Problems on Complex Hénon Maps 2

Fatou components: Volume decreasing (dissipative) case

Note: I expect that Problems A–D are probably very hard.

Problem A. *Can f have a wandering Fatou component?* Such a component would be a connected component Ω of the interior of K^+ such that $f^n(\Omega) \cap \Omega$ for all nonzero $n \in \mathbb{Z}$.

Problem 1. If Ω is a wandering component and q is a saddle point, is it necessary that $\Omega \cap W^u(q) \neq \emptyset$? Show that if q_1 and q_2 are saddle ponts, then $W^u(q_1) \cap \Omega \neq \emptyset$ if and only if $W^u(q_2) \cap \Omega \neq \emptyset$. *In other words, are wandering components "invisible" to unstable manifolds?*

Problem 2. Suppose that there is a wandering component Ω. If $\Omega \cap W^u(q) = \emptyset$, is there some computer picture that would somehow "capture" or "see" Ω?

Problem 3. Suppose Ω is a wandering component. Is it possible for Ω to be bounded? Can Ω have finite volume?

Problem B. Suppose $\Omega \subset \mathrm{int}(K^+)$ is a component with $f(\Omega) = \Omega$. *Is it possible that Ω is the basin of a rotational annulus $\mathcal{R}$?* In such a case, we would have a recurrent Fatou component without fixed points, and it would be biholomorphically equivalent to the product $A \times \mathbb{C}$, where $A = \{r < |\zeta| < R\}$ is an annulus.

Problem 4. A component of $\mathrm{int}(K^+)$ is necessarily polynomially convex, which is to say that if $S \subset \Omega$ is compact, then its polynomial hull is again inside Ω. *Is it possible to imbed $A \times \mathbb{C}$ into $\mathbb{C}^2$ so that the image is polynomially convex?* If it is not possible, then we would have a negative answer to Problem B.

Problem 5. Suppose that $\mathcal{D}$ is an invariant disk. That is, suppose there is a holomorphic imbedding $\varphi : \Delta \to \mathcal{D} \subset \mathbb{C}^2$ such that $f \circ \varphi(\zeta) = \varphi(\alpha\zeta)$ for all $\zeta \in \Delta$ and some $|\alpha| = 1$. It follows that the basin $W^s(\mathcal{D})$ is biholomorphically equivalent to $\Delta \times \mathbb{C}$. For almost every θ, there is a radial limit $\varphi^*(e^{i\theta}) := \lim_{r \to 1} \varphi(re^{i\theta})$. Let ν be the measure obtained by the pushing-forward under φ^* of the normalized circular measure $d\theta/(2\pi)$.

Show that the Lyapunov exponents of ν are 0 and $\log|\delta|$.

Show that for almost all θ, the stable set $W^s(\varphi^*(e^{i\theta}))$ is equivalent to $\mathbb{C}$?

Show that for almost all θ, $W^s(\varphi^*(e^{i\theta}))$ is dense in J^+?

Does it help if we assume that φ extends continuously to the boundary?

Problem 6. Suppose that $\mathcal{D}$ is an invariant disk, as in the previous problem, and let $\Omega \supset \mathcal{D}$ be the Fatou component containing it. Then there exists a map $\Phi : \Delta \times \mathbb{C} \to \Omega$ which conjugates f to a linear map.

Is it possible for Φ *to extend continuously to the boundary?* I expect that it does not.

Is Φ *bounded on the sets* $\Delta \times \{|w| < R\}$*?* If this is the case, then we have radial limits $\Phi_\theta(e^{i\theta}, \cdot) : \mathbb{C} \to \mathbb{C}^2$ for almost every θ. *Does it then follow that* $\Phi_\theta(\mathbb{C}) = W^s(\varphi(e^{i\theta}))$*?*

Problem C. Suppose that $\Omega = f(\Omega)$ is a non-recurrent Fatou component. Show that Ω is the basin of a semi-parabolic fixed point. (A fixed point is *semi-parabolic* if its multipliers are 1 and δ.) This has been proved recently by Lyubich and Peters in the case where $|\delta| < d^{-2}$. It will be a significant challenge to handle the more general case $|\delta| < 1$.

Problem D. Every Hénon map has infinitely many saddle cycles. It is known from Newhouse that there can be infinitely many attracting cycles. Is it possible for a Hénon map to have infinitely many cycles for which one of the multipliers has modulus 1?