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Laminar currents in $\mathbb{P}^2$

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Abstract. In this paper we study laminar currents in $\mathbb{P}^2$. Given a sequence of irreducible algebraic curves (C_n) converging in the sense of currents to T, we find geometric conditions on the curves ensuring that the limit current T is laminar. This criterion is then applied to meromorphic dynamical systems in $\mathbb{P}^2$, and laminarity of the dynamical "Green" current is obtained for a wide class of meromorphic self maps of $\mathbb{P}^2$, as well as for all bimeromorphic maps of projective surfaces.

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1. Introduction

On the two extreme sides of positive closed currents are smooth positive forms and currents of integration over varieties. One could suspect that the "general" closed positive current should be reminiscent of the geometric structure of currents of integration together with a measure theoretic structure arising from positivity. This was conjectured in the fundamental article [Su], where the foundations of the geometric theory of currents were settled.

There are examples, however, showing that this is not so simple: the famous Wermer example provides a positive closed $(1, 1)$ current in the bidisk whose support contains no analytic disk (see [DS]).

Laminar currents were introduced by Bedford, Lyubich and Smillie [BLS] as a class of geometric currents in dimension two, flexible enough for applications. A positive $(1, 1)$ current is *uniformly laminar*, if it is locally described by integration over families of disjoint graphs, it is *laminar* if it is an increasing limit of uniformly laminar currents (see section 2 for details). This definition fits elegantly into Pesin theory, providing a powerful tool in complex non uniformly hyperbolic dynamics.

In the present paper we are interested in constructing laminar currents in $\mathbb{P}^2$. Laminar currents appeared as currents of integration over entire curves in [BLS], [Ca], and as limits of dynamically defined rational divisors in [BS5]. It is well

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known that all positive closed currents in $\mathbb{P}^2$ are limits of sequences of rational divisors [De2]. Suppose $d_n^{-1}[C_n]$ converges in the sense of currents to T. Under which geometric conditions on C_n is T laminar? We prove (section 3) the following simple criterion, independent of holomorphic dynamics : for an irreducible algebraic curve C_n of degree d_n in $\mathbb{P}^2$ we denote by g_n the geometric genus of C_n (i.e. the genus of the normalization of C_n), and if $x \in Sing(C_n)$ is a singular point, we let $n_x(C_n)$ be the number of local irreducible components at x.

Theorem 1. *Suppose the C_n are irreducible curves such that $d_n^{-1}[C_n] \rightharpoonup T$. Then, using the notations above, if*

$$g_n + \sum_{x \in Sing(C_n)} n_x(C_n) = O(d_n)$$

then T is laminar.

Note that, by the genus formula, sequences of smooth curves do never satisfy the assumption of the theorem. Furthermore all currents in $\mathbb{P}^2$ are limits of smooth divisors so the condition of the theorem is not necessary.

The proof of the theorem is modeled on the results of Bedford, Lyubich and Smillie [BLS], [BS5], who proved laminarity of T when the curves C_n are iterates of some line by a polynomial automorphism of $\mathbb{C}^2$. This corresponds in this theorem to the case $g_n = 0$ and C_n is singular at one point I, with $n_I(C_n) = 1$.

This criterion can in turn be applied to obtain new examples of laminar currents in holomorphic dynamics on $\mathbb{P}^2$. Let $\mathcal{M}_d$ be the space of rational maps of degree d, which has the structure of a projective space. We say that a subset $\mathcal{U} \subset \mathcal{M}_d$ is Zariski residual if it contains a countable intersection of Zariski open sets. It is known [Si] that the existence of the dynamical "Green" current is valid on the Zariski residual set of dominating algebraically stable maps (for details see section 4).

Theorem 2. *There exists a Zariski residual set $\mathcal{U} \subset \mathcal{M}_d$, such that if $f \in \mathcal{U}$ and its topological degree $d_t(f)$ satifies $d_t(f) < d$, then the Green current of f exists and is laminar.*

Concerning the case of bimeromorphic maps ($d_t = 1$) of smooth connected projective surfaces, it is possible to state a more precise result. It is a combination of results of Cantat [Ca] and Diller-Favre [DF], that if such a map has positive topological entropy, then its Green current always exists (in a sense to be precised, see section 5).

Theorem 3. *Let a bimeromorphic map of positive topological entropy of a projective surface. Then its Green current is laminar.*

The precise outline of the paper is as follows : in section 2 we recall the definition of laminar currents, in section 3 we prove theorem 1. Sections 4 and 5 are respectively devoted to the proofs of theorems 2 and 3.

2. Laminar currents

In this section we collect some definitions and results from [BLS] (see also [BS5], [Ca]; some observations below are taken from Cantat's thesis). All definitions are local so we consider an open subset Ω of $\mathbb{C}^2$, and T is a positive (1, 1) current in Ω.

We let $\mathrm{Supp}(T)$ denote the (closed) support of T, $\|T\|$ the trace measure and $\mathbf{M}(T)$ the mass norm (for general references on positive currents see e.g. [LG], [De1]).

Definition 2.1. *T is uniformly laminar if for all $x \in \mathrm{Supp}(T)$ there exists open sets $V \supset U \ni x$, with V biholomorphic to the unit bidisk $\mathbb{D}^2$ so that in this coordinate chart $T|_U$ is the direct integral of integration currents over a measured family of disjoint graphs in $\mathbb{D}^2$, i.e. : there exists a measure λ on $\{0\} \times \mathbb{D}$, and a family (f_a) of holomorphic functions $f_a : \mathbb{D} \to \mathbb{C}$ such that $f_a(0) = a$, the graphs Γ_{f_a} of two different f_a's are disjoint, and*

$$T|_U = \int_{\{0\}\times\mathbb{D}} [\Gamma_{f_a} \cap U]\, d\lambda(a).$$

It is easily proven that the holonomy map is automatically continuous, and in particular there is an embedded lamination in $\mathrm{Supp}(T)$; moreover T is closed, since it is closed in all coordinate charts U. Unfortunately this definition is much too restrictive for dynamical purposes: for example there is no uniformly laminar current in $\mathbb{P}^2$ except integration currents on smooth curves. Indeed a uniformly laminar current not charging curves would have (homological) self intersection 0 [HM] (this observation first appeared in a slightly different form in [CLS]).

To avoid technicalities we give an adapted definition of laminar currents. The new terminology was suggested to us by the referee.

Definition 2.2. *T is laminar in Ω if there exists an increasing sequence of currents $(T^{(i)})_{i\geq 0}$, such that for every i there exists a finite subdivision $\mathcal{Q}_i$ of Ω (up to a set of zero $\|T\|$ measure) into disjoint open subsets and*

$$T^{(i)} = \sum_{Q\in\mathcal{Q}_i} T_Q^{(i)}$$

is the sum of the currents $T_Q^{(i)}$, uniformly laminar in $Q \in \mathcal{Q}_i$, and such that

$$\lim_{i\to\infty} T^{(i)} = T.$$

Note that replacing $\mathcal{Q}^{(i)}$ by the subdivision $\mathcal{Q}^{(1)} \wedge \cdots \wedge \mathcal{Q}^{(i)}$ consisting of the open sets $Q_1 \cap \cdots \cap Q_n$, $Q_i \in \mathcal{Q}^{(i)}$, one can always assume that the sequence of subdivisions increases also. This definition contains the one given in [BLS] :

direct integral of currents of integration over a measured family of disjoint disks in Ω (see [BLS] Proposition 6.2).

Some of the difficulties occuring when dealing with laminar currents are illustrated in the following examples.

Example 2.3. An interesting example of laminar current in $\mathbb{P}^2$ is studied in detail in Demailly [De2]: let $T = dd^c \max(\log^+ |z|, \log^+ |w|)$. Demailly proves that

$$T = \int_{S^1} [\{e^{i\theta}\} \times \mathbb{D}] d\lambda(\theta) + \int_{S^1} [\mathbb{D} \times \{e^{i\theta}\}] d\lambda(\theta) + \int_{S^1} [V_\theta] d\lambda(\theta),$$

where $\mathbb{D}$ is the unit disk, λ is the Lebesgue measure on the unit circle S^1, and $V_\theta = \left\{(z, w) \in \mathbb{C}^2,\ z = e^{i\theta} w, |z| > 1\right\}$. So T is a closed laminar current, with continuous local potential (in the whole of $\mathbb{P}^2$). Nevertheless $T \wedge T$ (the Lebesgue measure on the unit torus) has positive mass, and there is a set of positive $\|T\|$ measure of disks with transversely intersecting analytic continuations.

Let $T_r = dd^c \max(\log^+ \left|\frac{z}{r}\right|, \log^+ \left|\frac{w}{r}\right|)$, and consider the current $T' = \int_1^2 T_r dr$. T' is positive and closed, and can be decomposed as

$$\begin{aligned} T' = &\int_1^2 \int_{S^1} [\{re^{i\theta}\} \times \mathbb{D}] d\lambda(\theta) dr + \int_1^2 \int_{S^1} [\mathbb{D} \times \{re^{i\theta}\}] d\lambda(\theta) dr \\ &+ \int_1^2 \int_{S^1} [r V_\theta] d\lambda(\theta) dr. \end{aligned}$$

Let T_3' be the third term on the right side, and let ϕ be a (1,1) test form,

$$\begin{aligned} \langle T_3', \phi \rangle &= \int \left(\int_{r V_\theta} \phi \right) dr d\lambda(\theta) = \int \left(\int_{V_\theta} \mathbf{1}_{|z|>r} \phi \right) dr d\lambda(\theta) \\ &= \int \left(\int_{V_\theta} \alpha(|z|) \phi \right) d\lambda(\theta), \end{aligned}$$

where $\alpha(s) = \min(s - 1, 1)$ on $[1, +\infty)$. Approximating $\alpha(|z|)$ from below by locally constant functions proves that T' also satisfies our definition 2.2 of laminar currents. Observe that T' cannot be locally written as a direct integral of disjoint disks, because α is not locally constant.

An explicit computation proves that $T' \wedge T'$ is (a positive constant times) Lebesgue measure on the annulus $2V_\theta \backslash V_\theta$, integrated over θ. In particular $T' \wedge T'$ is absolutely continuous with respect to the trace measure of T', although T' is laminar.

Remark 2.4. It is possible to prove that if T is a laminar current in $\mathbb{P}^2$ and T is the limit of a sequence of curves satisfying the criterion of theorem 1, then if T has continuous potential in some open set Ω, $T \wedge T = 0$ in Ω. This implies that the Demailly example 2.3 cannot be approximated by such a sequence of curves. We postpone this issue to a future paper.

3. Laminar currents as limits of divisors in $\mathbb{P}^2$

It is well known that any positive closed current in $\mathbb{P}^2$ is the limit in the weak sense of currents, of a sequence of rational divisors [De2]. In this section we prove that under some geometric conditions on the (irreducible) divisors, the limit is a laminar current; this criterion turns out to be useful in some dynamical problems (see section 4).

We begin with some notation : (C_n) is a sequence of (possibly singular) irreducible curves in $\mathbb{P}^2$, with $d_n = \deg(C_n) \to \infty$, such that $\frac{1}{d_n}[C_n] \rightharpoonup T$, where $\rightharpoonup$ denotes the weak convergence of currents. We denote by:

- $\pi : \widehat{C}_n \to C_n$ the resolution of singularities of C_n;
- g_n the geometric genus of C_n, i.e. $g_n = \text{genus}(\widehat{C}_n)$;
- $\nu_x(C_n)$ the multiplicity of C_n at x; $\nu_x(C_n)$ is the number of intersection points of C_n with a generic line near x;
- $n_x(C_n)$ the number of local irreducible components of C_n at $x \in C_n$, that is, $n_x(C_n) = \#\pi^{-1}(x)$.

Theorem 3.1. *Let (C_n) be a sequence of irreducible curves in $\mathbb{P}^2$ of degree $d_n \to \infty$, such that the sequence of rational divisors $d_n^{-1}[C_n] \rightharpoonup T$. Then, using the notations above, if*

$$g_n + \sum_{x \in Sing(C_n)} n_x(C_n) = O(d_n) \tag{1}$$

then T is laminar.

The condition of the theorem is of course not necessary: take an arbitrary sequence of curves (C_n), and pick a sequence $r_n \to \infty$, such that $d_n^{-1}[h_{r_n}(C_n)]$ tends to the line at infinity in the sense of currents where $h_r(z, w) = (rz, rw)$. A less trivial example is the current $dd^c \max(\log^+ |z|, \log^+ |w|)$ considered in example 2.3, which is the limit of the sequence $\frac{1}{n}[D_n]$ where D_n is the smooth curve (in homogeneous coordinates $[z : w : t]$) of equation $z^n + w^n + t^n = 0$. Concerning this last current, we have in fact the following finer result (see remark 2.4): T is not approximable in the weak sense by a sequence of divisors $d_n^{-1}[C_n]$ satisfying (1).

The remaining of this section will be devoted to the proof of theorem 1. The basic idea to count good and bad components is similar to [BLS], [BS5], but this theorem is independent of any dynamical context.

Let $p \in \mathbb{P}^2$ and π_p be the central projection $\mathbb{P}^2 \backslash \{p\} \to \mathbb{P}^1$ ([GH]); we consider a subdivision $\mathcal{Q}$ of $\mathbb{P}^1$ into disjoint simply connected open sets (which we will call "squares") which have the same area with respect to the Fubini-Study metric $\omega_{\mathbb{P}^1}$; we also suppose that the boundaries of the squares are piecewise smooth.

Such a subdivision $\mathcal{Q}$ will be called *admissible*. Note that $\pi_p^{-1}(Q)$ is isomorphic to $Q \times \mathbb{C}$ ($Q \in \mathcal{Q}$).

We say that a connected component Γ of $C_n \cap \pi_p^{-1}(Q)$ is *good* if $\pi_p : \Gamma \to Q$ is a homeomorphism (i.e. Γ is a graph over Q), *bad* if not. The *multiplicity* of the bad component Γ is the topological degree $\deg(\pi_p|_\Gamma)$. The number of bad components of C_n for the projection π_p with respect to $\mathcal{Q}$ is by definition always counted with multiplicity. We denote this number by $bc_p(C_n, \mathcal{Q})$.

It is clear that if $p \notin C_n$ the component Γ over Q is bad if and only if for some $z \in Q$ the line (pz) is tangent to C_n or hits a singular point of C_n. The next lemma says that the number of bad components for a generic projection is maximal and does not depend on the subdivision provided it is fine enough. Such a lemma is necessary because we will have to make $n \to \infty$ with $\mathcal{Q}$ fixed, so we cannot find a subdivision which is fine enough for all curves C_n.

Lemma 3.2. *Fix p such that: $p \notin C_n$, p does not lie on any line joining two singular points of C_n, nor any line through a singular point and tangent to C_n at some smooth point (this is a Zariski dense condition).*

Let

$$bc_p(C_n) = \sup\left\{bc_p(C_n, \mathcal{Q}),\ \mathcal{Q} \textit{ admissible subdivision of } \mathbb{P}^1\right\},$$

then if $\mathcal{Q}$ is an admissible subdivision of $\mathbb{P}^1$ such that the critical points of $\pi_p|_{C_n}$ (including singular points of C_n) lie over different squares of $\mathcal{Q}$, then $bc_p(C_n, \mathcal{Q}) = bc_p(C_n)$.

Proof. We are looking for admissible subdivision maximizing the number of bad components, with p fixed as stated in the lemma. It is clear that the number of bad components decreases if some singular fiber $(pz) = \pi_p^{-1}(z)$ of the projection – that is a fiber tangent to C_n or meeting a singular point of C_n – lies over the boundary of a square Q. Then we can assume $\mathcal{Q}$ is chosen so that no singular fiber lies over ∂Q for $Q \in \mathcal{Q}$.

Now fix Γ a bad component of multiplicity d over $Q \in \mathcal{Q}$; suppose $\pi_p^{-1}(z_1^s), \dots, \pi_p^{-1}(z_k^s)$ are the singular fibers. We take an admissible subdivision of Q into squares $\{Q_i,\ 1 \le i \le \ell\}$ separating the singular fibers. Without loss of generality we assume for $1 \le i \le k$, $Q_i \ni z_i^s$. Bad components for the new subdivision are over $Q_1, \dots, Q_k$, with respective multiplicities $d_1, \dots, d_k$. We have to prove that $\sum_i d_i \ge d$.

Suppose this is false. Fix a regular fiber $\pi_p^{-1}(z^r)$ and paths γ_i in Q joining z_i^s to z^r. There are d local good plaques over a small neighborhood of z^r. For each $1 \le i \le k$ and $z \in \gamma_i$ near z_i^s there are d_i local plaques over z corresponding to the bad component over Q_i; follow them by analytic continuation along γ_i. We get no more than $\sum d_i < d$ plaques over z^r. The remaining $d - \sum d_i$ plaques correspond by analytic continuation along $-\gamma_i$ to good plaques at each singular fiber $\pi_p^{-1}(z_i^s)$, giving rise to global good components over Q, a contradiction.

Any further refinement of the subdivision will produce the same number of bad components. □

Proposition 3.3. *Let C_n be an irreducible curve in $\mathbb{P}^2$, and p a (Zariski-) generic point in $\mathbb{P}^2$. Then (notations as in the beginning of the section):*

$$bc_p(C_n) \leq 2\,(2g_n - 2 + 2d_n) + \sum_{x \in Sing(C_n)} n_x(C_n).$$

Proof. We first list the generic assumptions we make for choosing p:

- $p \notin C_n$,
- p does not lie on any line joining two singular points of C_n,
- p does not lie on any line through a singular point and tangent to C_n in some smooth point,
- p does not lie in any tangent line to a singular point of C_n.

then for an admissible subdivision $\mathcal{Q}$ separating the critical fibers as in the preceding lemma, the bad components of C_n are exactly:

- components through some singular point x of C_n : the multiplicity of such a component for π_p is $\nu_x(C_n)$, because the multiplicity of C_n at x is the number of points of intersection between C_n and a generic (as above) line near the singularity;
- components through smooth points x of C_n such that the tangent line $T_x C_n$ is the line (px) : the multiplicity of such a component is the local degree of $\pi_p|_{C_n}$ near x. If we choose coordinates so that $p = [0 : 1 : 0]$ lies in the line at infinity $(t = 0)$ (in homogeneous coordinates $[z : w : t]$) and C_n has (reduced) equation $(P_n = 0)$, then C_n has a vertical tangent at x and the local degree of π_p is the intersection multiplicity $I_x(P_n, \frac{\partial P_n}{\partial w})$ (this formula is false for singular points).

The remaining of the proof is now a careful examination of the Riemann-Hurwitz formula (see [GH]) for the projection $\pi_p \circ \pi : \widehat{C}_n \to \mathbb{P}^1$ (recall that $\pi : \widehat{C}_n \to C_n$ is the resolution of singularities). $\pi_p \circ \pi$ is a branched covering, let $\mathcal{R}$ be the set of its critical points, and $v(x)$ be its local degree near $x \in \mathcal{R}$. The Riemann-Hurwitz formula states:

$$\chi(\widehat{C}_n) = 2 - 2g_n = d_n \chi(\mathbb{P}^1) - \sum_{x \in \mathcal{R}} (v(x) - 1), \tag{2}$$

with $\chi(\mathbb{P}^1)$=2. We want to relate this to the number of bad components for the projection π_p. First, remark that there may be points $x \in \widehat{C}_n$ with $\pi(x) \in Sing(C_n)$ but $x \notin \mathcal{R}$ (e.g. if the singularity is an ordinary multiple point); for those x, $v(x) = 1$. Let $\mathcal{S} = \mathcal{R} \cup \pi^{-1}(Sing(C_n))$, we then have

$$\sum_{x \in \mathcal{S}} (v(x) - 1) = \sum_{x \in \mathcal{R}} (v(x) - 1).$$

We claim that

$$bc_p(C_n) = \sum_{x\in\mathcal{S}} v(x).$$

Indeed if $x \in \mathcal{S}\backslash\pi^{-1}(Sing(C_n))$, the local degree of $\pi_p \circ \pi$ at x is equal to that of π_p at $\pi(x)$, and if $y_0 \in Sing(C_n)$ the multiplicity of C_n at y_0, that is the number of intersection points between C_n and the line $\pi_p^{-1}\pi_p(y)$, y near y_0, is exactly

$$\sum_{x\in\pi^{-1}(y_0)} v(x).$$

Let $\mathcal{S}_1 = \mathcal{S}\backslash\pi^{-1}(Sing(C_n))$ and $\mathcal{S}_2 = \pi^{-1}(Sing(C_n))$. We infer

$$\sum_{x\in\mathcal{S}} v(x) = \sum_{x\in\mathcal{S}_1} v(x) + \sum_{x\in\mathcal{S}_2} v(x).$$

Now for $x \in \mathcal{S}_1$ we have $v(x) \geq 2$, so $v(x) \leq 2(v(x)-1)$, and on the other hand

$$\begin{aligned}\sum_{x\in\mathcal{S}_2} v(x) &= \sum_{x\in\mathcal{S}_2}(v(x)-1) + \#\pi^{-1}(Sing(C_n))\\ &= \sum_{x\in\mathcal{S}_2\cap\mathcal{R}}(v(x)-1) + \sum_{y\in Sing(C_n)} n_y(C_n).\end{aligned}$$

Thus

$$\sum_{x\in\mathcal{S}} v(x) \leq 2\sum_{x\in\mathcal{S}_1}(v(x)-1) + \sum_{x\in\mathcal{S}_2\cap\mathcal{R}}(v(x)-1) + \sum_{y\in Sing(C_n)} n_y(C_n),$$

and by noting that $\mathcal{S}_1 \subset \mathcal{R}$, i.e. all points in $\mathcal{S}_1$ are critical for $\pi_p \circ \pi$, and using the Riemann-Hurwitz formula (2) we get the desired estimate. □

Remark. The inequality $v(x) \leq 2(v(x)-1)$ used above might seem far from being sharp, however if we choose p outside the finitely many inflexive tangents to C_n, this is an equality.

Theorem 1 is now a consequence of the following proposition, which is a generalization of the reasoning of [BS5].

Proposition 3.4. *Let (C_n) be a sequence of curves in $\mathbb{P}^2$, of degree $d_n \to \infty$, such that the sequence ${d_n}^{-1}[C_n] \rightharpoonup T$. Assume that for (Baire-) generic p, $bc_p(C_n) = O(d_n)$. Then T is laminar.*

Note that the hypotheses on C_n are slightly weaker than in theorem 1, in particular we do not make any irreducibility hypothesis, so that this proposition could apply in various contexts not particularly involving proposition 3.3 (the reason for the baire genericity here is that we have to choose p in a countable intersection of Zariski open sets). The main difference with [BS5] is that we make no assumption on the support of T, leading to several difficulties.

Proof. We still consider the projection $\pi_p : \mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^1$, and we denote by $\omega_{\mathbb{P}^1}$ and $\omega_{\mathbb{P}^2}$ the respective Fubini-Study forms. We will construct a laminar current $T_\infty \leq T$, with

$$\langle T_\infty, \pi_p^* \omega_{\mathbb{P}^1} \rangle = \langle T, \pi_p^* \omega_{\mathbb{P}^1} \rangle.$$

Note that the form $\pi_p^* \omega_{\mathbb{P}^1}$ which is singular at the point p, is integrable with respect to all positive closed currents because in local coordinates it expresses as $dd^c \log |Z - p|$ (see the classical proof of the existence of the Lelong number at p).

Let $\mathcal{Q}$ be an admissible subdivision of $\mathbb{P}^1$, recall that the "squares" have the same $\omega_{\mathbb{P}^1}$-area. Then for a good component Γ over $Q \in \mathcal{Q}$

$$\langle [\Gamma], \pi_p^* \omega_{\mathbb{P}^1} \rangle = \int_\Gamma \pi_p^* \omega_{\mathbb{P}^1} = \int_{\pi_p(\Gamma)} \omega_{\mathbb{P}^1} = \mathrm{Area}_{\mathbb{P}^1}(Q)$$

and for a bad component Γ of multiplicity m,

$$\int_\Gamma \pi_p^* \omega_{\mathbb{P}^1} = m \mathrm{Area}_{\mathbb{P}^1}(Q).$$

Let $\mathcal{G}(Q, n)$ be the set of good components of C_n over Q, and

$$T_{\mathcal{Q},n} = \frac{1}{d_n} \sum_{Q \in \mathcal{Q}} \sum_{\Gamma \in \mathcal{G}(Q,n)} [\Gamma] \leq T_n = \frac{1}{d_n} [C_n].$$

By lemma 3.2, $bc_p(C_n)$ dominates the number of bad components for the subdivision $\mathcal{Q}$, and we get

$$\langle T_n - T_{\mathcal{Q},n}, \pi_p^* \omega_{\mathbb{P}^1} \rangle \leq \frac{bc_p(C_n)}{d_n} \mathrm{Area}_{\mathbb{P}^1}(Q) \leq C \mathrm{Area}_{\mathbb{P}^1}(Q).$$

We need a normal families argument to get some laminar structure on cluster values of the sequences of currents. For this, we remark that $\mathbf{M}_{\mathbb{P}^2}(T_{\mathcal{Q},n}) \leq 1$ ($\mathbf{M}_{\mathbb{P}^2}$ denotes the mass norm with respect to $\omega_{\mathbb{P}^2}$), hence

$$\# \left\{ \Gamma \in \bigcup_{Q \in \mathcal{Q}} \mathcal{G}(Q, n),\ \mathbf{M}_{\mathbb{P}^2}([\Gamma]) > 1 - \eta \right\} \leq \frac{d_n}{1 - \eta} \tag{3}$$

where η is some fixed positive constant (s.t. $1 - \eta \gg \mathrm{Area}_{\mathbb{P}^1}$(Q)).

Lemma 3.5. *The family of holomorphic functions $f : Q \to \mathbb{C}$ such that*

$$\mathbf{M}_{\mathbb{P}^2}([\Gamma_f]) = \int_{\Gamma_f} \omega_{\mathbb{P}^2} \leq 1 - \eta$$

is normal, where Γ_f is the graph of f in $Q \times \mathbb{C} \subset \mathbb{P}^2$, and η is some fixed positive constant.

Proof of the lemma. Suppose the result is false. By the Zalcmann lemma there exist a sequence $Q \ni x_n \to x \in Q$, a sequence of positive numbers ρ_n converging to zero, and a sequence of holomorphic functions in Q satisfying the volume assumption, such that $f_n(x_n + \rho_n z)$ converges uniformly on compact sets in $\mathbb{C}$ to a non constant entire map h. Thus the graph of $\zeta \mapsto f_n(x_n + \zeta)$ over the disk $D(0, \rho_n)$ is close to the graph Γ_n of the map $\zeta \mapsto h(\zeta/\rho_n)$. As $n \to \infty$ the cluster set of the sequence of graphs (Γ_n) contains the vertical line $\{x\} \times \mathbb{C}$, which is impossible because of the area bound. An alternate approach for this lemma is to use Bishop's Theorem [Bi]. $\square$

Let $\mathcal{G}'(Q, n) \subset \mathcal{G}(Q, n)$ the set of components of volume $\leq 1 - \eta$ in $\mathbb{P}^2$. By dropping the components of $\mathcal{G}(Q, n) \backslash \mathcal{G}'(Q, n)$ we get a new current $T'_{Q,n}$, and by (3) we have

$$\langle T_n - T'_{Q,n}, \pi_p^* \omega_{\mathbb{P}^1} \rangle \leq (C + 1/(1 - \eta))\mathrm{Area}_{\mathbb{P}^1}(Q) = C'\mathrm{Area}_{\mathbb{P}^1}(Q). \quad (4)$$

Now we extract a subsequence n_j such that $T'_{Q,n_j} \rightharpoonup T_Q \leq T|_{Q\times\mathbb{C}}$ for every Q, where $T'_{Q,n} = T'_{Q,n}|_{Q\times\mathbb{C}}$. We have to show that T_Q is uniformly laminar. The proof is very similar to [BS5] so we only sketch it.

Let L_x be the line $\pi_p^{-1}(x)$, $x \in Q$, and

$$\lambda_{Q,n_j}(x) = \frac{1}{d_{n_j}} \sum_{\Gamma \in \mathcal{G}'(Q,n_j)} [\Gamma \cap L_x] = T'_{Q,n_j} \wedge [L_x];$$

it is a consequence of the theory of slicing currents that the sequence $\lambda_{Q,n_j}(x)$ converges weakly for almost all $x \in Q$. As $\cup_n \mathcal{G}'(Q, n)$ is a normal family, the family of graphs meeting some compact subset of a line L_x is equicontinuous. Then if φ is a test function in $Q \times \mathbb{C}$, the family $\int_{L_x} \varphi\, \lambda_{Q,n}(x)$ is equicontinuous as a function of x. Thus we get that $\lambda_{Q,n_j}(x)$ converges weakly to a measure $\lambda_Q(x)$ for *all* $x \in Q$.

It remains to prove that $\mathcal{G}'(Q, n_j)$ "converges" to a lamination by graphs in $Q \times \mathbb{C}$. The fact that λ_Q is a transverse measure is then a consequence of weak convergence and equicontinuity.

Lemma 3.6 ([BS5]). *Fix $x_0 \in Q$. Let $\mathcal{G}'(Q)$ be the set of holomorphic $f : Q \to \mathbb{C}$ such that f is the limit of some sequence of $f_{n_j} \in \mathcal{G}(Q, n_j)$ and $f(x_0) \in$* Supp$(\lambda_Q(x_0))$. *Then the graphs of $\left\{f,\ f \in \mathcal{G}'(Q)\right\}$ form a lamination in $Q \times \mathbb{C}$.*

Proof of the lemma. one has to show

- for each $y \in \mathrm{Supp}(\lambda_Q(x_0))$, there is a unique $f \in \mathcal{G}'(Q)$, s.t. $f(x_0) = y_0$;
- two distinct graphs are disjoint.

(this is more subtle than the Hurwitz Theorem since good components of different $\mathcal{G}'(Q, n_j)$ may intersect, we use the convergence of currents instead)

It suffices to show the following fact : "if a sequence $f_{n_j} \in \mathcal{G}'(Q, n_j)$ satisfies $f_{n_j}(x_0) \to y_0 \in \mathrm{Supp}(\lambda_Q(x_0))$ then the sequence converges". Suppose not: there exists two subsequences $f_{n^i_j} \to f^i$, $f^i(x_0) = y_0$, $i = 1, 2$, and $f^1 \neq f^2$. If y_0 is not an atom of $\lambda_Q(x_0)$, we can assume $(f^1)'(x_0) \neq (f^2)'(x_0)$: indeed we take a sequence $g_{n^1_j} \in \mathcal{G}'(Q, n^1_j)$ such that $g_{n^1_j}(x_0)$ converges to y'_0 near y_0. By extracting a further subsequence if necessary, we get a limit function g^1 whose graph does not intersect that of f^1 by the Hurwitz theorem, and so has transversal intersection with Γ_{f_2} (e.g.[BLS] Lemma 6.4).

As $y_0 \in \mathrm{Supp}(\lambda_Q(x_0))$, $\lambda_Q(x_0)(B(y_0, \varepsilon)) > \alpha$, and $\lambda_{Q,n^i_j}(x_0)(B(y_0, \varepsilon)) > \alpha/2$ for j large enough. Moreover all graphs of $\mathcal{G}'(Q, n^i_j)$ near $f_{n^i_j}(x_0)$ have slope close to $(f^i)'(x_0)$. This contradicts the convergence $T_{Q,n_j} \rightharpoonup T_Q$.

If y_0 is an atom of $\lambda_Q(x_0)$ of mass α, there is a sequence of intervals I_{n_j} shrinking to y_0 with mass more than $3\alpha/4$ for λ_{Q,n_j}. If there are two distinct limiting graphs for points in I_{n_j}, we contradict the convergence of currents again. □

We have thus far proven that T_Q is a uniformly laminar current on $Q \times \mathbb{C}$, and let

$$T_{\mathcal{Q}} = \sum_{Q \in \mathcal{Q}} T_Q \leq T.$$

By successively refining $\mathcal{Q}$ we get an increasing sequence $T_{\mathcal{Q}^\ell}$ converging to some laminar current T_∞. Because of the estimate (4), we have $\langle T_\infty, \pi_p^* \omega_{\mathbb{P}^1} \rangle = \langle T, \pi_p^* \omega_{\mathbb{P}^1} \rangle$.

We claim that for generically chosen p, this last relation forces $T = T_\infty$. Indeed, let $S = T - T_\infty \leq T$, S is a positive current, thus is representable by integration. This means that there is a positive measure ν_S (which is in fact the trace measure $\|S\|$ of S), and a measurable field of (1,1) vectors S_x such that $S = \int \langle S_x, \cdot \rangle d\nu_S$. If $\langle S, \pi_p^* \omega_{\mathbb{P}^1} \rangle = 0$, the (1,1) vector S_x is a.e. tangent to the pencil of lines through p. Now the set of points $p \in \mathbb{P}^2$ such that there is a set of positive $\|T\|$ measure of x such that T_x is directed by the pencil through p is at most countable. It suffices to choose p outside this at most countable exceptional set to achieve the desired result. □

Observe that there is no need in proposition 3.4 for the $[C_n]$ to be closed currents. More precisely we have shown:

Proposition 3.7. *Let (C_n) be a sequence of complex submanifolds, possibly with boundary, in $\mathbb{P}^2$. Assume that ${d_n}^{-1}[C_n] \rightharpoonup T$, where T is a positive, closed current. Assume also that for a generic projection π_p, and any admissible subdivision $\mathcal{Q}$ of $\mathbb{P}^1$, the total mass with respect to $\pi_p^* \omega_{\mathbb{P}^1}$ of the set of bad components of $[C_n]$ is $\leq C d_n \mathrm{Area}_{\mathbb{P}^1}(\mathcal{Q})$ (C is some fixed constant).*

Then T is laminar.

This proposition applies for example for entire curves in $\mathbb{P}^2$, in which case the control of the number of bad components comes from the Ahlfors' Covering Theorem (see [BLS], [Ca]). This yields the following corollary, which solves a question posed by Cantat (in the special case of $\mathbb{P}^2$):

Corollary 3.8. *Let $f : \mathbb{C} \to \mathbb{P}^2$ be an injective holomorphic mapping. Let*

$$A(r, f) = \frac{[f(D(0, R))]}{\mathbf{M}_{\mathbb{P}^2}([f(D(0, R))])}.$$

Then all closed currents in $\mathbb{P}^2$ which are limits of subsequences $A(r_j, f)$ are laminar.

4. Applications to holomorphic dynamics in $\mathbb{P}^2$

In this section we prove Theorem 2. We will define a Zariski-generic set $\mathcal{U}$ (a countable intersection of Zariski open sets) in the space $\mathcal{M}_d$ of rational maps of degree d in $\mathbb{P}^2$, such that if $f \in \mathcal{U}$ and $d_t(f) < d$, then the dynamical "Green" current of f exists and is laminar. All rational maps considered are supposed to be *dominating*, i.e. with generically nonvanishing Jacobian determinant. We roughly describe the equations defining $\mathcal{U}$. The existence of the Green current requires an assumption (algebraic stability) to control the "algebraic growth" of iterates of f. We also have to understand the "topological growth" of preimages of a generic line, which leads to an additional hypothesis (H). Both hypotheses (AS) and (H) lead to a generic set in $\mathcal{M}_d$ (prop. 4.1).

Let us be more specific. We consider a rational self map of $\mathbb{P}^2$ of degree d, given by its graph $\Gamma_f \subset \mathbb{P}^2 \times \mathbb{P}^2$, Γ_f is an irreducible, possibly singular surface. We let X be the minimal desingularization of Γ_f, with the natural projections π_1, $\pi_2 : X \to \mathbb{P}^2$

$$\begin{tikzcd} & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "f"] & & \mathbb{P}^2 \end{tikzcd}$$

π_1 is a composition of point blow-ups and π_2 is a holomorphic map with the same topological degree as f.

Let $I(f^\infty) = \cup_{n \geq 1} f^{-n} I(f)$ be the total indeterminacy set of f (for background on iteration of rational maps on $\mathbb{P}^2$ see [Si]). The first hypothesis is classical, and is a necessary condition for the Green current to describe the asymptotic distribution of the preimages of a generic hypersurface of $\mathbb{P}^2$; namely we assume f is algebraically stable (AS) which means that $I(f^\infty)$ is at most countable. In this case it follows from [RS] and [Si] that there exists a current T describing the asymptotic distribution of preimages of hyperplanes. More precisely we have:

suppose f is an AS rational self map of $\mathbb{P}^2$, then there exists a pluripolar set E in $\check{\mathbb{P}}^2$ (the dual space of $\mathbb{P}^2$), such that if $L \notin E$,

$$\frac{1}{d^n}(f^n)^*[L] \rightharpoonup T.$$

The second hypothesis is the following: for a generically finite holomorphic map $h : X \to Y$ between complex manifolds, we define $\mathcal{E}(h)$ to be the set of points in X where h is not locally finite. $\mathcal{E}(h)$ is a subvariety of X [Fi]. We say that f satisfies (H) if

$$\pi_2(\mathcal{E}(\pi_1) \cap \mathcal{E}(\pi_2)) \cap I(f^\infty) = \emptyset. \tag{H}$$

We give another equivalent version of (H). Let $\varpi : X \to \Gamma_f$ denote the resolution of singularities, which is a composition of finitely many point blow-ups and η_1, η_2 be the natural projections $\Gamma_f \to \mathbb{P}^2$, s.t. $\pi_i = \eta_i \circ \varpi$. We claim that $\varpi(\mathcal{E}(\pi_1) \cap \mathcal{E}(\pi_2)) \subset \Gamma_f$ is a finite set of points, which means that the curves of $\mathcal{E}(\pi_1) \cap \mathcal{E}(\pi_2)$ come from the resolution of singularities of Γ_f. Indeed if a curve $D \subset \Gamma_f$ projects to some point p by η_1, i.e. $D \subset (\{p\} \times \mathbb{P}^2) \cap \Gamma_f$, then as η_2 restricted to $\{p\} \times \mathbb{P}^2$ is 1-1, $\eta_2(D)$ is a curve, i.e. D is not contracted by η_2. So (H) is equivalent to

$$\eta_2(\varpi(\mathcal{E}(\varpi)) \cap I(f^\infty) = \emptyset.$$

In case $Sing(\Gamma_f)$ is zero dimensional, $\varpi(\mathcal{E}(\varpi)) = Sing(\Gamma_f)$, this means that up to the determination of the singularities of Γ_f, and the knowledge of $I(f^\infty)$ (which is necessary in order to know if f is AS), it is practically possible (though in fact probably rather difficult) to determine whether f satisfies (H) or not.

The next proposition shows that (H) is satisfied in many interesting cases.

Proposition 4.1. *(i) An algebraically stable birational map satisfies (H);*
(ii) (H) and (AS) are generically satisfied in the (algebraic) set of rational maps of $\mathbb{P}^2$, and in the set of polynomial mappings of $\mathbb{C}^2$.

Remark that it is unclear whether (H) and (AS) are generic in subsets of maps with fixed topological degree.

Proof. *(i)* suppose $x \in \pi_2(\mathcal{E}(\pi_1) \cap \mathcal{E}(\pi_2)) \cap I(f^\infty)$, say $x \in I(f^n)$. As f is birational, $\pi_2(\mathcal{E}(\pi_2)) = I(f^{-1})$, hence $x \in I(f^n) \cap I(f^{-1})$. In particular, $f^{-1}(x)$ is not finite, contradicting that f is AS.

(ii) The set $\mathcal{M}_d$ of dominating meromorphic maps of degree d in $\mathbb{P}^2$ is a Zariski open subset of a projective space $\mathbb{P}^N$. The subset of algebraically stable maps is a countable intersection $\mathcal{A}_d = \cap_n A_n$ (A_n is defined by $\{f \in \mathcal{M}_d$ s.t. $I(f^n)\}$ is finite) of Zariski open proper subsets of $\mathcal{M}_d$ [Si]. We show that maps satisfying (H) are generic in $\mathcal{A}_d$. The set of maps f such that

$$\pi_2(\mathcal{E}(\pi_1) \cap \mathcal{E}(\pi_2)) \cap I(f^n) \neq \emptyset$$

is algebraic in A_n and its complement is not empty since there are maps in $\mathcal{A}_d$ satisfying (H) (e.g. Hénon maps of degree d).

The case of polynomial maps of degree d in $\mathbb{C}^2$ is similar. □

Example. we present an explicit family of maps $\mathbb{P}^2 \to \mathbb{P}^2$, non birational, whose generic element satisfies AS and (H). Let $g = (g_1, g_2) = (p(z) - aw, az)$ be a Hénon map of degree $d(g) \geq 3$. Let $f = (g_1^2, g_2^2)$; f is a polynomial map, $d(f) = 2d(g) \geq 6$ and $d_t(f) = 4$. We have $I(f) = I(g) = [0 : 1 : 0]$ in homogeneous coordinates $[z : w : t]$, and $f((t = 0)\backslash I) = [1 : 0 : 0] =: q \neq I$ so $I(f^\infty) = I(f)$ and f is AS.

Let L be a linear automorphism $\mathbb{C}^2 \to \mathbb{C}^2$ and denote also by L its extension $\mathbb{P}^2 \to \mathbb{P}^2$. We get $d(L \circ f) = d(f)$, $d_t(L \circ f) = d_t(f)$, $I(L \circ f) = I(f) = I$ and if $L(q) \neq I$, $L \circ f$ is AS and $I((L \circ f)^\infty) = I$. We now prove that for generic L, $L \circ f$ satisfies (H).

Let $id \otimes L$ be the map $\mathbb{P}^2 \times \mathbb{P}^2 \ni (x, y) \mapsto (x, Ly)$; one checks that $\Gamma_{L\circ f} = (id \otimes L)(\Gamma_f)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. Recall that the condition (H) concerns a finite subset $\mathcal{S}(f) := \varpi(\mathcal{E}(\pi_1) \cap \mathcal{E}(\pi_2))$ of Γ_f. Hence $\pi_2(\mathcal{S}(L \circ f)) = \pi_2((id \otimes L)(\mathcal{S}(f))) = L\pi_2(\mathcal{S}(f))$. As $\pi_2(\mathcal{S}(f))$ is a finite subset of the line at infinity and $I((L \circ f)^\infty) = I$ is fixed, for generic L we get $\pi_2(\mathcal{S}(L \circ f)) \neq I$ and we are done. □

Theorem 2 is a consequence of proposition 4.1 and the following proposition.

Proposition 4.2. *Let f be a rational self map of $\mathbb{P}^2$, of algebraic degree d, and topological degree $d_t < d$. We assume that f is AS and satisfies the hypothesis (H) above. Then the Green current T is laminar.*

Note that the result holds in particular for all AS birational maps. In the next section we will show that for birational maps the "AS" hypothesis is in fact unnecessary.

Proof. We want to use theorem 1 with the sequence of currents $d_n^{-1}(f^n)^*[L] = d_n^{-1}[f^{-n}(L)]$. Let $C_n = f^{-n}(L)$. We have to show that

$$g_n + \sum_{x \in Sing(C_n)} n_x(C_n) = O(d^n) \tag{5}$$

with notations as in the first section. We will estimate the two terms separately, for generic L. Let us first discuss the genericity assumptions made on L:

(G1) we choose L s.t. the convergence $d_n^{-1}(f^n)^*[L] \rightharpoonup T$ holds;
(G2) the hypothesis (H) says that $\cup_{n \geq 0} f^n(\pi_2(\mathcal{E}(\pi_1) \cap \mathcal{E}(\pi_2)))$ is an at most countable set. We take L missing this countable number of points.

We will prove the estimate (5) by induction on n. This leads us to introduce the following diagram, which illustrates the step from $n-1$ to n in terms of the graphs of the rational maps f^n. Suppose we have constructed a surface X^{n-1} (which is a proper modification of the graph of f) with the following diagram:

$$\begin{tikzcd} & X^{n-1} \arrow[dl, "\pi_{1,n-1}"'] \arrow[dr, "\pi_{2,n-1}"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "f^{n-1}"] & & \mathbb{P}^2 \end{tikzcd}$$

Then we define X^n such that we have the following diagram, where X^n is smooth and minimal among the possible graphs:

$$\begin{tikzcd} & & X^n \arrow[dl, "\eta"'] \arrow[dr, "\omega"] \arrow[ddll, bend right, "\pi_{1,n}"'] \arrow[ddrr, bend left, "\pi_{2,n}"] & & \\ & X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & X^{n-1} \arrow[dl, "\pi_{1,n-1}"'] \arrow[dr, "\pi_{2,n-1}"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "f"'] & & \mathbb{P}^2 \arrow[rr, dashed, "f^{n-1}"'] & & \mathbb{P}^2 \end{tikzcd}$$

Such a X^n always exists since there is a natural rational map $X \to X^{n-1}$ birationally equivalent to f; X^n is not *a priori* a minimal desingularization of the graph of f^n. Note that all arrows are holomorphic, $\pi_{1,n}$, π_1, η are compositions of points blow-ups, $d_t(\omega) = d_t$, $d_t(\pi_{2,n-1}) = d_t^{n-1}$. This allows us to state the last (Zariski) generic hypothesis on L

(G3) by definition, $\pi_{2,n}(\mathcal{E}(\pi_{2,n})) \subset \mathbb{P}^2$ is finite for all n, we take L missing the union of these sets. Moreover, Bertini's theorem says that $\pi_{2,n}^{-1}(L)$ is smooth, irreducible and of mutiplicity 1 for generic L (see [GH], and [FL] for irreducibility). We choose such an L.

The proof of the estimate (5) splits up into two lemmas.

Lemma 4.3. *Let f be as in theorem 2, L a line in $\mathbb{P}^2$ satisfying hypotheses (G2) and (G3), and $C_n = f^{-n}(L)$, then*

$$\sum_{x \in Sing(C_n)} n_x(C_n) = O(d^n).$$

Before we begin the proof, we want to give a heuristic argument, which gives a true proof for birational mappings. We have seen that $\sum n_x(C_n) =: N_n$ is the number of points in $\pi^{-1}(Sing(C_n))$, where π is the resolution of singularities. Now if f is birational, $f^{-n}|_L : L \to C_n$ is a (non minimal a priori) resolution of singularities. The hypothesis (G3) ensures that $L \cap I((f^{-1})^\infty) = \emptyset$ and so $\sum n_x(C_n)$ is not greater than the number of points in $L \cap \mathcal{E}(f^{-n})$, and $\mathcal{E}(f^{-n}) = Crit(f^{-n})$. To conclude, note that the critical set of a rational self mapping of $\mathbb{P}^2$ has degree at most $3d - 3$, where d is the degree of f, and that for an AS birational map f and f^{-1} have the same degree. This argument gives in fact the proof of theorem 2 for birational maps (since $g_n = 0$ in this case).

For non invertible maps, given C_{n-1}, with set of singularities $\mathcal{S}$, we want to analyze $Sing(C_n)$. The set $Sing(C_n)$ contains points of $f^{-1}(\mathcal{S})$, which has cardinality not greater than $d_t \cdot \#\mathcal{S}$, preimages of points of $f(I(f)) \cap C_{n-1}$, and possibly other points (the preimage of a smooth curve, even by a holomorphic map need not in general be smooth). In fact the sense of the Bertini argument in (G3) is precisely that the latter set is empty. We have to estimate the number of local irreducible components at these points. We thus can expect a formula such as $N_n \leq d_t N_{n-1} + cd^{n-1}$ which is indeed the case.

We also recall for future reference some properties of divisors and intersection products on compact surfaces (see [GH]): a *divisor* is a formal linear combination of subvarieties with integer coefficients, we write $D \geq D'$ if $D' - D$ is an *effective divisor*, that is a divisor with nonnegative coefficients. The *intersection product* $C \cdot C'$ of two curves is the sum of intersection multiplicities at common points; the product $\cdot$ is extended by bilinearity to divisors, and depends only on cohomology classes in $H^2(X, \mathbb{Z})$, where X is the ambient surface. Given a holomorphic map $h : X \to Y$ there are natural pull back and push forward operations h^* and h_* on divisors, which satisfy $h^*D \cdot h^*D' = d_t(h)(D \cdot D')$, and $h^*D \cdot D' = D \cdot h_*(D')$ provided these expressions make sense.

Proof of lemma 4.3. By (G3) $\widehat{C}_n := \pi_{2,n}^{-1}(L)$ is smooth and irreducible. As $\pi_{1,n}$ is a composition of point blow-ups, $\pi_{1,n} : \widehat{C}_n \to C_n$ is a resolution of singularities. Hence

$$\sum_{x \in Sing(C_n)} n_x(C_n) \leq \#\widehat{C}_n \cap \mathcal{E}(\pi_{1,n}).$$

We introduce $E(\pi_{1,n})$ the exceptional divisor, which is the sum (with coefficients equal to 1) of irreducible components of $\mathcal{E}(\pi_{1,n})$, and $\#\widehat{C}_n \cap \mathcal{E}(\pi_{1,n}) \leq \widehat{C}_n \cdot E(\pi_{1,n})$ (generically equal).

Now $\pi_{1,n} = \pi_1 \circ \eta$, so $\mathcal{E}(\pi_{1,n}) \subset \mathcal{E}(\eta) \cup \eta^{-1}\mathcal{E}(\pi_1)$. Hence

$$E(\pi_{1,n}) \leq \eta^* E(\pi_1) + E_\eta$$

since of course all coefficients in the second member are ≥ 1. From this we infer

$$E(\pi_{1,n}) \cdot \widehat{C}_n \leq \eta^* E(\pi_1) \cdot \widehat{C}_n + E(\eta) \cdot \widehat{C}_n \tag{6}$$

because the pull back of a curve from $\mathbb{P}^2$ intersects positively any effective divisor.

We claim that $E(\eta) \cdot \widehat{C}_n \leq \omega^* E(\pi_{1,n-1}) \cdot \widehat{C}_n$. Indeed

$$E(\eta) = \sum_{i=1}^{k} V_i$$

with V_i a rational curve; suppose without loss of generality that $\widehat{C}_n$ cuts $V_1, \ldots V_l$. By (G3), $\pi_{2,n}$ is a branched covering near $\widehat{C}_n$ and so for $1 \leq i \leq l$, $\omega(V_i)$ is not a point. As $\pi_2 \circ \eta = \pi_{1,n-1} \circ \omega$, and $V_i \subset \mathcal{E}(\pi_2 \circ \eta)$, we have $\omega(V_i) \subset \mathcal{E}(\pi_{1,n-1})$, $1 \leq i \leq l$. This implies

$$\omega^*(E(\pi_{1,n-1})) \geq \sum_{i=1}^{l} V_i$$

which yields the desired result. We deduce

$$\begin{aligned} E(\eta) \cdot \widehat{C}_n \leq \omega^* E(\pi_{1,n-1}) \cdot \widehat{C}_n &= \omega^* E(\pi_{1,n-1}) \cdot \omega^*(\widehat{C}_{n-1}) \\ &= d_t (E(\pi_{1,n-1}) \cdot \widehat{C}_{n-1}), \end{aligned} \tag{7}$$

with $\widehat{C}_{n-1} = \pi_{2,n-1}^{-1}(L)$.

On the other hand, we have $\eta^* E(\pi_1) \cdot \widehat{C}_n = cd^{n-1}$. Indeed

$$\eta^* E(\pi_1) \cdot \widehat{C}_n = E(\pi_1) \cdot \eta(\widehat{C}_n)$$

with $\eta(\widehat{C}_n)$ an irreducible curve in X which projects down to C_{n-1} by π_2 and to C_n by π_1. By the hypothesis (G2), $\pi_2^{-1}(C_{n-1})$ is irreducible so $\eta(\widehat{C}_n) = \pi_2^{-1}(C_{n-1})$, and we know the cohomology class $\{C_{n-1}\}$ of C_{n-1} in $\mathbb{P}^2$, $\{C_{n-1}\} = d^{n-1}\{L\}$. From this we can evaluate the intersection product in cohomology

$$\begin{aligned} \eta^* E(\pi_1) \cdot \widehat{C}_n &= \{E(\pi_1)\} \cdot \{\eta(\widehat{C}_n)\} = \{E(\pi_1)\} \cdot \pi_2^* \{C_{n-1}\} \\ &= d^{n-1} \{E(\pi_1)\} \cdot \pi_2^* \{L\}. \end{aligned} \tag{8}$$

We can thus conclude from (6), (7), (8) that $N_n = E(\pi_{1,n}) \cdot \widehat{C}_n$ satisfies $N_n \leq d_t N_{n-1} + cd^{n-1}$, $d_t < d$, and it is then a standard result that $N_n = O(d^n)$. □

It seems more difficult to find a heuristic argument for the next lemma, so we systematically use the terminology of divisors.

Lemma 4.4. *Let f be as in theorem 2, L a line in $\mathbb{P}^2$ satisfying hypotheses (G2) and (G3), $C_n = f^{-n}(L)$, and g_n the geometric genus of C_n, then $g_n = O(d^n)$.*

Proof. We apply the Riemann-Hurwitz formula for the branched covering $\pi_{2,n} : \widehat{C}_n \to L$, and use induction again. We first write down the formula in the language of divisors [GH].

Let $h : X \to Y$ be a dominating holomorphic map of smooth surfaces. We define the ramification divisor R_h to be the divisor locally defined by the holomorphic function $Jac(h)$ (Jacobian determinant). If h is a branched covering, the order of vanishing of $Jac(h)$ can be interpreted in terms of the local topological degree of h near the divisor. If $C \subset Y$ and $C' \subset X$ are smooth curves s.t. $h^{-1}(C) = C'$, then the Riemann Hurwitz formula between Euler characteristics reads (this holds without the branched covering assumption)

$$\chi(C') = d_t \chi(C) - R_h \cdot C'.$$

If $k : Y \to Z$ is another map, the usual chain rule reads

$$R_{k\circ h} = h^* R_k + R_h.$$

The Riemann Hurwitz formula for $\pi_{2,n} : \widehat{C}_n \to L$ then states

$$2 - 2g_n = \chi(\widehat{C}_n) = d_t^n \chi(L) - R_{\pi_{2,n}} \cdot \widehat{C}_n,$$

then as $d_t < d$ we only have to show $R_{\pi_{2,n}} \cdot \widehat{C}_n = O(d^n)$.

By the chain rule,

$$R_{\pi_{2,n}} = \omega^* R_{\pi_{2,n-1}} + R_\omega,$$

hence

$$R_{\pi_{2,n}} \cdot \widehat{C}_n = \omega^* R_{\pi_{2,n-1}} \cdot \widehat{C}_n + R_\omega \cdot \widehat{C}_n = d_t(R_{\pi_{2,n-1}} \cdot \widehat{C}_{n-1}) + R_\omega \cdot \widehat{C}_n \qquad (9)$$

As before, it only remains to see that $R_\omega \cdot \widehat{C}_n = cd^n$ to get the desired result.

By the genericity hypothesis (G3) we know that ω is a branched covering near $\widehat{C}_n$. In particular if an irreducible component R of R_h intersects $\widehat{C}_n$, the order of vanishing of the Jacobian along R is $\leq d_t - 1$ since it is exactly $e - 1$, where e is the local degree in the neighborhood of R. Moreover the chain rule for the commutative diagram $\pi_{1,n-1} \circ \omega = \pi_2 \circ \eta$ yields

$$\omega^* R_{\pi_{1,n-1}} + R_\omega = \eta^* R_{\pi_2} + R_\eta,$$

so

$$R_\omega \leq \eta^* R_{\pi_2} + R_\eta.$$

By dropping all components of R_η which do not meet $\widehat{C}_n$, we can write

$$R_\omega \cdot \widehat{C}_n \leq \eta^* R_{\pi_2} \cdot \widehat{C}_n + D \cdot \widehat{C}_n,$$

with D some divisor supported in $\mathcal{E}(\eta)$ with coefficients $\leq d_t - 1$ by the discussion above, that is $D \leq (d_t - 1)E(\eta)$. By the preceding lemma

$$E(\eta) \cdot \widehat{C}_n \leq E(\pi_{1,n}) \cdot \widehat{C}_n \leq C^{st} d^n.$$

To conclude, $\eta^* R_{\pi_2} \cdot \widehat{C}_n$ can be estimated in cohomology in X as in the preceding lemma, using (G2),

$$\eta^* R_{\pi_2} \cdot \widehat{C}_n = R_{\pi_2} \cdot \eta(\widehat{C}_n) = \{R_{\pi_2}\} \cdot \{\eta(\widehat{C}_n)\} = d^{n-1}\{R_{\pi_2}\} \cdot \pi_2^* \{L\}$$

and we are done. □

Remark. It might be possible to get rid of the hypothesis (H), by a more precise control of the multiplicities of the curves C_n at the indeterminacy points, but this seems difficult and is beyond the scope of this article.

5. Birational invariance and applications.

"Birational invariance" is the following easy proposition:

Proposition 5.1. *Let T be a positive closed current in the (compact) surface X and $h : X \to Y$ a birational map.*

Then T is laminar iff h_*T is laminar.

Proof. by the structure theorem for birational maps [GH] it suffices to show the following:

i if $\pi : X \to Y$ is a point blow up and T a positive closed current on X, T laminar $\Rightarrow \pi_*T$ laminar;
ii if $\pi : X \to Y$ is a point blow-up and T a positive closed current on Y, T laminar $\Rightarrow \pi^*T$ laminar.

The first point goes as follows: we can write $T = T_1 + c[E]$, where E is the exceptional divisor of the blow up ($\pi(E) =: p$), and T_1 gives no mass to E. T_1 is in fact the trivial extension through E of $T|_{X\setminus E}$, which is of course laminar (restriction of a laminar current). Now $\pi : X\setminus E \to Y\setminus\{p\}$ is a biholomorphism, and $\pi_*(T|_{X\setminus E}) = \pi_*(T_1|_{X\setminus E}) = (\pi_*T_1)|_{Y\setminus\{p\}}$ so $(\pi_*T_1)|_{Y\setminus\{p\}}$ is laminar. We conclude that π_*T itself is laminar because a neighborhood U of p can be covered up to a set of $\|\pi_*T\|$-measure 0 by countably many disjoint open subsets of $U\setminus\{p\}$.

For the second part, write $\pi^*T = T_1 + \nu(T,0)[E]$ where T_1 is the trivial extension through E of $(\pi^*T)|_{X\setminus E}$. T_1 is laminar because $T|_{Y\setminus\{p\}}$ is, and we get that T is laminar by subdividing into smaller subdisks all disks in T intersecting E. □

We now prove theorem 3 :
Let M be a connected smooth projective surface, and f a birational selfmap of M, of positive topological entropy. It is a result of J. Diller and C. Favre [DF] that there exists a proper modification $\pi : \widehat{M} \to M$ (π is a composition of point blow ups), such that f lifts to an algebraically stable map $\hat{f} : \widehat{M} \to \widehat{M}$ (of positive entropy also). Either $\hat{f}$ is an automorphism, in which case the results of Cantat [Ca] yield the existence of a natural laminar Green current on $\widehat{M}$, or $\hat{f}$ has indeterminacy points, and the existence of the Green current is proved in [DF]; moreover in the latter case $\widehat{M}$ is birational to $\mathbb{P}^2$. We define the Green current of f to be $T(f) = \pi_*T(\hat{f})$. By proposition 5.1 laminarity of $T(f)$ is equivalent to laminarity of $T(\hat{f})$, so we focus on $\hat{f}$. Abusing notation we write f for $\hat{f}$ and M for $\widehat{M}$.

We only need to treat the case when f has indeterminacy points. We define $\lambda_1 = \lambda_1(f)$, the first dynamical degree of f to be the spectral radius of the linear map $f^* : H^{1,1}(M) \to H^{1,1}(M)$. In $\mathbb{P}^2$, $\lambda_1 = d$ (see [DF], [RS], [Gu] for details and references). One has $h_{top}(f) \leq \log \lambda_1$ so $\lambda_1 > 1$ [Fr]. Let L be an element of the linear system of hyperplane sections of M. Bertini's theorem implies that a generic L is smooth and irreducible. Moreover for generic L

$$\frac{1}{\lambda_1^n}(f^n)^*[L] \rightharpoonup cT,$$

where c is some positive constant independent of L ([DF] Theorem 6.5, see also [Si] Theorem 1.10.1). Without loss of generality we assume $c = 1$. Let $C_n = f^{-n}(L)$, and g_n, $\sum n_x(C_n)$ be defined as in section 3. We want to prove

$$g_n + \sum_{x \in Sing(C_n)} n_x(C_n) = O(\lambda_1^n).$$

As f is birational $g_n = g(f^{-n}(L))$ is independent of n so we only bound the second term.

We adapt lemma 4.3 to this context. First note that the divisor L is *ample* (because as a line bundle it is the restriction of the hyperplane bundle [GH]), in particular its intersection product with any effective divisor is nonnegative. If we draw the same diagram as in the preceding section, replacing $\mathbb{P}^2$ by M, we get that the intersection product of $\pi_{2,n}^*(L)$ with any effective divisor in X^n is nonnegative. If moreover we choose L satisfying the generic assumptions (G2): L misses the countable set $\cup_{n\geq 0} f^n(\pi_2(\mathcal{E}(\pi_1) \cap \mathcal{E}(\pi_2)))$ (by algebraic stability of f^{-1}) and (G3): L misses $\cup_n \pi_{2,n}(\mathcal{E}(\pi_{2,n}))$ and $\pi_{2,n}^{-1}(L)$ is smooth and irreducible for every n, we can read again lemma 4.3, replacing $\mathbb{P}^2$ by M and d by λ_1 ($d_t = 1$).

To conclude, we apply theorem 1 in $\mathbb{P}^2$. Let $h : M \to \mathbb{P}^2$ be a birational map, and consider the sequence of curves $h(f^{-n}(L)) = h(C_n)$, at the level of currents we have

$$\frac{1}{\lambda_1^n}[h(C_n)] = h_* \left(\frac{1}{\lambda_1^n}[C_n] \right) \rightharpoonup h_* T.$$

We can decompose $[h(C_n)] = [\Gamma_n] + [E_n]$ where $[\Gamma_n]$ is the (irreducible) proper transform of C_n, and $[E_n]$ is some divisor subordinate to $\mathcal{E}(h^{-1})$.

The sequence $(1/\lambda_1^n)[E_n]$ converges to a divisor, which does not affect the laminarity of $h_* T$, and it remains to prove that the sequence of curves Γ_n satisfy the hypotheses of theorem 1. The term g_n is invariant by the birational transformation h, and

$$\sum_{x \in Sing(\Gamma_n)} n_x(\Gamma_n) \leq \sum_{x \in Sing(C_n)} n_x(C_n) + \# C_n \cap \mathcal{C}(h).$$

the first term is $O(\lambda_1^n)$ by the preceding discussion, and the second is $O(\lambda_1^n)$ ($\mathcal{C}(h)$, the critical set, contains the indeterminacy points of h by definition) because the spectral radius of the action f^* on $H^2(M, \mathbb{Z})$ is $\leq \lambda_1$. Thus $h_* T$ is laminar, and so is T by proposition 5.1. □

Remark. With the same method it is possible to derive laminarity of the Green current for AS rational maps of rational surfaces, satisfying $d_t < \lambda_1$ and (H).

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