SADDLEDROP: A TOOL FOR STUDYING DYNAMICS IN $\mathbb{C}^2$

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Abstract. In this expository note, we discuss the mathematics behind the computer program, SaddleDrop. Based upon ideas of a group at Cornell University, this program draws parameter space pictures for the complex Hénon map. The difficulty of this task is explained as we contrast the well-developed theory of one variable complex dynamics with the two variable Hénon case.

Introduction

Some of the most beautiful results of complex dynamics arise from studying families of maps which depend on parameters. One such example is the quadratic family $f_c(x) = x^2+c$, which is parameterized by the complex plane; for each $c \in \mathbb{C}$, there is a map $f_c : \mathbb{C} \to \mathbb{C}$. We can color each point c in the parameter space according to different dynamical properties of the map f_c. For this particular example, coloring the parameter space generates the Mandelbrot set, one of the most important objects in complex dynamics. There is a simple algorithm to draw this set, but it requires a crucial theorem from one variable dynamics.

The Hénon map $H_{a,c}(x, y) = (x^2 + c - ay, x)$ is a polynomial diffeomorphism of $\mathbb{C}^2$, where $(a, c) \in \mathbb{C}^2$, $a \neq 0$, are complex parameters. We can color the parameter space for this family just as before for the quadratic polynomials f_c: we color each pair $(a, c) \in \mathbb{C}^2$ based upon certain dynamical properties of the map $H_{a,c}$. However, it is much more difficult to know what to draw in this case. We not only face the issue of drawing in $\mathbb{C}^2$, but the theory from one variable dynamics that was so important in the Mandelbrot set example, begins to break.

As an undergraduate at Cornell University, K. Papadantonakis worked with J. H. Hubbard, J. Smillie, and E. Bedford to write a computer program called SaddleDrop which was one of the first to draw parameter space pictures for the Hénon map. In this note, we explain the mathematics behind the algorithm used by the program.

We first introduce some preliminaries from the field of complex dynamics, and a key result from the one variable case. We then discuss the Mandelbrot set in greater detail and explain the simple algorithm used to draw it. The remainder of the paper is devoted to two variable complex dynamics: we discuss the analogs of the theory of one variable dynamics when they exist, the challenges that arise when they do not, and the program SaddleDrop which conquers some of these difficulties.

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One Variable Preliminaries

We are interested in iterating maps and asking questions about what happens to certain points in the domain as we continue to iterate. For this section we suppose that $f : \mathbb{C} \to \mathbb{C}$ is a polynomial. Many of the following definitions and remarks hold for arbitrary spaces and maps, but we will not discuss those here.

To any point $p \in \mathbb{C}$, we can associate the sequence of all forward images of p: $f^n(p)$, where f^n is the composition of f with itself n times.

Definition 1. *Let $p \in \mathbb{C}$. The sequence of all forward images of p is called the orbit of p under f, and we denote it as $O^+(p)$. That is, $O^+(p) := \{f^n(p) : n \geqslant 0\}$. If f is invertible, we can define $O^-(p)$ to be the sequence of inverse images of the point p.*

It is quite reasonable to wonder how the sequences $O^+(p)$ behave for a given p. For example, there may be some special values of p where $O^+(p)$ is periodic. These points are naturally called periodic points of the map f. To every periodic point, we associate the corresponding periodic cycle.

Definition 2. *A point $p \in \mathbb{C}$ is periodic if $f^n(p) = p$ for some $n \geqslant 1$. The smallest such n is called the period of p.*

Definition 3. *If $p \in \mathbb{C}$ is periodic of period n, then the set $\{p, \dots, f^{n-1}(p)\}$ is called a periodic cycle of length n, or a periodic n-cycle.*

There are different types of periodic n-cycles. Their dynamical properties are classified by the value of the derivative of f^n, at a point p_i in the cycle. Note that the chain rule guarantees that the value of $(Df^n)(p_i) = (Df^n)(p_j)$, where p_i and p_j both belong to the periodic cycle. Hence this classification of periodic cycles is well defined.

Definition 4. *If p is a periodic point of period n and $|(Df^n)(p)| < 1$, then $\{p, \dots, f^{n-1}(p)\}$ is called an attracting cycle.*

An attracting cycle attracts an open set of points called a basin. Not all cycles are attracting of course; some are repelling.

Definition 5. *If p is a periodic point of period n and $|(Df^n)(p)| > 1$, then $\{p, \dots, f^{n-1}(p)\}$ is called a repelling cycle.*

We omit the case where p is a periodic point of period n and $|(Df^n)(p)| = 1$. This is significantly different from those listed above and will not be discussed here.

As p varies, we can partition $\mathbb{C}$ into two subsets based upon whether the sequence $O^+(p)$ is bounded.

Definition 6. *Let $K^+(f) := \{p \in \mathbb{C} : O^+(p)$ is bounded$\}$. If f is a polynomial, the set $K^+(f)$ is called the filled Julia set.*

The filled Julia set and its complement are dynamically interesting; these sets are both forward invariant; that is, $f(K^+(f)) \subset K^+(f)$, and $f(\mathbb{C} \setminus K(f)) \subset \mathbb{C} \setminus K(f)$, so we can consider f restricted to each as a separate dynamical system.

Example 1. *Let $f(z) = z^2$, and suppose $p \in \mathbb{C}$. We can compute a formula for $|f^n(p)|$, namely*

$$f^n(p) = p^{2^n}, \text{ and if } p = re^{i\theta}, \text{ then } |f^n(p)| = r^{2^n}.$$

As $n \to \infty$ there are three possible outcomes:

- *If $p \in \mathbb{D}$, then $r < 1$ and $|f^n(p)| \to 0$*
- *If $p \in \partial\mathbb{D}$, then $r = 1$ and $|f^n(p)| = 1$*
- *If $p \in \mathbb{C} \setminus \overline{\mathbb{D}}$, then $r > 1$ and $|f^n(p)| \to \infty$*

We see that if $p \in K^+(f) \iff p \in \overline{\mathbb{D}}$, so $K^+(f) = \overline{\mathbb{D}}$. Note for this example that $p = 0$ is an attracting fixed point of f, and $p = 1$ is a repelling fixed point. The basin of attraction for the fixed point at $p = 0$ is $\mathbb{D}$.

One may be tempted to think that the formula for $f^n(p)$ always lends itself so nicely to determining $K^+(f)$; this example is deceptive. The set can be amazingly complicated and almost always has fractal boundary. We can easily draw pictures of K^+ on the computer. For each point $p \in \mathbb{C}$ we ask if $O^+(p)$ is bounded. If yes, we color the p black, and if no, we color p grey[1]. This generates what is known as a dynamical space picture for f.

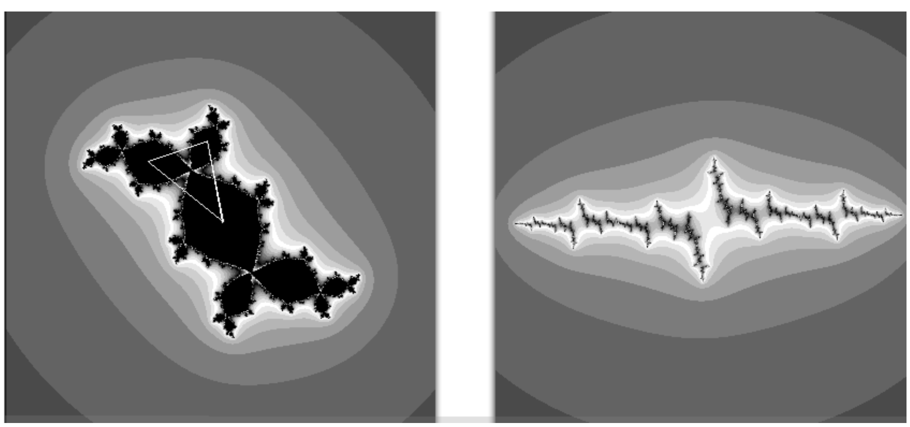

Figure 1. Above are the sets $K^+(f)$ for two different quadratic polynomials. One is connected and the other is a Cantor set. The one on the left contains an attracting 3-cycle given by the vertices of the triangle. The set on the left is called the "Douady rabbit".

[1]In this paper, we use grey for our pictures, but the computer program used to generate the pictures uses other colors.

It is quite natural to wonder what $K^+(f)$ looks like, and we can ask questions of a topological nature: Is it connected? If not, how many connected components does it have? We could just draw the computer picture to get a visual answer, but there is a very important theorem (due to Fatou) which is the key to rigorously answering the connectedness question question.

Theorem 1. *Let f be a polynomial of degree $d \geqslant 2$. If the filled Julia set $K^+(f)$ contains all of the finite critical points of f then both $K^+(f)$ and $\partial K^+(f)$ are connected.*

This theorem is incredibly useful. In the case where f is a quadratic polynomial, there is only one critical point to consider, so determining whether $K^+(f)$ is connected in that case is precisely asking what happens to the critical point under forward iteration. This lays the foundation for drawing parameter space pictures for polynomials; we will focus on the quadratics.

The Mandelbrot Set, M

We would like to draw a parameter space picture for the family of quadratic polynomials. It may appear that a quadratic polynomial, $f(x) = a_1x^2 + a_2x + a_3$, depends on three complex parameters, however, this polynomial is topologically conjugate to a unique polynomial of the form $g(x) = x^2 + c$ through an affine change of variables. Dynamically, it suffices to consider conjugacy classes of maps since orbits are preserved under conjugation. Hence the parameter space for the entire family of quadratic polynomials is $\mathbb{C}$.

Let $f_c(x) = x^2 + c$. As discussed in the introduction, it is desirable to color parameter space according to the dynamical properties of the map f_c. In particular, we would like to color the parameter plane according to properties of $K^+(f_c)$. Color $c \in \mathbb{C}$ black if the corresponding $K^+(f_c)$ is connected and color $c \in \mathbb{C}$ grey if the corresponding $K^+(f_c)$ is disconnected. The theorem above provides an easy algorithm for this task: we must follow the critical point of f_c, which is $z_0 = 0$. For each $c \in \mathbb{C}$ we have a polynomial f_c, and we must determine if $O^+(z_0) \subset K^+(f_c)$. This simple procedure generates the Mandelbrot set $M := \{c \in \mathbb{C} : K^+(f_c) \text{ is connected}\}$.

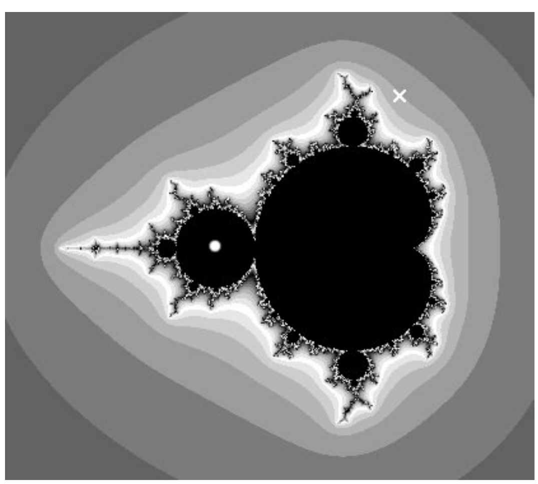

Figure 2. The Mandelbrot Set M with two parameters marked.

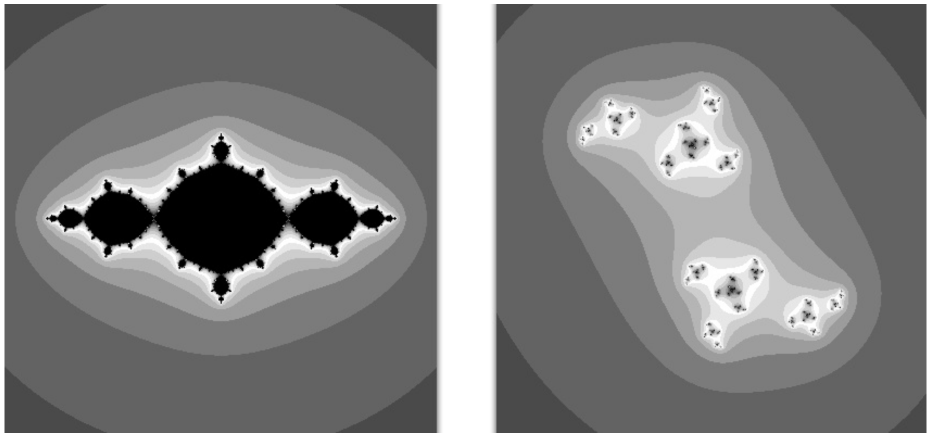

FIGURE 3. The set on the left is $K^+(f_c)$ for the polynomial $f_c(x) = x^2 - 1$. There is a white dot marking this parameter value in figure 2. The set on the right is $K^+(f_c)$ for the polynomial $f_c(x) = x^2 + 0.23 + 0.91i$. There is a white 'x' marking this parameter value in $\mathbb{C} \setminus M$ above.

Notice that there are grey curves in $\mathbb{C} \setminus M$ drawn above in figure 2. If $c \in \mathbb{C} \setminus M$, then $O^+(z_0) \to \infty$; we can color c according to how quickly z_0 escapes to infinity. Color $c \in \mathbb{C} \setminus M$ dark grey if z_0 escapes to infinity quickly, and color $c \in \mathbb{C} \setminus M$ light grey if z_0 escapes relatively slowly. We therefore have a "rate of escape" function defined on $\mathbb{C} \setminus M$ in parameter space. The grey curves in $\mathbb{C} \setminus M$ are level curves of this function.

In the quadratic case, there is a nice dichotomy: either $K^+(f_c)$ is connected or it is a Cantor set. If the filled Julia set is a Cantor set, then the dynamics of f_c on $K^+(f_c)$ is conjugate to the horseshoe dynamics of the one-sided shift map σ on the space of infinite sequences of two symbols. We mention this as an interesting fact; the theory behind it is omitted, and we will not require this result for any future discussion. More information can be found in [BuS] or [BH].

The Green Function

We refer the reader to figure 3 for this portion of the discussion. Consider the complement of $K^+(f_c)$; it is shaded in grey in both pictures. This set contains all points p that escape to infinity under iteration of f_c. Similar to the case of $\mathbb{C} \setminus M$ above, it is natural to ask how quickly these points escape: the darker the grey, the faster the point escapes to infinity. We therefore have a function, $G : \mathbb{C} \setminus K^+(f) \to (0, \infty)$, which measures the rate of escape of these points. This function is called a Green function, or a potential function for $K^+(f_c)$. Notice that there are curves faintly drawn in the grey regions of both figures; these are level curves of G. If we set $G := 0$ on $K^+(f_c)$, the Green function is defined on all of $\mathbb{C}$. For the polynomial $f_c(x) = x^2 + c$, the following gives a formula for G.

Definition 7. *The map $G : \mathbb{C} \to [0, \infty)$ associated to the filled Julia set $K^+(f_c)$ is defined as*

$$G(x) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ |f_c^n(x)|, \text{ where } \log^+ |z| = \max\{0, \log |z|\}$$

and is the Green function of $K^+(f_c)$. This function is subharmonic everywhere and harmonic on $\mathbb{C} \setminus \partial K^+(f_c)$. The level curves of this function discussed above are called equipotentials.

Critical points of G

From the formula for G, one can derive the equation $G(f_c(x)) = 2G(x)$, which implies that f_c maps equipotentials to equipotentials. The function G has critical points, and this equation reveals exactly what they are. The critical points we consider are contained in $\mathbb{C} \setminus K^+(f_c)$. We do not consider critical points of G for $z \in K^+(f_c)$ since $G = 0$ there.

$$G(f_c(z)) = 2G(z) \implies DG(f_c(z))f_c'(z) = 2DG(z)$$

Suppose $z \in \mathbb{C} \setminus K^+(f_c)$ is a critical point of G. We then have $DG(z) = 0$. Therefore if z is a critical point of G, then either of the following holds:

- $f_c'(z) = 0,$ which implies $z = z_0$
- $DG(f_c(z)) = 0,$ which implies $f_c(z)$ is also a critical point of G

Note that the first possibility implies $z_0 = 0$ is a critical point of G; this is indeed true if $z_0 \notin K^+(f_c)$.

Proposition 1. *If $z \in \mathbb{C} \setminus K^+(f_c)$ is a critical point of G, then there exists $m \geqslant 0$ such that $f_c^m(z) = z_0$.*

Proof. As mentioned above, if z is a critical point of G, then either $z = z_0$, or $f_c(z)$ is also a critical point of G. If $z = z_0$, then the proposition is true with $m = 0$. Suppose then that $z \neq z_0$. We therefore must have that $f_c(z)$ is also a critical point of G. Since $f_c(z)$ is a critical point of G, then either $f_c(z) = z_0$, and the proposition holds for $m = 1$, or $f_c(z) \neq z_0$, and $f_c(f_c(z))$ is a critical point by the second possibility above. We can continue in this way, and we see that if there is no m such that $f_c^m(z) = z_0$, then we must have that $f_c^m(z)$ is a critical point of G for every m. Since $z \in \mathbb{C} \setminus K^+(f_c)$, $O^+(z)$ is unbounded and $f_c^m(z) \to \infty$ as $m \to \infty$. We therefore have a sequence of critical points of G which are arbitrarily close to infinity. However, analysis of the formula for G reveals that $G(w) \sim \log|w|$ as $w \to \infty$. The function $\log|w|$ does not have critical points arbitrarily close to infinity, and we have arrived at a contradiction. □

The equation $G(f_c^n(z)) = 2^n G(z)$, and proposition 1 imply that the critical points of G are precisely the critical point, $z_0 = 0$ of f_c as well as all inverse images of this point under the polynomial f_c as long as these points are not in $K^+(f_c)$. If one of these points were contained in $K^+(f_c)$, then all of them would be, and G would have no critical points. We therefore observe that for the case $f_c(x) = x^2 + c$, $K^+(f_c)$ is connected if and only if G has no critical points, by theorem 1. This is a very important fact.

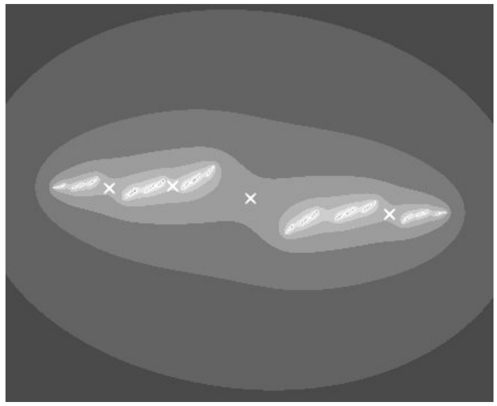

FIGURE 4. An example of a disconnected $K^+(f_c)$. The white marks are placed at a few critical points of G.

TWO VARIABLE PRELIMINARIES

We now advance to complex dynamics in two variables. The map we consider is the Hénon map, given by the formula

$$H_{a,c} : \mathbb{C}^2 \to \mathbb{C}^2, \quad \text{where} \quad H_{a,c} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}.$$

The Jacobian of $H_{a,c}$ equals a, and H^{-1} is given by

$$H_{a,c}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \frac{y^2+c-x}{a} \end{pmatrix}.$$

Note that this map is a polynomial diffeomorphism of $\mathbb{C}^2$ as long as $a \neq 0$. We can define two variable analogs for the Hénon map of the one variable objects previously discussed.

Definition 8. *Let $\mathbf{p} \in \mathbb{C}^2$. The sequence of all forward images of $\mathbf{p}$ is called the orbit of $\mathbf{p}$ under $H_{a,c}$, and we denote it as $O^+(\mathbf{p})$. The Hénon map is invertible, so we can define $O^-(\mathbf{p})$ to be the sequence of all inverse images of $\mathbf{p}$.*

Definition 9. *A point $\mathbf{p}$ is periodic if $H^n_{a,c}(\mathbf{p}) = \mathbf{p}$ for some $n \geqslant 1$. The smallest such n is called the period of $\mathbf{p}$.*

Definition 10. *If $\mathbf{p}$ is periodic of period n, then the set $\{\mathbf{p}, \dots\ H^{n-1}_{a,c}(\mathbf{p})\}$ is called a periodic cycle of length n.*

Definition 11. *Let $\mathbf{p}$ be a periodic point of period n, and let λ_1 and λ_2 be the eigenvalues of $DH^n_{a,c}(\mathbf{p})$. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then $\{\mathbf{p}, \dots\ H^{n-1}_{a,c}(\mathbf{p})\}$ is an attracting cycle.*

Definition 12. *Let $\mathbf{p}$ be a periodic point of period n, and let λ_1 and λ_2 be the eigenvalues of $DH^n_{a,c}(\mathbf{p})$. If $|\lambda_1| < 1$ and $|\lambda_2| > 1$, then $\{\mathbf{p}, \dots\ H^{n-1}_{a,c}(\mathbf{p})\}$ is a saddle cycle.*

We can also define repelling cycles as those periodic cycles where the eigenvalues λ_1 and λ_2 are greater than 1 in modulus. We shall be most concerned with saddle cycles. We can again partition our dynamical space into different sets depending upon the orbits of points.

Definition 13. *Let $K^+(H_{a,c}) := \{\mathbf{p} \in \mathbb{C} : O^+(\mathbf{p})$ is bounded$\}$.*

Definition 14. *Let* $K^-(H_{a,c}) := \{\mathbf{p} \in \mathbb{C} : O^-(\mathbf{p})$ *is bounded*$\}$.

Definition 15. *Let* $K(H_{a,c}) := K^+(H_{a,c}) \cap K^-(H_{a,c})$.

As in the one variable case the sets above and their complements are invariant sets and are therefore dynamically interesting, and we can ask topological questions: for a given pair of parameters (a, c), which of these sets above is connected? The answers to these questions are a bit complicated. Previously, we saw pictures of $K^+(f_c)$ for some quadratic polynomial examples (see figures 3 and 4). In the one variable case, the sets $K^+(f_c)$ and its complement were contained in $\mathbb{C}$; we could draw this copy of $\mathbb{C}$ on a computer screen, and observe visually whether these sets were connected.

The Algorithm: Dynamical Space

The dynamical space for $H_{a,c}$ is $\mathbb{C}^2$. We face two challenges. The first is how do we draw computer pictures in $\mathbb{C}^2$? One possible way is to slice $\mathbb{C}^2$ with a complex line, say l, and just as in the one variable case, we could examine which points in this line escape to infinity under forward iteration of the Hénon map. This line is a one-dimensional object and thus lends itself nicely to computer pictures. We could then color the points in our computer picture based on whether or not they escape. This provides us with a picture of the intersection of $K^+(H_{a,c})$ with the complex line.

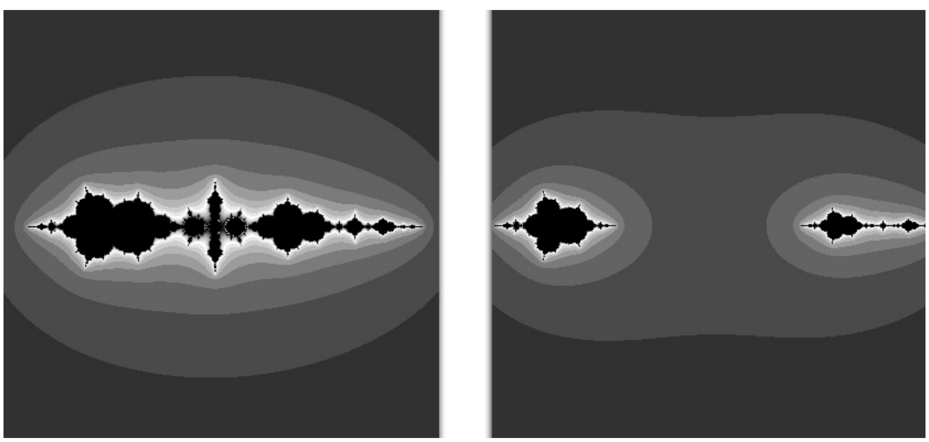

Figure 5. Both pictures depict the set $K^+(H_{a,c})$ for the parameter values $a = 0.3$, $c = -1.17$. On the left is a slice of $\mathbb{C}^2$ with the line $y = 1$ and on the right is a slice of $\mathbb{C}^2$ with the line $y = 4$.

The pictures above are a little disconcerting. In one picture, $K^+(H_{a,c})$ appears to be connected, but in the other picture it does not. It is possible for a set to be disconnected in a complex line l but to be connected in $\mathbb{C}^2$. And it is equally possible for a set to be connected in a complex line l but to be disconnected in $\mathbb{C}^2$. These pictures reveal nothing about the connectedness of $K^+(H_{a,c})$. This is our second challenge: is there a one-dimensional object in $\mathbb{C}^2$ we can draw, which provides topological information about any of the sets above? We will focus our attention on the set $K(H_{a,c})$.

Definition 16. *If* $\mathbf{p}$ *is a saddle fixed point of* $H_{a,c}$*, then the unstable manifold of* $\mathbf{p}$ *is the set* $W^u(\mathbf{p}) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 : H_{a,c}^{-n} \begin{pmatrix} x \\ y \end{pmatrix} \to \mathbf{p} \text{ as } n \to \infty \right\}$.

This set is an immersed one-dimensional manifold in $\mathbb{C}^2$. Moreover, note that it is an invariant set for $H_{a,c}$ and for $H_{a,c}^{-1}$, and is therefore a dynamical object of interest. Such a set exists for all $H_{a,c}$ that have a saddle fixed point.

Proposition 2. *For every pair of parameters* (a, c)*, with* $|a| \neq 1$*,* $H_{a,c}$ *has a saddle fixed point, except for those on the curve* $(a + 1)^2 = 4c$*.*

This result in [HP] guarantees that there is a saddle fixed point for practically any parameter pair (a, c), and for each saddle fixed point $\mathbf{p}$, there is an unstable manifold $W^u(\mathbf{p})$. This is a one-dimensional object, but it is all wound up in $\mathbb{C}^2$; drawing it with a computer would be rather difficult if it were not for the following result due to J. H. Hubbard.

Proposition 3. *Let* $\mathbf{p}$ *be a saddle fixed point of* $H_{a,c}$*. Let* $(\lambda, \mathbf{v})$ *be the eigenvalue-eigenvector pair of* $DH_{a,c}(\mathbf{p})$*, where* $|\lambda| > 1$ *is the expanding eigenvalue. The map* $\gamma : \mathbb{C} \to W^u(\mathbf{p})$*, given by*

$$\gamma(z) = \lim_{n\to\infty} H_{a,c}^n \left(\mathbf{p} + \frac{z}{\lambda^n} \mathbf{v} \right)$$

is an analytic injective immersion, where $H_{a,c}(\gamma(z)) = \gamma(\lambda z)$*.*

This has been a tremendously useful fact. For any saddle fixed point $\mathbf{p}$, the map γ gives a parameterization of $W^u(\mathbf{p})$ by $\mathbb{C}$. We can draw this copy of $\mathbb{C}$ on a computer. For each point $z \in \mathbb{C}$, we calculate $\gamma(z) \in W^u(\mathbf{p})$. We then examine $H_{a,c}^n(\gamma(z))$ as $n \to \pm\infty$ (since we are attempting to draw $K(H_{a,c})$, we must consider both forward and backward orbits). We color the original point z grey if either $O^+(\gamma(z))$ or $O^-(\gamma(z))$ is unbounded, and color z black if $O^+(\gamma(z))$ and $O^-(\gamma(z))$ are both bounded, or equivalently if $\gamma(z) \in K(H_{a,c})$. Note however, that $W^u(\mathbf{p}) \subset K^-(H_{a,c})$, so $O^-(\gamma(z))$ is necessarily bounded; in fact, $O^-(\gamma(z)) \to \mathbf{p}$ since $\gamma(z) \in W^u(\mathbf{p})$. We therefore need only investigate whether $O^+(\gamma(z))$ is bounded.

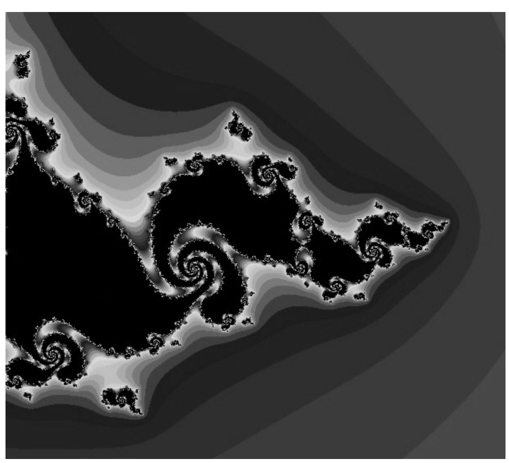

FIGURE 6. This is a copy of $\mathbb{C}$ colored as described above. In black we see $\gamma^{-1}(K(H_{a,c}))$, for the parameter values $a = 0.3, c = -0.375 + 0.6125i$.

We have found another one-dimensional object to draw in $\mathbb{C}^2$. Observe that in the figure above $K(H_{a,c}) \cap W^u(\mathbf{p})$ is disconnected. But is $K(H_{a,c})$ disconnected inside $\mathbb{C}^2$? How does the connectedness of $K(H_{a,c}) \cap W^u(\mathbf{p})$ relate to the connectedness of $K(H_{a,c})$ inside $\mathbb{C}^2$? Moreover, it is certainly possible for a Hénon map, $H_{a,c}$ to have two saddle fixed points. We may wonder how the connectedness of $K(H_{a,c}) \cap W^u(\mathbf{p}_1)$ compares with the connectedness of $K(H_{a,c}) \cap W^u(\mathbf{p}_2)$, where $\mathbf{p}_1$ and $\mathbf{p}_2$ are the two points in question. The following two theorems by Bedford and Smillie provide very promising answers to these questions in [BS6].

Theorem 2. *The set $K(H_{a,c})$ is connected if and only if $K(H_{a,c}) \cap W^u(\mathbf{p})$ is connected for some saddle point $\mathbf{p}$.*

Theorem 3. *$K(H_{a,c}) \cap W^u(\mathbf{p}_1)$ is connected for the saddle point $\mathbf{p}_1$ if and only if $K(H_{a,c}) \cap W^u(\mathbf{p}_2)$ is connected for the other saddle point $\mathbf{p}_2$.*

We have succeeded: we found a one-dimensional object to draw that provides topological information about $K(H_{a,c})$. The first part of the algorithm is complete.

The Algorithm: Parameter Space

Drawing a parameter space picture for the Hénon map is quite difficult. It is unclear where to even begin. The parameter space is $\mathbb{C}^2$, so again we face challenges of drawing in this space on a computer. In this case, slicing the parameter space $\mathbb{C}^2$ with a complex line will suffice. We choose to fix $a = a_0$, (where $0 < |a_0| < 1$) and examine a c parameter plane, which is just a copy of $\mathbb{C}$. We color each $c \in \mathbb{C}$ according to the properties of $K(H_{a_0,c})$. Alternatively, we could have fixed $c = c_0$ and examined a parameter planes as well, however, the c parameter planes provide analogs of the Mandelbrot set. Recall the formula for the Hénon map

$$H_{a,c}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}.$$

Notice that if we fix $a = a_0$, and vary c, the parameter we are changing is the translation; this translation parameter space is the analog of the parameter space $\mathbb{C}$ for the polynomials $f_c(x) = x^2 + c$. Indeed, if $a = 0$, we obtain the Mandlebrot set M in the c parameter plane.

Ideally, we would like to color points in $\mathbb{C}^2$ according to the properties of $K(H_{a_0,c})$. We could color a point $(a_0, c) \in \mathbb{C}^2$ black if $K(H_{a_0,c})$ is connected, and we could color it grey if $K(H_{a_0,c})$ is disconnected. But is there a systematic way to determine if $K(H_{a_0,c})$ is connected?

This is where we encounter one of the fundamental differences between complex dynamics in one variable and complex dynamics in several variables. Theorem 1 provided a very simple algorithm for drawing the Mandelbrot set: the connectedness of $K^+(f_c)$ was precisely related to the orbit of the critical point z_0 of the polynomial f_c. Unfortunately, we cannot even hope that such a statement will be as helpful to us for $H_{a,c}$. One immediate observation is that $H_{a,c}$ is a diffeomorphism and therefore has no critical points. Regardless of this, there is no direct analog of theorem 1 in higher dimensions. We must find another criterion with which to draw parameter space.

Fix a parameter pair, (a, c). Consider $W^u(\mathbf{p})$; let $\mathbf{z} \in W^u(\mathbf{p})$. Either $O^+(\mathbf{z})$ is bounded or it is not, so either $\mathbf{z} \in K(H_{a,c})$ or it is not. If it is not, then $||H^n_{a,c}(\mathbf{z})|| \to \infty$ as $n \to \infty$, and we can therefore measure how quickly $\mathbf{z}$ escapes, providing an analog of the Green function.

Definition 17. *The map $G^+ : \mathbb{C}^2 \to [0, \infty)$ associated to $K^+(H_{a,c})$ defined as*

$$G^+\begin{pmatrix} x \\ y \end{pmatrix} = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \left\| H^n_{a,c}\begin{pmatrix} x \\ y \end{pmatrix} \right\|$$

is the Green function of $K^+(H_{a,c})$. This function is plurisubharmonic everywhere and pluriharmonic (harmonic on any complex line) on $\mathbb{C}^2 \setminus \partial K^+(H_{a,c})$.

Note that this function is identically 0 on $K^+(H_{a,c})$. We can examine $G^+ \circ \gamma$ where γ is the parameterization of $W^u(\mathbf{p})$ by $\mathbb{C}$. Observe that $K^+(H_{a,c}) \cap W^u(\mathbf{p}) \subset K(H_{a,c})$; in fact, $K^+(H_{a,c}) \cap W^u(\mathbf{p}) = K(H_{a,c}) \cap W^u(\mathbf{p})$, so G^+ restricted to $W^u(\mathbf{p})$ is defined on $W^u(\mathbf{p}) \setminus K^+(H_{a,c}) = W^u(\mathbf{p}) \setminus K(H_{a,c})$. In our computer pictures, we color $\gamma(z) \notin K(H_{a,c})$ according to how fast it escapes under forward iteration. In the pictures below, we see faint grey curves drawn in the complement of $K(H_{a,c})$. These are level curves of the function G^+. Remember that these pictures are really $\gamma^{-1}(W^u(\mathbf{p}))$.

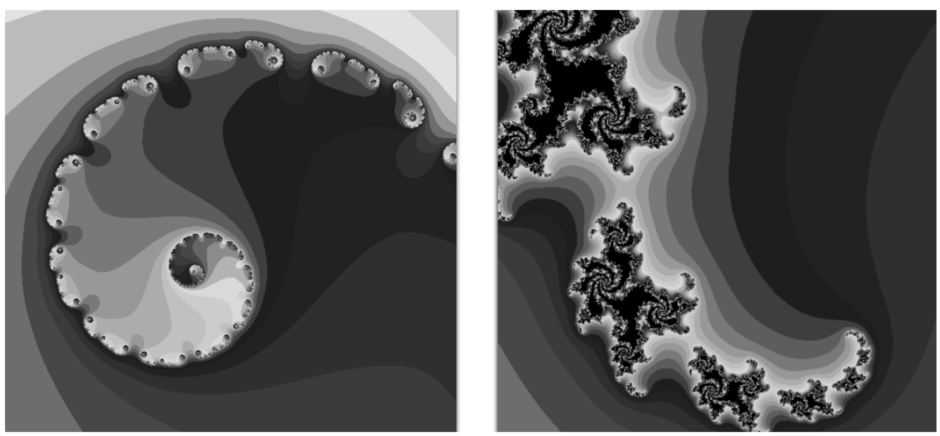

FIGURE 7. These are unstable manifolds for two different pairs of parameter values. For the map on the left, it appears that $K(H_{a,c}) \cap W^u(\mathbf{p})$ is a Cantor set, and for the map on the right, $K(H_{a,c}) \cap W^u(\mathbf{p})$ is disconnected but not a Cantor set.

Although there are no critical points of our map $H_{a,c}$, there are sometimes critical points of $G^+ \circ \gamma$; that is, for some parameter values there are critical points of G^+ restricted to $W^u(\mathbf{p})$. Recall that for the case $f_c(x) = x^2 + c$, $K^+(f_c)$ is connected if and only if there are no critical points of the Green function G in the dynamical space $\mathbb{C}$. This suggests that the critical points of G^+ in $W^u(\mathbf{p})$ may be related to the connectedness of $K(H_{a,c})$. This is indeed the case; Bedford and Smillie assert in [BS6] that $K(H_{a,c})$ is connected if and only if for μ almost every $\mathbf{p}$, $G^+|_{W^u(\mathbf{p})\setminus K^+(H_{a,c})}$ contains no critical points. The measure μ is a special harmonic measure that will not be discussed here.

This result is somewhat encouraging; it provides a preliminary criterion for drawing a parameter space picture. Fix a parameter pair (a_0, c_0) and find a saddle fixed point, $\mathbf{p}$. Draw the dynamical space $W^u(\mathbf{p})$. In $W^u(\mathbf{p})$, find a critical point of G^+ if there are any. Color (a_0, c_0) in parameter space according to the 'behavior' of this critical point. Note that these critical points of G^+ are contained in $W^u(\mathbf{p}) \setminus K(H_{a_0,c_0}) = W^u(\mathbf{p}) \setminus K^+(H_{a_0,c_0})$, which means that they have unbounded forward orbits, but bounded backward orbits. We can therefore measure how quickly these critical points escape forward to infinity. More specifically, color (a_0, c_0) according to how fast the critical point escapes. Move through parameter space, coloring each parameter pair (a, c) according to how fast a critical point of G^+ in $W^u(\mathbf{p}) \setminus K(H_{a,c})$ escapes.

There are some issues that arise with this algorithm, namely, if there are any critical points of G^+ in $W^u(\mathbf{p})$, then there are infinitely many. How can we be sure that this algorithm is well defined? If α_1 and α_2 are two critical points in $W^u(\mathbf{p})$, how do we color (a_0, c_0) according to behavior of the critical points if the critical points display two different behaviors? The way we color the parameter pair fundamentally depends on which critical point we choose; the algorithm is not well defined. In fact, essentially different parameter space pictures can arise if a different critical point is chosen. This is actually part of the richness of the subject: we do not fully understand this phenomenon yet, but it is very tantalizing and continues to create more and more questions about $H_{a,c}$.

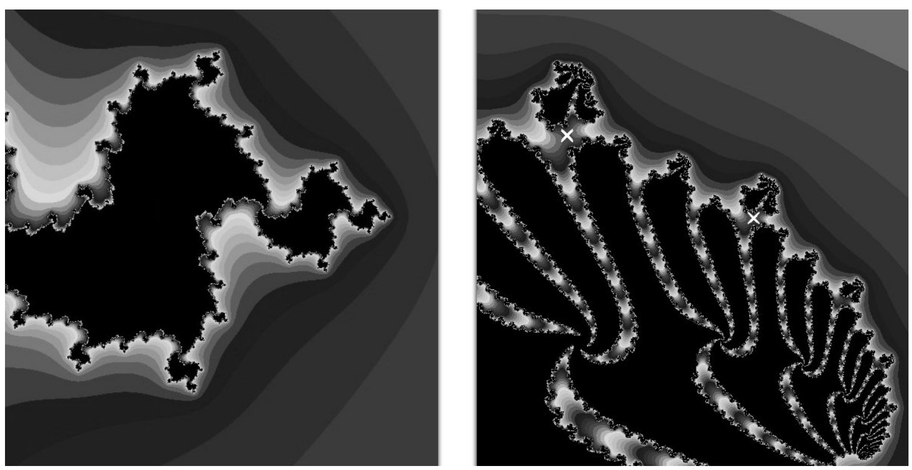

FIGURE 8. On the left is a picture of $W^u(\mathbf{p})$ for a parameter pair (a, c) such that $W^u(\mathbf{p}) \setminus K(H_{a,c})$ contains no critical points of G^+. On the right is a picture of $W^u(\mathbf{p})$ for a parameter pair (a, c) such that $W^u(\mathbf{p}) \setminus K(H_{a,c})$ contains infinitely many critical points of G^+. Two of them are marked with a white 'x'.

Another issue that surfaces with this algorithm surrounds the idea of 'moving around in parameter space', coloring each pair (a, c) according to how fast a critical point of G^+ escapes. Let $\mathbf{p}_1$, $\mathbf{p}_2$ be saddle fixed points for the parameter pairs (a_1, c_1) and (a_2, c_2) respectively. As we move from parameters (a_1, c_1) to (a_2, c_2) does the critical point we select for (a_1, c_1) in $W^u(\mathbf{p}_1)$ influence the choice of the one we select for (a_2, c_2) in $W^u(\mathbf{p}_2)$? Is there any continuity of the critical point selection as the parameter pair is varied? Is there any systematic way of choosing a corresponding critical point as we move in parameter space? The answer is yes: one systematic approach is to use Newton's method. This is the algorithm in the program SaddleDrop.

SaddleDrop

The program SaddleDrop was written by Karl Papadantonakis in 2000 at Cornell University. He collaborated with J. H. Hubbard, J. Smillie, and E. Bedford to draw a parameter space for the Hénon map using the algorithm outlined above. To summarize the points above and to be a bit more precise, we outline the algorithm:

GOAL : Draw parameter space pictures for the Hénon map.

- Fix a parameter. Suppose we fix $a = a_0$.
- Choose a single value of the other parameter; suppose we pick $c = c_0$.
- For the parameter pair (a_0, c_0), find a saddle fixed point $\mathbf{p}$ of H_{a_0,c_0}.
- Draw $W^u(\mathbf{p})$ as discussed in the dynamical space algorithm.
- Find a critical point, α, of G^+ in $W^u(\mathbf{p}) \setminus K(H_{a_0,c_0})$. If there are no critical points, change the initial parameter pair recalculating $\mathbf{p}$ and $W^u(\mathbf{p})$ until a critical point of G^+ is found in $W^u(\mathbf{p}) \setminus K(H_{a_0,c_0})$.
- For the parameter pair (a_0, c_0), measure how fast α escapes to ∞. Color (a_0, c_0) accordingly.
- Move to new parameter pair (a_0, c'), keeping a_0 but changing c_0 to c'. Let $\mathbf{p}'$ be the saddle fixed point for this new parameter pair.
- Use Newton's method to find a corresponding critical point α' in $W^u(\mathbf{p}')$.
- Color (a_0, c') according to how fast α' escapes.
- Do this for all parameter pairs (a_0, c), where $c \in \mathbb{C}$.

The end result will be a c parameter space picture for $H_{a_0,c}$.

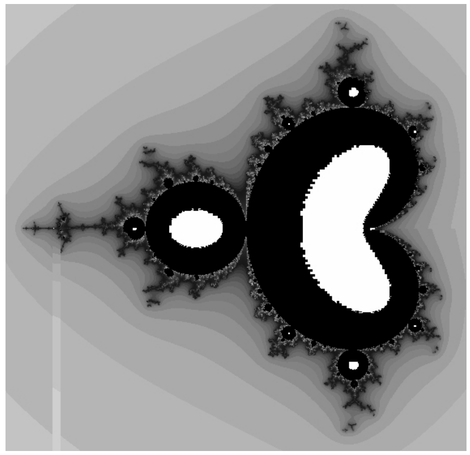

Figure 9. Above is the c parameter space picture for $a = -0.2$ obtained with SaddleDrop.

Notice that the above picture somewhat resembles a squished Mandelbrot set. This is not so surprising as the Hénon map can be viewed as a two variable perturbation of the polynomial $f_c(x) = x^2 + c$, for small enough values of $|a|$. The picture is colored according to the behavior of one particular critical point. For the grey c values surrounding the black set in the middle, the critical point found by Newton's method escapes to infinity.

The black and white c parameters are rather interesting. For the black c values in the middle, Newton's method converged to a critical point that escaped very slowly; slower than some set tolerance. For the parameter values that are colored white, Newton's method did not converge to a critical point of G^+, so the program could not color these parameter values according to the behavior of a critical point since it could not find one. Perhaps two critical points of G^+ coalesced for these c parameter values, thus confusing Newton's method there. Note also there are lines of discontinuity visible in part of the grey region; these are byproducts of Newton's method which is used as we move through parameter space.

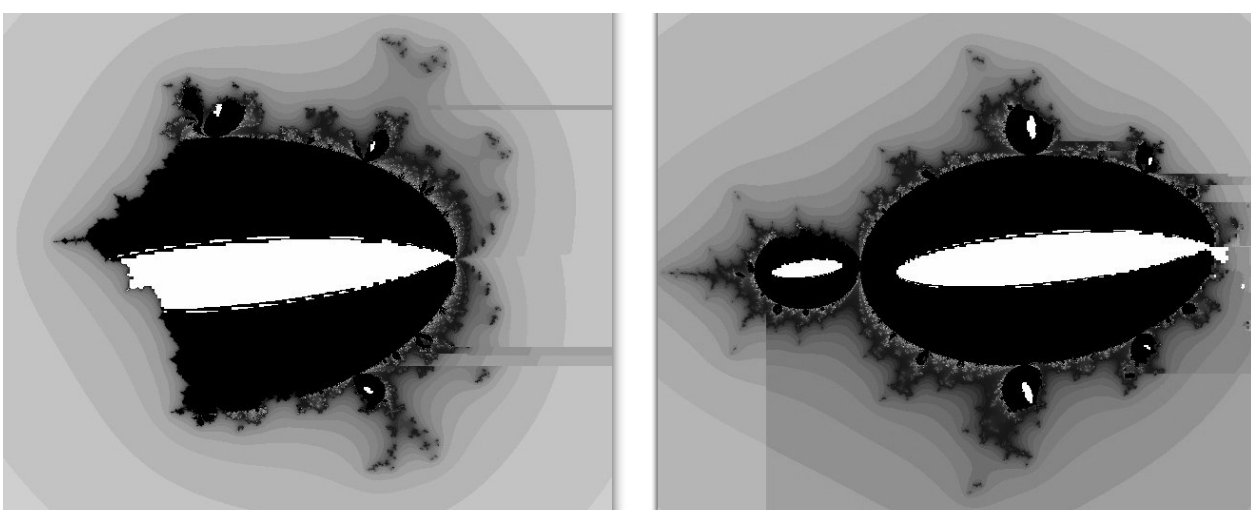

Figure 10. Above are two c parameter space pictures for $a_0 = 0.37 + 0.04i$. Each was obtained by following around a critical point of G^+ in $W^u(\mathbf{p})$.

The pictures above appear to be very different. The picture on the left was obtained by following around a critical point in $W^u(\mathbf{p})$, while the picture on the right was obtained by following around a different critical point. The parameter pictures vary greatly with the critical point originally chosen. How the parameter space picture depends on the critical point is quite interesting. We do not understand this yet but have some conjectures. This is one of the many questions that have arisen as a result of this program. We have also found some strange phenomena in parameter space; an example of this is fingering, see figure 11.

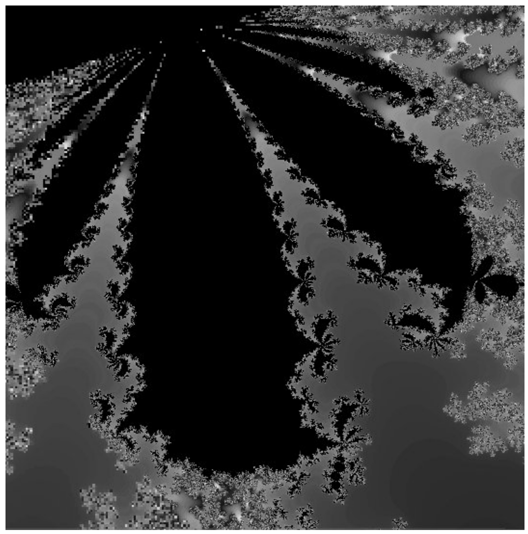

FIGURE 11. A zoom-in of a c parameter space; note the finger-like appearance of the black set.

SaddleDrop has been an extremely important contribution to the study of Hénon maps. There is still much to understand about the program; it has given us some of the first glimpses of parameter space for $H_{a,c}$ and continues to raise many fruitful questions about complex dynamics in more than one variable.[2]

[2]SaddleDrop is available for download at http://www.math.cornell.edu/~dynamics.

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