsto\lambda_i$) that

$$R^*\left(\overline{\mathcal{A}}_g^{\Sigma}\right)\simeq A^*(Y_g),$$

but the degrees of 0-cycles differ by the constant multiple:

$$c(\Gamma) = (-1)^{g(g+1)/2}\frac{1}{2^g}\prod_{k=1}^{g}\zeta(1-2k).$$

This is the reciprocal of a large integer ($1/24$ for $g=1$ is familiar).

Theorem. $R^*(\mathcal{A}_g)\simeq A^*(Y_{g-1})$.

Proof. By the localization sequence, we know that $R^*(\mathcal{A}_g)$ is a quotient of $A^*(Y_g)$. It turns out that the kernel is (u_g). To see this, apply GRR[4] to $\mathcal{O}_{\mathcal{X}_g}$ on the universal family $\pi:\mathcal{X}_g\to\mathcal{A}_g$. On an abelian variety X, we have $H^i(\mathcal{O}_X)\simeq\wedge^i H^1(\mathcal{O}_X)$, so by relative Serre duality the LHS reads:

$$\mathrm{ch}(\pi_*[\mathcal{O}_{\mathcal{X}_g}]) = \mathrm{ch}(1-\mathbb{E}^\vee+\wedge^2\mathbb{E}^\vee-\cdots\pm\wedge^g\mathbb{E}^\vee) = c_g(\mathbb{E})\cdot\mathrm{Td}(\mathbb{E})^{-1}.$$

The other hand, the RHS is $\pi_*\mathrm{Td}(\mathcal{X}_g/\mathcal{A}_g)=0$, since $T_{\mathcal{X}_g/\mathcal{A}_g}$ is pulled back from the base. To show that there are no further relations, we use the algebraic fact that $A^*(Y_{g-1})$ is a Gorenstein ring with socle $u_1^{g(g-1)/2}$, so it suffices to show that $\lambda_1^{g(g-1)/2}\neq 0$. To do this, use ampleness of λ_1 (Baily-Borel) combined with the existence of a complete subvariety of dimension $g(g-1)/2$ over fields of characteristic p, namely the locus of abelian varieties with p-torsion rank 0. The p-rank is generically g, and drop in rank occurs with codimension 1. Since semi-abelian varieties with non-trivial $\mathbb{G}_m$ part have positive p-rank, this locus is complete. □

[4]Applying GRR to the relative theta divisor instead, one can reprove the relation above on $\mathcal{A}_g$, but we avoid that computation by appealing to Hirzebruch-Mumford Proportionality.

20. 4/22/20 - Interactions between $\mathcal{K}_g$ and $\mathcal{A}_g$

Both $\mathcal{K}_g$ and $\mathcal{A}_g$ are examples of Shimura varieties $\Gamma\backslash G/K$. As such, they have a rich collection of Shimura subvarieties corresponding to subgroups $G' \subset G$. If $K' = K \cap G'$ and $\Gamma' = \Gamma \cap G'$, then we have

$$\Gamma'\backslash G'/K' \hookrightarrow \Gamma\backslash G/K.$$

For this to be a subvariety, we need K' maximal and Γ' arithmetic (use Baily-Borel).

In the case of K3 surfaces, let $V \subset \Lambda_g \otimes \mathbb{Q}$ be a $\mathbb{Q}$-subspace such that the pairing is non-degenerate there. This implies that $\Lambda_g \otimes \mathbb{Q} \simeq V \oplus V^\perp$. Recall that $\mathcal{D} = O(2,19)/O(2) \times O(19)$ is the space of positive definite 2-planes in $(\Lambda_g) \otimes \mathbb{R}$. Define a subdomain of codimension equal to $\dim(V)$:

$$\mathcal{D}_V = \{Z \in \mathcal{D} : Z = Z \cap V + Z \cap V^\perp\},$$

$G_V \subset G$ the stabilizer of V and $\Gamma_V = \Gamma \cap G_V$. In the case where V is a negative definite line, then $\Gamma_V\backslash\mathcal{D}_V$ is a Noether-Lefschetz divisor and $G_V = O(2,18)$. In the case where V is a negative definite subspace of dimension m, then $\Gamma_V\backslash\mathcal{D}_V$ is a *higher NL-locus*, parametrizing polarized K3 surfaces with Picard rank m, and $G_V = O(2,19-m)$. These are classified (modulo Γ) by their Gram matrices M with respect to a basis in $\Lambda_g^\vee$, and they satisfy a modularity theorem[5] as well:

Theorem. (Kudla-Millson, Zhang) For $m \geq 1$, let $S_m(t)$ be the set of positive semi-definite $m \times m$ matrices of rank t with $\mathbb{Q}$-coefficients. Then the power series

$$F(q) = (-\lambda)^m + \sum_{M \in S_m(1)} (-\lambda)^{m-1}[\Gamma_M\backslash\mathcal{D}_M]q^M + \cdots + \sum_{M \in S_m(m)} [\Gamma_M\backslash\mathcal{D}_M]q^M.$$

is a Siegel modular form (weight 21/2, level $2g-2$) with coefficients in $A^m(\widetilde{K}_g)$.

Definition. A Siegel modular form is a section of $\lambda^{\otimes k}$ on $\mathcal{A}_g$ ($g = m$, $k = 21/2$ in the theorem) which satisfies a polynomial growth condition at the cusps. More explicitly, they are holomorphic functions on the space of symmetric $g \times g$ matrices τ with positive definite imaginary part. The notation q^M denotes $e^{\pi i \operatorname{tr}(M\tau)}$.

The proof uses a technique called theta lifting, which is a way of passing between automorphic forms for symplectic and orthogonal groups. The key fact is that if W is a symplectic vector space and V is an orthogonal vector space, then $W \otimes V$ is naturally symplectic, and we have $Sp(W) \times O(V) \subset Sp(W \otimes V)$. The Siegel theta function on $Sp(W \otimes V)$ restricts to a function Θ, which serves as a correspondence between automorphic forms for the two groups:

$$Sp(W) \leftarrow Sp(W) \times O(V) \rightarrow O(V).$$

A second interaction between $\mathcal{K}_g$ and $\mathcal{A}_g$ stems from the exceptional isomorphism

$$SO(2,3)^+ \simeq Sp(4,\mathbb{R}).$$

[5] We have only given the statement for $\mathcal{K}_g$ with coefficients in Chow groups, but Kudla-Millson prove a statement for all $O(p,q)$ symmetric spaces with coefficients in cohomology.

This implies that higher NL-loci of dimension 3 are Shimura varieties for $Sp(4,\mathbb{R})$. Using the structure theorem for $R^*(\mathcal{A}_2)$ from last time, we have

$$\lambda_1^2 = 2\lambda_2 = 0 \in R^2(\mathcal{A}_2).$$

Theorem. (v.d.Geer-Katsura) If $\Gamma\backslash D$ is a Shimura variety for $O(2,n)$, then

$$\lambda^{n-1} = 0 \in A^*(\Gamma\backslash\mathcal{D})_{\mathbb{Q}}$$

In particular, any complete subvariety of $\mathcal{K}_g$ has dimension ≤ 17.

Proof. We argue by induction on the dimension n, with the base case $n = 3$ coming from the exceptional isomorphism above. Write λ as a linear combination $\sum c_i H_i$ of NL-divisors H_i which are themselves Shimura varieties.

$$\lambda^{n-1} = \sum c_i H_i \lambda^{n-2} = \sum c_i \iota_*(\lambda_{H_i}^{n-2}) = 0.$$

Here ι denotes the inclusion the relevant H_i. $\square$

A third interaction is the *Kuga-Satake construction*, which produces an enormous abelian variety from a K3 surface. More precisely, it produces a weight one PHS of dimension 2^{n-2} from a K3-type weight two PHS H of dimension n. The even Clifford algebra $\mathrm{Cl}^+(H_{\mathbb{R}})$ is a real vector space of dimension 2^{n-1}, and one uses the K3 decomposition to produce a complex structure on it by hand, and then a PHS of weight one. This gives an injection

$$\mathcal{K}_g \hookrightarrow \mathcal{A}_{2^{19},\delta}$$

where δ is a non-principal polarization type depending g. The construction is useful for importing arithmetic and cycle theoretic results about abelian varieties to K3 surfaces. Examples include the Tate Conjecture, Shafarevich Conjecture, versions of the Hodge conjecture, etc.

21. 4/27/20 - Quillen K-Theory

The last topic for the course will be higher Chow groups and their cousins, higher algebraic K-groups. The ultimate goal is to extend the localization sequence toward the left. We would like to define groups $A_*(X, n)$ such that for any nested pair of schemes (X, Z) with $U = X \setminus Z$ open, we have a long exact sequence:

$$\cdots \to A_*(Z,1) \to A_*(X,1) \to A_*(U,1) \to A_*(Z,0) \to A_*(X,0) \to A_*(U,0) \to 0,$$

where $A_*(X) = A_*(X, 0)$ recovers the usual Chow groups. Bloch defined the higher Chow groups in 1986. He was primarily motivated by the parallel story in algebraic K-theory due to Quillen around 1972. We have already encountered the Grothendieck group $K_0(X) = K_0(\mathrm{Coh}(X))$, which is (rationally) isomorphic to the Chow group $A_*(X, 0)$ via the natural transformation τ_X of Grothendieck-Riemann-Roch. A similar natural isomorphism $\tau_X(n)$ exists between $K_n(X)_{\mathbb{Q}}$ and $A_*(X, n)_{\mathbb{Q}}$.

Definition. Let $\mathfrak{C}$ be a small category. The *nerve* of $\mathfrak{C}$ is defined to be the simplicial set $N(\mathfrak{C})$ whose p-simplices are compositions

$$X_0 \to X_1 \to \dots X_p.$$

The classifying space $B\mathfrak{C}$ is the geometric realization of the nerve.

Example. If G is a group, we can form the category $\underline{G}$ with a single object. Then the classifying space of the category matches usual notion: $B\underline{G} = BG$.

Definition. If X is an object of $\mathfrak{C}$, we can define the homotopy groups:

$$\pi_n(\mathfrak{C}, X) := \pi_n(B\mathfrak{C}, X).$$

This notion of classifying space has a number of nice properties:

- $B\mathfrak{C} = B\mathfrak{C}^{op}$.
- $B(\mathfrak{C} \times \mathfrak{C}') = B(\mathfrak{C}) \times B(\mathfrak{C}')$.
- A functor $F : \mathfrak{C} \to \mathfrak{C}'$ induces a cellular map $BF : B\mathfrak{C} \to B\mathfrak{C}'$.
- A natural transformation $F \Rightarrow G$ induces a homotopy from BF to BG.
- If F has a right or left adjoint, then BF is a homotopy equivalence.
- If $\mathfrak{C}$ has an initial or final object, then $B\mathfrak{C}$ is contractible.

Definition. Let $\mathfrak{C}$ be an abelian category. Quillen's Q-construction gives a category $Q\mathfrak{C}$ whose objects are the same as those of $\mathfrak{C}$. A morphism in $Q\mathfrak{C}$ from $M \to N$ is defined by a pair: $M' \hookrightarrow M$, $M' \twoheadrightarrow N$. The composition of morphisms is given by:

$$\begin{tikzcd}
M & & \\
M' \arrow[u] \arrow[r] & N & \\
M' \times_N N' \arrow[u] \arrow[r] & N' \arrow[u] \arrow[r] & L,
\end{tikzcd}$$

where the vertical arrows are monos and the horizontal arrows are epis.

Note: 0 is the final object of $\mathfrak{C}$, but not of $Q\mathfrak{C}$ because you lose uniqueness.

Theorem. The fundamental group $\pi_1(Q\mathfrak{C})$ is isomorphic to $K_0(\mathfrak{C})$.

Proof. Given $M \in K_0(\mathfrak{C})$, we have two natural morphisms in $M \to 0$ in $Q\mathfrak{C}$, $M \supset M \to 0$ and $M \supset 0 \to 0$, which give a loop γ_M in $BQ\mathfrak{C}$. Given a short exact sequence $0 \to M' \to M \to M'' \to 0$, the composition of the loops $\gamma_{M'}$ and $\gamma_{M''}$ is homotopic to γ_M (draw a picture of a 2-sphere with three disks cut out). Conversely, send a loop (based at 0) consisting of arrows with alternating orientation in $Q\mathfrak{C}$ to the alternating sum of the vertices as elements of $K_0(\mathcal{C})$. $\square$

Definition. The algebraic K-groups of an abelian category $\mathfrak{C}$ are defined by:

$$K_n(\mathfrak{C}) = \pi_{n+1}(Q\mathfrak{C}, 0).$$

Definition. If R is a regular ring, define $K_n(R) = K_n(\mathrm{Mod}^f(R))$. If R is not regular then we instead use the subcategory generated by projective R-modules.

Theorem. (Dévissage) Let $\mathfrak{B} \subset \mathfrak{C}$ be a full abelian subcategory such that any object $M \in \mathfrak{C}$ admits a finite filtration whose successive quotients are in $\mathfrak{B}$. Then the induced map $BQ\mathfrak{B} \to BQ\mathfrak{C}$ is a homotopy equivalence, so $K_n(\mathfrak{B}) \simeq K_n(\mathfrak{C})$.

Proof. By the long exact sequence of homotopy groups, it suffices to show that the homotopy fiber of $BQ\mathfrak{B} \to BQ\mathfrak{C}$ is contractible. Recall that the homotopy fiber of a continuous map of pointed spaces $f : B \to C$ is the space of pairs $(b \in B, \gamma : f(b) \to 0)$. To avoid alternating arrow orientations, Quillen proves that a certain homotopy fiber category is contractible for any choice of basepoint. $\square$

Example. Let G be a reductive group. Taking $\mathfrak{C}$ the category of G-modules and $\mathfrak{B}$ the collection of irreps α for G, we obtain

$$K_n(\mathfrak{C}) = \bigoplus_{\alpha} K_n(\mathrm{End}(\alpha)),$$

and by Schur's Lemma each $\mathrm{End}(\alpha)$ is a division ring.

Example. Let $I \subset R$ be a nilpotent ideal. Then $K_n(R/I) \simeq K_n(R)$.

Theorem. (Localization) Let $\mathfrak{S} \subset \mathfrak{C}$ be a *Serre subcategory*, that is a full abelian subcategory that is closed under extensions. Then there is a fibration sequence $BQ\mathfrak{S} \to BQ\mathfrak{C} \to BQ(\mathfrak{C}/\mathfrak{S})$ which induces a long exact sequence on homotopy:

$$\cdots \to K_{n+1}(\mathfrak{C}/\mathfrak{S}) \to K_n(\mathfrak{S}) \to K_n(\mathfrak{C}) \to K_n(\mathfrak{C}/\mathfrak{S}) \to \dots$$

Recall: The localized category $\mathfrak{C}/\mathfrak{S}$ is obtained by formally inverting all morphisms whose kernel and cokernel lie in $\mathfrak{S}$.

Corollary. Let R be a Dedekind domain with fraction field F. Taking $\mathfrak{C}$ the category of R-modules and $\mathfrak{S}$ to be the category of torsion R-modules, we obtain:

$$\cdots \to K_{n+1}(F) \to \bigoplus_{\mathfrak{m}} K_n(R/\mathfrak{m}) \to K_n(R) \to K_n(F) \to \dots$$

We have used the Dévissage Theorem to describe $K_n(\mathfrak{S})$ in terms the residue fields.

22. 4/29/20 - Bloch's Formula

Last time, we showed that $\pi_1(BQ\mathfrak{C}, 0)$ was the Grothendieck group. Is there a similarly concrete description of $K_1(\mathfrak{C}) = \pi_2(BQ\mathfrak{C}, 0)$?

Theorem. For a commutative ring R, $K_1(R) \simeq GL(R)^{ab}$.

Proof. Construct a map $GL(R) \to \pi_2(BQ\mathfrak{C}, 0)$ by sending an element $T \in GL_n(R)$ to the sphere in $BQ\mathrm{Mod}^f(R)$ built by gluing together two cones over the loop $A : R^{\oplus n} \to R^{\oplus n}$, obtained by composing T with the two canonical Q-morphisms $R^{\oplus n} \supset 0 \to 0$ and $R^{\oplus n} \supset R^{\oplus n} \to 0$. Since π_2 is abelian, this map factors through $GL(R)^{ab}$. We leave the isomorphism fact as an exercise. $\square$

Theorem. (Whitehead) For R a local ring or Euclidean domain, $GL(R)^{ab} \simeq R^\times$.

Theorem. (Milnor) For $R = \mathcal{O}_F$ with F a number field, $GL(R)^{ab} \simeq R^\times$.

For a scheme X, let $\mathcal{M}^r(X) \subset \mathrm{Coh}(X)$ be the Serre subcategory of coherent sheaves whose support has codimension $\geq r$. Then the localization $\mathcal{M}^r(X)/\mathcal{M}^{r+1}(X)$ has a dévissage over all points $x \in X$ of codimension r:

$$\cdots \to \bigoplus_{x \in X^{(r)}} K_{n+1}(F(x)) \to K_n(\mathcal{M}^{r+1}(X)) \to K_n(\mathcal{M}^r(X)) \to \bigoplus_{x \in X^{(r)}} K_n(F(x)) \to \dots$$

Here we have fixed the codimension r. If instead we use the full filtration $\mathcal{M}^\bullet(X)$ of $\mathrm{Coh}(X)$, the long exact sequences above combine to produce an exact couple whose spectral sequence[6] relates the K-theory of functions fields with the K-theory of $\mathrm{Coh}(X)$, filtered by K-theory of $\mathcal{M}^r(X)$.

$$E_1^{pq} = \bigoplus_{x \in X^{(p)}} K_{-p-q}(F(x)) \Rightarrow K_{-q}(X).$$

Lemma. (Quillen) The map $K_n(\mathcal{M}^{r+1}(X)) \to K_n(\mathcal{M}^r(X))$ vanishes if X is a regular local scheme over a field (this vanishing is called *Gersten's Conjecture*).

Corollary. For X as above, the following *Gersten sequence* is exact:

$$0 \to K_n(X) \to \bigoplus_{x \in X^{(0)}} K_n(F(x)) \to \bigoplus_{x \in X^{(1)}} K_{n-1}(F(x)) \to \cdots \to \bigoplus_{x \in X^{(n)}} K_0(F(x)) \to 0.$$

Proof. The vanishing of the differential implies that the $E_2 = E_\infty$ in the spectral sequence, and furthermore the coniveau filtration on $K_n(\mathrm{Coh}(X))$ is the stupid one. This means that the only nonvanishing terms in E_2 are $E_2^{0q} \simeq K_q(X)$. This is the 0-th cohomology of the E_1-page complex which is otherwise exact. $\square$

Theorem. (Bloch's Formula) Given a smooth variety X, consider the sheaf $\mathcal{K}_n(X)$ of abelian groups associated to the presheaf $U \mapsto K_n(\mathcal{O}(U))$. The classical Chow groups are computed by sheaf cohomology:

$$A^n(X, 0) = H^n(X, \mathcal{K}_n(X)).$$

[6]The construction of the spectral sequence is very analogous to the Serre spectral sequence.

Proof. By sheafifying the Gersten sequence, we have a resolution

$$0 \to \mathcal{K}_n(X) \to \bigoplus_{x \in X^{(0)}} i_{x*}K_n(F(x)) \to \bigoplus_{x \in X^{(1)}} i_{x*}K_{n-1}(F(x)) \to \cdots \to \bigoplus_{x \in X^{(n)}} i_{x*}K_0(F(x)) \to 0.$$

Each sheaf in the resolution is a sum of skyscraper sheaves, so this is a flasque resolution. As a result,

$$H^n(X, \mathcal{K}_n(X)) \simeq \operatorname{coker}\left(\bigoplus_{x \in X^{(n-1)}} K_1(F(x)) \to \bigoplus_{x \in X^{(n)}} K_0(F(x)) \right)$$

The algebraic K-theory of fields is far simpler than the general case: $K_0(F) \simeq \mathbb{Z}$ and $K_1(F) \simeq F^\times$. So the map in question goes from

$$\bigoplus_{x \in X^{(n-1)}} F(x)^\times \to Z^n(X),$$

and is actually the map div from Lecture 3. Quillen reduces to the case of a DVR, where $K_1(F) \to K_0(R/\mathfrak{m})$ is the ord function. $\square$

Remark. Bloch's Formula is essentially a definition for $n = 0, 1$: $H^0(X, \mathbb{Z}) \simeq \mathbb{Z}$, $H^1(X, \mathbb{G}_m) = \operatorname{Pic}(X) \simeq A^1(X)$, and it was first proved by Bloch for $n = 2$ using dilogarithms. Quillen proved the statement for all n.

The *Steinberg group* is the universal central extension of $E(R) \subset GL(R)$, and its center is $K_2(R) = H_2(E(R), \mathbb{Z})$: $0 \to K_2(R) \to St(R) \to E(R) \to 1$.
More explicitly, $St(R)$ is the free group on X_{ij}^a, where $i, j \in \mathbb{N}$ and $a \in R$, modulo the relations satisfied by elementary matrices:

$$\begin{aligned} X_{ij}^a X_{ij}^b &= X_{ij}^{a+b}; && \\ [X_{ij}^a, X_{jk}^b] &= X_{ik}^{ab}, && i \neq k; \\ [X_{ij}^a, X_{kl}^b] &= 1, && i \neq l,\ j \neq k. \end{aligned}$$

The kernel of $St(R) \to E(R)$ is a quotient of $R^* \otimes R^*$, by the Steinberg relations:

$$\{a, b\} = \{b, a\}^{-1}; \qquad \{a, b\}\{a', b\} = \{aa', b\}; \qquad \{a, 1-a\} = 1.$$

For X/k a regular scheme with function field F, there is a well-defined pairing

$$d\log \wedge d\log : F^\times \otimes F^\times \to \Omega^2_{F/k}.$$

It satisfies the Steinberg relations, so it descends to $K_2(F)$. When restricted to $\mathcal{K}_2(X)$ via the Gersten resolution, it lifts to a map $\mathcal{K}_2(X) \to \Omega^2_{X/k}$. Bloch proves that the induced map

$$H^2(X, \mathcal{K}_2(X)) \to H^2(X, \Omega^2_{X/k})$$

is the cycle class map via his isomorphism $H^2(X, \mathcal{K}_2(X)) \simeq A^2(X)$.

Remark. There are several equivalent definitions of higher algebraic K-theory: the $+$-construction, the Q-construction, and then S_*-construction. They each give an infinite loop space (or spectrum), and the algebraic K-groups are equal to its homotopy groups. The Whitehead bracket $\pi_n \otimes \pi_n \to \pi_{n+m-1}$ gives a product on the algebraic K-groups:

$$K_n \otimes K_m \to K_{n+m}.$$

23. 5/4/20 - Higher Chow Groups

We start by defining an algebraic version of an n-simplex (without the positivity assumption on coordinates). For each $n \geq 0$, consider the affine space

$$\Delta^n := \operatorname{Spec} k[t_0, t_1, \dots, t_n]/\left(1 - \sum t_i\right) \simeq \mathbb{A}^n_k.$$

For each ordered map $\rho : \{0, 1, \dots, m\} \to \{0, 1, \dots, n\}$, define $\tilde{p} : \Delta^m \to \Delta^n$ by

$$\tilde{\rho}^*(t_i) = \sum_{\rho(j)=i} t_j.$$

If ρ is injective, $\tilde{\rho}$ is called a face map (facet if $m = n-1$). The higher cycle groups $Z(X, n) \subset Z(X \times \Delta^n)$ are the subgroups generated by cycles which intersect each face $X \times \Delta^m$ dimensionally transversely. If ∂_i denotes the pullback to the i-th facet, we can define an algebraic boundary map

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i : Z(X, n) \to Z(X, n-1).$$

Definition. The higher Chow groups $A(X, n)$ are defined by taking the homology:

$$Z(X, n) = H_n(Z(X, *))$$

Bloch verifies the functoriality properties (proper pushforward, flat pullback) and the property that $A^*(X, n) \simeq A^*(X \times \mathbb{A}^1, n)$ directly from the definitions.

Theorem. (Localization) For a quasi-projective scheme X over k and a closed subscheme $Y \subset X$ with complement U, there is a long exact sequence

$$\cdots \to A_*(Y, n) \to A_*(X, n) \to A_*(U, n) \to A_*(Y, n-1) \to \dots$$

Proof. Since higher Chow groups are the homology of a complex, we expect the long exact sequence to come from a short exact sequence of cycle groups. In fact the sequence

$$0 \to Z(Y, n) \to Z(X, n) \to Z(U, n)$$

is only left exact for degrees $n > 0$. Nonetheless, the injective map of complexes

$$Z(X, *)/Z(Y, *) \to Z(U, *)$$

is a quasi-isomorphism. The proof of this is quite technical and uses higher moving lemmas, relative to the facets. $\square$

Corollary. For any vector bundle $E \to X$, the flat pullback $A^*(X, n) \to A^*(E, n)$ is an isomorphism.

Theorem. (Gysin pullback) The functor $A^*(-, n)$ is contravariant for morphisms of smooth, quasi-projective varieties.

Proof. Let $f : X \to Y$ be a morphism of smooth, quasi-projective varieties. Define the subcomplex

$$Z_*^f(Y) \to Z_*(Y)$$

to be generated by cycle classes dimensionally transverse to f. Using higher moving lemmas, Bloch proves that this is a quasi-isomorphism. Define the Gysin pullback on transverse cycles in the obvious way, using f^{-1} with multiplicities. $\square$

Using triangulations of product simplices, Bloch defines an external product

$$A^*(X,n) \otimes A^*(Y,m) \to A^*(X \times Y, n+m).$$

Composing with the Gysin pullback via the diagonal, we get an internal product

$$A^*(X,n) \otimes A^*(X,m) \to A^*(X, n+m),$$

which generalizes the intersection product.

Theorem. For any projective bundle $\mathbb{P}E \to X$, if $\zeta \in A^1(\mathbb{P}E, 0)$ is the relative hyperplane class, then we have an isomorphism (for $r+1 = \operatorname{rk} E$).

$$A^*(\mathbb{P}E, n) \simeq \bigoplus_{i=0}^{r} \zeta^i \cdot A^*(X,n) \simeq A^*(\mathbb{P}E, 0) \underset{A^*(X,0)}{\otimes} A^*(X,n),$$

Proof. Using the localization sequence with induction on $\dim(X)$, it suffices to prove the statement for trivial bundles. This is done using the localization sequence again and induction on r. $\square$

Theorem. In the codimension 1 case, we have

$$\begin{aligned} A^1(X,n) &= 0, \quad n \geq 2; \\ A^1(X,1) &= H^0(X, \mathcal{O}_X^\times); \\ A^1(X,0) &= \operatorname{Pic}(X). \end{aligned}$$

Proof. Define the *algebraic n-sphere* to be

$$S^n = \Delta^n \underset{\partial\Delta^n}{\cup} \Delta^n,$$

where $\partial\Delta^n$ is the union of the facets. Bloch proves that

$$A^1(X,n) \simeq \operatorname{Pic}(X \times S^n)/\operatorname{Pic}(X).$$

The theorem follows easily from there (note that S^0 is two points). $\square$

Exercise. If X is an algebraic surface, then $A^2(X,1)$ is generated by formal sums

$$\sum_i (C_i, f_i),$$

where $C_i \subset X$ is a curve, $f_i \in k(C_i)^\times$, and $\sum_i \operatorname{div}(f_i) = 0$ as an element of $Z_0(X)$.

Theorem. (Totaro) $A^2(\operatorname{Spec} k, 2) \simeq K_2(k)$. More generally,

$$A^n(\operatorname{Spec} k, n) \simeq K_n^M(k),$$

the Milnor K-groups, which form the simplest part of the higher K-theory of fields.

There is a higher Chern character map $\operatorname{ch}(n) : K_n(X) \to A^*(X,n)$ due to Gillet. It is completely formal, and uses the Quillen $+$-construction of algebraic K-theory, adapted to the world of simplicial schemes. Multiplying by the Todd class of X, $\tau(n) : K_n(X) \to A^*(X,n)$ satisfies a higher Grothendieck-Riemann-Roch for proper morphisms $X \to Y$, and gives an isomorphism over $\mathbb{Q}$.

24. 5/6/20 - Motives and L-Functions

A famous conjecture of Birch and Swinnerton-Dyer says that for an elliptic curve E defined over $\mathbb{Z}$, its Hasse-Weil L-function $L_E(s)$ has a zero at $s = 1$ of order determined by the rank of the Mordell-Weil group $E(\mathbb{Q})$:

$$\mathrm{ord}_{s=1} L_E(s) = \mathrm{rk}\, E(\mathbb{Q}).$$

The L-function is an infinite Euler product over primes p of *local factors* related to the number of points of E modulo p. The L-function is closely related to the zeta function, which encodes the full motive of E instead of only the weight 1 part:

$$Z_E(s) = \zeta(s)\zeta(s-1)L_E(s)^{-1}.$$

How can we generalize this to varieties which are not group schemes? Recall that $A_0(E) \simeq \mathrm{Pic}(E)$ fits into the short exact sequence

$$0 \to \mathrm{Pic}^0(E) \to \mathrm{Pic}(E) \xrightarrow{\deg} \mathbb{Z} \to 0,$$

and $\mathrm{Pic}^0(E) \simeq E(\mathbb{Q})$ after choosing an origin. Every variety has Chow groups.

Conjecture. (Lichtenbaum-Soulé) If X is a smooth variety defined over $\mathbb{Z}$, then for $d = \dim(X)$ and $Z_X(s)$ its zeta function, we have

$$-\mathrm{ord}_{s=n} Z_X(s) = \sum_{i \geq 0} (-1)^i\, \mathrm{rk}\, A_n(X, i).$$

In general we do not even know whether this sum is finite!

Remark. It is important to use cycles over $\mathbb{Q}$ in the definition for the Chow groups; over $\mathbb{C}$ they are often infinite rank.

The general philosophy is that motivic cohomology groups (defined by Voevodsky) are supposed to go here, but higher Chow groups are a good approximation.

$$H^p_{\mathcal{M}}(X, \mathbb{Q}(q)) \simeq A^q(X, 2q-p) \otimes \mathbb{Q}.$$

Beilinson's Conjecture predicts the order of vanishing of an L-function at certain integer points s, and the Bloch-Kato Conjecture predicts the leading coefficient at those points. Both statements are formulated in terms of a regulator:

$$R(p,q) : H^p_{\mathcal{M}}(X, \mathbb{Q}(q)) \otimes \mathbb{C} \to H^p_{\mathcal{D}}(X, \mathbb{Q}(q)) \otimes \mathbb{C}.$$

Conjecture. (Beilinson) If X be a smooth projective variety defined over $\mathbb{Z}$, then for $d = \dim(X)$ and $L_X(s)$ its L-function in weight d, we have

$$\mathrm{ord}_{s=d+1-q} L_X(s) = \dim \ker(R(d+1, q)).$$

for $d+1 < 2q$. The critical case $d+1 = 2q$ (ordinary Chow group) is slightly larger:

$$\mathrm{ord}_{s=q} L_X(s) = \mathrm{rk}\, A^q(X)_{hom}.$$

In that case, the Beilinson regulator fits into the middle of the diagram

$$\begin{tikzcd}
0 \arrow[r] & A^q(X)_{hom} \arrow[r] \arrow[d, "AJ"'] & H^{2q}_{\mathcal{M}}(X, \mathbb{Z}(q) \arrow[r] \arrow[d, "R(2q,q)"'] & Hdg(q) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & \mathrm{Jac}(X) \arrow[r] & H^{2q}_{\mathcal{D}}(X, \mathbb{Z}(q)) \arrow[r] & Hdg(q) \arrow[r] & 0.
\end{tikzcd}$$

Definition. The Deligne cohomology is defined by a hypercohomology group:

$$H^p_{\mathcal{D}}(X, \mathbb{Z}(q)) \simeq \mathbb{H}^p(X, 0 \to \mathbb{Z}_X(q) \to \mathcal{O}_X \to \Omega^1_X \to \dots \Omega^{q-1}_X \to 0).$$

Its key property is the long exact sequence:

$$\dots \to H^p_{\mathcal{D}}(X, \mathbb{Z}(q)) \to H^p(X, (2\pi i)^q \mathbb{Z}) \to H^p(X, \mathbb{C})/F^q \to H^{p-1}_{\mathcal{D}}(X, \mathbb{Z}(q)) \to \dots$$

which produces the short exact sequence above, assuming the Hodge Conjecture.

The Bloch-Kato conjecture gives the leading coefficient of $L_X(s)$ at the special points. It is too complicated to state here, but in the case where $X = \mathcal{O}_F$ for a number field F it recovers the class number formula:

$$\lim_{s \to 1} (s-1)\zeta_F(s) = \frac{2^{r_1} \cdot (2\pi)^{r_2} \cdot R_F \cdot h_F}{\omega_F \sqrt{D_F}},$$

as well as the Dirichlet unit theorem:

$$\operatorname{rk} \mathcal{O}_F^\times = r_1 + r_2 - 1.$$

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