# The Secant Method for Root Finding, Viewed as a Dynamical System

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### Abstract

The secant method is a procedure for approximating the zeros of a function. We will explore the behavior of this procedure in the case of univariate rational functions for arbitrary starting points, whether or not there is convergence to a zero.

## Introduction

Let a function $f : \mathbb{C} \to \mathbb{C}$ be given. The secant method for finding a root $\alpha$, $f(\alpha) = 0$, is as follows. Given two initial points $z_1, z_2 \in \mathbb{C}$, we determine a third point $z_3 \in \mathbb{C}$ by the condition that $(z_3, 0)$ is the point where the (secant) line between $(z_1, f(z_1))$ and $(z_2, f(z_2))$ crosses the horizontal coordinate axis. Inductively, we determine $z_{n+1}$ starting with $z_{n-1}$ and $z_n$. This procedure may be written $(z_n, z_{n+1}) = S_f(z_{n-1}, z_n)$ where

$$S_f(x, y) = (y, R(x, y)) : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2, \quad \text{with} \tag{1}$$

$$R(x, y) = \frac{yf(x) - xf(y)}{f(x) - f(y)} = x - \frac{(x - y)f(x)}{f(x) - f(y)}. \tag{2}$$

If the initial points for the secant method are the same, $z_1 = z_2$, then the first step of the Secant Method essentially coincides with Newton's method, but after the first step, the two methods are different.

If $f(z)$ is a rational function, then $S_f$ is a rational mapping. A point $\alpha$ is a simple zero of $f$ if and only if $\hat{\alpha} := (\alpha, \alpha)$ is a regular fixed point of $S_f$. The secant map $S_f$ may be considered to be a root-finding algorithm: *If $\alpha$ is a simple root of $f$, then if the starting points $z_1$ and $z_2$ are sufficiently close to $\alpha$, then the iterates of $S_f(z_1, z_2)$ converge to $\hat{\alpha}$ at the super-exponential rate $\phi \sim 1.61803$, where $\phi$ is the golden ratio* (see Theorem 4.2).

On the other hand, if $\alpha$ is a root of $f$ with multiplicity greater than one, then $\hat{\alpha}$ is a point of indeterminacy for $S_f$. We show (Theorem **??**) that in this indeterminate case there is a nonempty basin $\mathcal{B}(\hat{\alpha})$ of points which are attracted to $\hat{\alpha}$ in forward time, although the rate of convergence is merely geometric.

Since we consider $S_f$ in its capacity as a root-finding algorithm, we must ask about the size of $\tilde{\mathcal{B}} := \bigcup \mathcal{B}(\hat{\alpha}_j)$, where $\alpha_j$ ranges over all roots of $f$. A root-finding method can hope that $\tilde{\mathcal{B}}$ might have full measure, or at least be dense in $\mathbb{C}^2$. However, we show that this is "almost never" the case. Namely, by Corollary 3.3, we have: *For a generic polynomial $f$, there is an open set of points which do not converge to a root.*

There are two cases in which the secant map takes an especially attractive form: when $f$ is a quadratic polynomial, and when $f$ is a rational fraction of degree 1. In Section 4, we describe the dynamics of $S_f$ in these cases.

Finally we discuss $S_f$ as a global dynamical system of the complex plane $\mathbb{C}^2$. If $f = p/q$ is rational, $S_f$ is a dominant rational map and has topological degree $d_t := \max(\deg(p) - 1, \deg(q))$ (see Proposition 1.1). We show that in most cases the dynamical degree $\delta = \delta(S_f)$ is greater than $d_t$ (Theorem 5.1), and thus $S_f$ is a map of "small topological degree". We then record a few conclusions that follow from the general theory.

Another root finder, Newton's method, has been intensively studied as a global dynamical system on $\hat{\mathbb{C}}$. In complex dimension 2, the Newton method for the roots of a pair of quadratics was studied by Hubbard and Papadopol [7]. The resulting rational map has algebraic degree 3 and topological degree 4. This is a map of "large topological degree" and will be expected to have dynamical behavior that is qualitatively different from that of $S_f$.

After our work was completed, we learned of independent work by Garijo and Jarque [6], who discuss related questions about the secant map, as applied in the real domain. One of their results is that there is a polynomial $p$ for which $S_p$ has an attracting cycle of period 4; and thus $S_p$ is not "generally convergent" in this case.

# 1   Basic properties

The general theory of iteration of rational maps has been studied intensively in recent years (see [8], [5], [3]). Everything we will do in Section 1 has been done in [2] for the case where $f$ is a polynomial with distinct (simple) zeros. Here we will deal with the case that $f$ may have poles and, most importantly, multiple zeros. Following [2], we start by extending $S_f$ to a rational map $S_f : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Throughout this paper, we will assume that $f(z) = \frac{p(z)}{q(z)}$ *is a rational function which is not linear.*    (Otherwise, $S_f$ is constant.)   Further, we may and will assume that $p$ and $q$ are monic, and we write $S(x,y) = S_f(x,y) = (y, R_f(x,y))$, where

$$R_f(x,y) = R_f(y,x) = \frac{y\,q(y)\,p(x) - x\,q(x)\,p(y)}{q(y)\,p(x) - q(x)\,p(y)} = x - \frac{(x-y)p(x)q(y)}{p(x)q(y) - p(y)q(x)} \tag{3}$$

We record here how the Secant Map changes as we perform scaling, translation and inversion on the map $f$:

$$\begin{aligned}
S_{\lambda f} &= S_f \\
\tilde{\sigma}^{-1} \circ S_f \circ \tilde{\sigma} &= S_{f \circ \sigma} \quad \text{with } \sigma(x) = \lambda x, \ \ \tilde{\sigma}(x,y) = (\lambda x, \lambda y) \\
\tilde{\tau}^{-1} \circ S_f \circ \tilde{\tau} &= S_{f \circ \tau} \quad \text{with } \tau(x) = x + a, \ \ \tilde{\tau}(x,y) = (x + a, y + a) \\
R_{1/f} - x &= (x - R_f) \circ \rho \quad \text{with } \ \rho(x,y) = (y,x) \\
S_g &= j \circ S_f \circ j \quad \text{with } \ g(t) = t f(1/t), \ \ j(x,y) = (1/x, 1/y).
\end{aligned} \tag{4}$$

By the first equation, we may and will assume that both $p$ and $q$ are monic.

**Proposition 1.1.** . *Let us set $d_t := \max\{\deg(p) - 1, \deg(q)\}$. It follows that for $y$ fixed, the degree of $R(x,y)$ in $x$ is $d_t$, and $S_f$ has topological degree $d_t$, i.e., if $(x_0, y_0)$ is a generic point, then $S^{-1}(x_0, y_0)$ consists of $d_t$ points.*

*Proof.*   Let us write the expression for $R(x,y)$ in the center of (3) as $\tilde{N}(x,y)/\tilde{D}(x,y)$. It is clear that $\tilde{N}(x,y)$ and $\tilde{D}(x,y)$ both vanish along the line $\{x = y\}$, and thus the GCD of $\tilde{N}$ and $\tilde{D}$ will have the form $(y - x)h(x,y)$. We show that $h(x,y)$ is constant by observing that $\tilde{N} - y\tilde{D} = (y - x)q(x)p(y)$ and $x\tilde{D} - \tilde{N} = (x - y)q(y)p(x)$. Thus if $(y - x)h$ divides both $\tilde{N}$ and $\tilde{D}$, then $h$ divides both $q(x)p(y)$ and $q(y)p(x)$. But then $h$ must be constant since $p$ and $q$ have no common factors.

Now we claim that the numerator $\tilde{N}(x,y)$ has degree $d_t + 1$ in $x$. The most difficult case is where $\deg(p(x)) = \deg(x\,q(x)) = d_t + 1$. In this case, the highest degree term in $x$ in the expression $\tilde{N}$, is $x^{d_t+1}(y\,q(y) - p(y))$. The coefficient $y\,q(y) - p(y)$ cannot be the zero polynomial because $p$ and $q$ are relatively prime. Thus the degree of $N(x,y) := \tilde{N}(x,y)/(y - x)$ in $x$ is $d_t$.

Now we claim that $\tilde{D}$ has degree $\bar{d} := \max(\deg(p), \deg(q))$ in $x$. The hardest case is when $\bar{d} = \deg(p) = \deg(q)$. In this case the term of highest degree in $\tilde{D}$ is $x^{\bar{d}}(q(y) - p(y))$. Thus for generic $y$, the degree of $\tilde{D}$ is $\bar{d}$, and so the degree of $D := \tilde{D}/(y - x)$ in $x$ is $\bar{d} - 1 \le d_t$. It follows that the degree in $x$ of $N - cD$ is $d_t$ for all but finitely many $c \in \mathbb{C}$.

The number of solutions $\{(x,y) : S(x,y) = (y, R(x,y)) = (x_0, y_0)\}$ is the same as the number of solutions of $\{x : R(x,x_0) = y_0\} = \{x : N(x,x_0) = y_0 D(x,x_0)\}$. It follows that or generic $(x_0, y_0)$, the number of solutions is $d_t$.   $\square$

Let us observe that the cohomology group $H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z})$ is generated by the (Poincaré dual) class $H = \{y = c\}$ of a horizontal line and $V = \{x = c\}$ of a vertical line. The intersection product is given by $H \cdot H = V \cdot V = 0$, and $H \cdot V = V \cdot H = 1$. If $C \subset \mathbb{P}^1$ is a curve, then its Poincaré dual is the cohomology class defined by

$$\{C\} = mH + nV \in H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z})$$

where $m$ (resp. $n$) is the number of times $C$ intersects a general vertical (resp. horizontal) line.

The *indeterminacy set* $\mathcal{I}(S)$ is the finite (possibly empty) set of points where $S$ fails to be continuous.

**Proposition 1.2.** $S_f$ *has* $2d_t(\max(\deg(p), \deg(q)) - 1)$ *points of indeterminacy, counted with multiplicity.*

*Proof.*   As in the proof of Proposition 1.1, we write $R(x,y) = N/D$ where $N$ and $D$ are obtained by dividing the numerator and denominator of the central expression in (3) by $(y - x)$. By Proposition 1.1, $N$ and $D$ have no common factor, and the degrees in $x$ are $d_t$ and $\max(\deg(p), \deg(q)) - 1$, respectively. We will compute the cardinality of

$$\mathcal{I}(S_f) = \{(x,y) \in \mathbb{P}^1 \times \mathbb{P}^1 : N(x,y) = D(x,y) = 0\}.$$

By the symmetry of $N$ and $D$, we see that the degrees in $y$ are the same as the degrees in $x$. Thus, as elements of $H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z})$, we have that the class of $\{N = 0\}$ is given by $d_t(H + V)$, and the class of $\{D = 0\}$ is $(\max(\deg(p), \deg(q)) - 1)(H + V)$. Taking the intersection product, then, we have $\#\{N(x,y) = D(x,y) = 0\} = 2d_t(\max(\deg(p), \deg(q)) - 1)$.   $\square$

If $\mathcal{C}$ is a curve, we define its *strict transform*  $S(\mathcal{C})$ to be the closure of $S(\mathcal{C} - \mathcal{I}(S))$. A curve is said to be *exceptional*  if $S(\mathcal{C})$ is a point. It is clear that any exceptional curve is of the form $\{y = c\}$, for some constant $c \in \mathbb{C} \cup \{\infty\}$. In order to work with points at infinity, we will use the coordinate system $(u,v) = j(x,y) = (1/x, 1/y)$. We define $a_\infty = \deg(q) - \deg(p)$ so that $f(1/t) = t^{a_\infty}h(t)$, where $h(t)$ is holomorphic at $t = 0$, and $h(0) \ne 0$. Thus $a_\infty$ gives the order of zero or pole of $f$ at $\infty$.

**Proposition 1.3.** *If $c \in \mathbb{C}$, $f(c) = 0$, then $S(\{y = c\}) = (c,c)$. If $f(\infty) \in \mathbb{C}$ (i.e., if infinity is not a pole of $f$), then $S(\{y = \infty\}) = (\infty, \infty)$. For all other cases, $\{y = c\}$ is not exceptional and is mapped to the vertical line $\{x = c\}$. If $c = \infty$, then we have the following: If $a_\infty < 0$, then $\mathbb{P}^1 \times \{\infty\}$ is not exceptional; if $a_\infty \ge 0$, then $\mathbb{P}^1 \times \{\infty\}$ is exceptional and maps to $(\infty, \infty)_{x,y}$.*

*Proof.* A horizontal line $\{y = c\}$, $c \in \mathbb{C}$, is exceptional if and only if $x \mapsto R(x, c)$ is constant. If $c \in \mathbb{C}$ is not a pole of $f$, then

$$R(x, c) = \frac{cf(x) - xf(c)}{f(x) - f(c)}$$

If this expression is constant, and if $f(c) \neq 0$, then $f(x)$ must be linear, which case we have excluded. Thus if $c \in \mathbb{C}$ is not a pole, then $\{y = c\}$ is exceptional if and only if $f(c) = 0$.

If $c = \infty$ is not a pole, then $\lim_{x \to \infty} f(x) = \alpha \in \mathbb{C}$ exists. Thus $\lim_{y \to \infty} R(x, y) = \lim_{y \to \infty} \frac{yf(x) - \alpha x}{f(x) - \alpha} = \infty$ for generic $x$, so we conclude that $S(\{y = \infty\}) = (\infty, \infty)$, and so $\{y = \infty\}$ is exceptional.

Now suppose that $c \in \mathbb{C}$ is a pole of $f = p/q$. Then $q(c) = 0$ and $p(c) = \alpha \neq 0$. Using the expression for $R(x, y)$ in (3), we have $\lim_{y \to c} R(x, y) = \lim_{y \to c} \frac{-xq(x)\alpha}{-q(x)\alpha} = x$, so $\{y = c\}$ is not exceptional. Finally, if $c = \infty$ is a pole, i.e., if $a_\infty \leq -1$, then the degree of $p$ is greater than the degree of $q$. If $\deg(p) > \deg(q) + 1$, then the previous argument shows that $\lim_{y \to \infty} R(x, y) = x$. If $\deg(p) = \deg(q) + 1$, then since $p$ and $q$ are monic, the previous argument shows that $\lim_{y \to \infty} R(x, y) = x - f(x)$, which is nonconstant since $f$ is nonlinear. $\qquad\square$

Let $x_0$ be a point which is not a pole of $f$. Then we say that $x_0$ is a *critical point* if $f'(x_0) = 0$. We say that the critical point is *simple* if $f''(x_0) \neq 0$. If $x_0$ is a pole, then we consider the coordinate $X = 1/x$ at $x = \infty$. The condition of $f$ having a pole at $x_0$ is equivalent to $F := 1/f$ having a zero at $x_0$. We say that $x_0$ is a *critical point* for $f$ at a pole $x_0$ if $F'(x_0) = 0$.

Let us record a computation which will be useful in the sequel. Suppose

$$f(x) = (x - \alpha)^a (1 + x h(x))$$

where $h(x)$ is analytic in a neighborhood of $x = \alpha$. By (4) there is no loss of generality if we assume $\alpha = 0$. Then for $a \geq 1$, we have

$$
\begin{aligned}
f(x) - f(y) &= x^a(1 + xh(x)) - y^a(1 + yh(y)) \\
&= (x^a - y^a)(1 + xh(x)) + y^a(xh(x) - yh(y)) \\
&= (x - y)\left[ (x^{a-1} + \cdots + y^{a-1})(1 + xh(x)) + y^a \left( \frac{xh(x) - yh(y)}{x - y} \right) \right]
\end{aligned}
\tag{5}
$$

Using a similar expansion for $yf(x) - xf(y)$ in the case $a \geq 2$, we find

$$R(x, y) = y \frac{N_1(x, y)}{D_1(x, y)} = y \frac{x(x^{a-2} + \cdots + y^{a-2})(1 + xh(x)) + x\,y^{a-1}\left( \frac{xh(x) - yh(y)}{x - y} \right)}{(x^{a-1} + \cdots + y^{a-1})(1 + xh(x)) + y^a \left( \frac{xh(x) - yh(y)}{x - y} \right)} \tag{6}$$

**Proposition 1.4.** *The behavior of $S_f$ on the diagonal inside $\mathbb{C}^2$ is as follows:*

*a)  If $x_0$ is a non-critical point for $f$, then $(x_0, x_0)$ is not indeterminate for $S_f$, and $S_f(x_0, x_0) = (x_0, x_0 - f(x_0)/f'(x_0))$.*

*b)  If $x_0$ is a critical point for $f$ with $f(x_0) \neq 0$, then $(x_0, x_0)$ is not indeterminate for $S_f$, and $S_f(x_0, x_0) = (x_0, \infty)$.*

*c)  If $f$ has a multiple zero or multiple pole at $x_0$, then $(x_0, x_0)$ is indeterminate for $S_f$.*

*Finally, for the behavior at the diagonal point at infinity:*

*d)  $S_f$ is indeterminate at $(\infty, \infty)$ if and only if $|1 + a_\infty| \geq 2$.*

*Proof.* The first two statements follow from the expression on the right hand side of (1). For the third statement, we note that the indeterminacy locus of $S_f$ is the same as the set where the fraction $R$ is indeterminate. By (4), it is sufficient to consider the case where $x_0$ is a multiple zero of $f$. Now we have $a \geq 2$ and may write $R(x, y)$ as in (6), so that the numerator is $yx(x^{a-2} + \cdots + y^{a-2})$ plus terms of order $> a$, while the denominator is $(x^{a-1} + \cdots + y^{a-1})$ plus terms of order $> a - 1$. The expression $yx(x^{a-2} + \cdots + y^{a-2})$ vanishes on $a$ lines passing through the origin, and these are distinct from the $a - 1$ lines where $(x^{a-1} + \cdots + y^{a-1})$ vanishes, which means that the numerator and denominator have no common factors. Thus the polar locus of $R$ intersects the zero locus of $R$ at the origin. Thus $R$ is indeterminate there.

To discuss the point $(\infty, \infty)$ we use the involution $j(x, y) = (1/x, 1/y)$ which is a biregular map taking $(x, y) = (\infty, \infty)$ to $(0, 0)$. Conjugating $S_f$ with $j$, we obtain a new map $S_g$ (last line of (4)) which is the secant method for the function $g(t) := tf(1/t)$. The function $g$ is meromorphic at $t = 0$ and vanishes to order $1 + a_\infty$. It follows from (c) in Proposition 1.4 that if $g$ has a multiple zero or pole at $t = 0$, then $S_g$ is indeterminate; otherwise it is regular. This corresponds to $|1 + a_\infty| \geq 2$. $\qquad\square$

A direct calculation shows the following two results.

**Proposition 1.5.** *If $\alpha \in \mathbb{C}$ is a simple zero for $f$, then $(\alpha, \alpha)$ is a fixed point of $S_f$, and the differential is*

$$DS_f(\alpha, \alpha) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{f''(\alpha)}{2f'(\alpha)} \begin{bmatrix} 0 & 0 \\ y - \alpha & x - \alpha \end{bmatrix} + O_2(x - \alpha, y - \alpha).$$

*In particular, the eigenvalues of $DS_f(\alpha, \alpha)$ are 0 and 0.*

This also says something about the behavior at $(\infty, \infty)$. If $f$ is regular at infinity, and $f(\infty) \neq 0$, then $g(t) = tf(1/t)$ has a simple zero at $t = 0$. By (4), $S_f$ in a neighborhood of $(\infty, \infty)$ is locally conjugate to the map $S_g$ at $(0, 0)$, and $DS_g$ has the form given by Proposition 1.5.

**Proposition 1.6.** *If $\alpha \in \mathbb{C}$ is a simple pole for $f$, then $(\alpha, \alpha)$ is a fixed point for $S_f$, and $DS_f(\alpha, \alpha) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so $(\alpha, \alpha)$ is of saddle type.*

## 2 Convergence at multiple roots

We have seen in Proposition 1.4 that if $\alpha$ is a zero of $f$ with multiplicity $a \geq 2$, then $\hat{\alpha} = (\alpha, \alpha)$ is a point of indeterminacy for $S_f$. By (4) we will suppose that $\alpha = 0$. The local behavior of $S_f$ at $\hat{\alpha}$ will be more clear if we work in the blowup $\pi : M \to \mathbb{C}^2$ at $(0,0) \in \mathbb{C}^2$. By *blow up*, we mean that $M$ is a complex manifold, and $\pi$ is a holomorphic map such that if we let $E := \pi^{-1}(0,0)$ denote the exceptional divisor of the blowup, then $E \cong \mathbb{P}^1$, and $\pi : M - E \to \mathbb{C}^2 - \{(0,0)\}$ is biholomorphic. We now work with the induced map $\hat{S} := \pi^{-1} \circ S_f \circ \pi : M \dashrightarrow M$.

We can represent the manifold $M$ explicitly in a neighborhood of $E$ by using two coordinate charts. One of them is the coordinate system $(s,t)$ such that $\pi(s,t) = (s, st) = (x,y)$ on an affine neighborhood in $M$ in which we have $E = \{s = 0\}$. Since $\pi^{-1}(x,y) = (x, y/x) = (s,t)$, it follows from (6) that the induced map is given by

$$\hat{S} : \ (s,t) \to (st, \psi(t)(1 + sg(s,t))), \quad \psi(t) = \frac{1 + t + \cdots + t^{a-2}}{1 + t + \cdots + t^{a-2} + t^{a-1}}. \tag{7}$$

where $g(s,t)$ is analytic in a neighborhood of $(0,0)$. Thus the behavior of $\hat{S}$ near $E$ is modeled by the map

$$(s,t) \to (s_1, t_1) = (st, \psi(t)) \tag{8}$$

whose iterates have the form

$$(s_n, t_n) = (st_0 \cdots t_{n-1}, \psi^n(t)) \tag{9}$$

with $t_j = \psi^j(t_0)$. The fixed points of $\psi$ are defined by the condition $\psi(t) = t$, which is equivalent to

$$1 = t^{a-1} + t^a$$

The following is elementary:

**Proposition 2.1.** *If $a \geq 2$, then $\psi$ has a fixed point $0 < t_a < 1$, and $-1 < \psi'(t_a) < 0$.*

Now we observe that the differential of $\hat{S}$ at $E = \{s = 0\}$ is

$$D\hat{S}(0,t) = \begin{bmatrix} t & 0 \\ \psi(t)g(0,t) & \psi'(t) \end{bmatrix}.$$

Thus the eigenvalues of $D\hat{S}(0, t_a)$ are strictly less than one in modulus. We say that a point $p_0$ *has a basin $\mathcal{B}$ for $f$* if for every $q \in \mathcal{B}$, the iterates $f^n(q)$ converge to $p_0$ uniformly in a neighborhood of $q$. In general, $p_0$ need not belong to $\mathcal{B}$. It follows that $(0, t_a) \in E$ has a basin of attraction for $\hat{S}$, which in fact contains a neighborhood of $(0, t_a)$. Pushing down by $\pi$ to a neighborhood of $(0,0) \in \mathbb{C}^2$, we have:

**Theorem 2.2.** *Let $t_a$ be as in Proposition 2.1. For sufficiently small $r > 0$, the set $\{(x,y) \in \mathbb{C}^2 : |x| < r, |t_a - y/x| < r\}$ is contained in the basin of $(0,0)$ for $S_f$. Thus for a multiple zero $\alpha$ of $f$, there is a conical neighborhood of $\hat{\alpha}$ in which the iterates of $S_f$ converge to $\hat{\alpha}$ at the asymptotic rate $O(|\psi'(t_a)|^n)$.*

## 3 Critical 3-cycles

Let $f$ be a polynomial of degree at least 3, and let $x_0$ be a critical point: $f'(x_0) = 0$. If $f(x_0) \neq 0$ then

$$(x_0, x_0) \to (x_0, \infty) \to (\infty, x_0) \to (x_0, x_0), \tag{10}$$

where the first map is given by (b) of Proposition 1.4, and the other two maps follow directly from the formula (3). Using (4), we may perform a translation and assume that $x_0 = 0$. Further, we introduce coordinate changes $j_1(x,y) = (1/x, y)$ and $j_2(x,y) = (x, 1/y)$ and define

$$S_1 := j_2 \circ S, \ \ S_2 := j_1 \circ S \circ j_2, \ \ S_3 := S \circ j_1$$

so that each $S_j$ is a holomorphic map from a neighborhood of the origin to another neighborhood of the origin. Since $j_1^2$ and $j_2^2$ are identity maps, we have the decomposition $S_f^3 = S_3 \circ S_2 \circ S_1$, which we may use to give the local behavior at $(x_0, x_0) = (0,0)$.

**Theorem 3.1.** *Let $f$ be a polynomial of degree $d \geq 3$, and let $x_0$ be a simple critical point which is not a zero: $f(x_0) \neq 0$, $f'(x_0) = 0$, and $f''(x) \neq 0$. Then the 3-cycle $(x_0, x_0)$ is both semi-parabolic and semi-superattracting, in the sense that $DS^3(x_0, x_0)$ has eigenvalues 0 and 1. Further, if $x_0 = 0$, then the local behavior at $(0,0)$ is conjugate to*

$$S^3(x,y) = (x,0) + \frac{(-a_2)^{d-1}}{a_d}(2x + y)^{d-1}(-2,1) + O_d \tag{11}$$

*where $O_d$ indicates terms bounded by $(|x| + |y|)^d$.*

*Proof.* As observed above, we may use (4) to assume that $x_0 = 0$. Further, by (4) we may assume that $f(0) = 1$. Thus

$$f(z) = 1 + a_2 z^2 + \cdots + a_d z^d,$$

where $a_2, a_d \neq 0$. We have

$$S_1(x,y) = (y, -a_2 x - a_2 y + G) + O_d$$

$$S_2(x,y) = \left(y, x - \frac{y^{d-1}}{a_d}\right) + O_d, \quad S_3(x,y) = \left(y, y - \frac{x^{d-1}}{a_d}\right) + O_d$$

where $G = G(x,y)$ is a polynomial in which all the terms have degree at least 2. To see the computation for $S_2$, observe that the second coordinate of $S_2$ is

$$R(x, 1/y) = \frac{f(x)/y - xf(1/y)}{f(x) - f(1/y)}$$

We define $\check{f}(t) = t^d f(1/t) = a_d + a_{d-1}t + \cdots + t^d$. If we multiply the numerator and denominator by $y^d$ we find

$$R(x, 1/y) = \frac{y^{d-1}f(x) - x\check{f}(y)}{y^d f(x) - \check{f}(y)} \equiv \frac{y^{d-1} - x\check{f}(y)}{-\check{f}(y)} = x - \frac{y^{d-1}}{\check{f}(y)} \equiv x - \frac{y^{d-1}}{a_d}$$

where the '$\equiv$' indicates that we are working modulo $O_d$, i.e., homogeneous terms of degree $d$ and higher. The computation of $S_3$ is similar.

Now

$$S_2(S_1(x,y)) = \left(-a_2(x+y) + G, y + \frac{-(-a_2)^{d-1}}{a_d}(x+y)^{d-1}\right) + O_d$$

so

$$S^3(x,y) = S_3 \circ S_2 \circ S_1 = (y,y) + \frac{-(-a_2)^{d-1}}{a_d}(x+y)^{d-1}(1,2) + O_d$$

Finally, if we set $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then $M^{-1} \circ S^3 \circ M$ will have the desired form. $\qquad \square$

*Remark* 1. The expressions for $S_2$ and $S_3$ that are given in the proof are not valid for rational maps that are not polynomials.

Now we want to show the existence of a basin for the map (11). By (4) we may scale coordinates so that $(-a_2)^{d-1}/a_d = 1$, so if we drop the $O_d$ terms from $S^3$, we have the map

$$F_0: \quad (x,y) \to \left(x - 2(2x+y)^{d-1}, (2x+y)^{d-1}\right) \qquad (12)$$

**Theorem 3.2.** *If $f$ is a polynomial of degree at least 3, and if $x_0$ is a simple critical point for $f$ with $f(x_0) \neq 0$, then the 3-cycle $(x_0, x_0)$ has a nonempty basin.*

*Proof.* For notational convenience, we will use (4) to move the point $(x_0, x_0)$ to the origin $(0,0)$. For small $c, r > 0$ we define

$$\mathcal{D}_{c,r} := \{(x,y) \in \mathbb{C}^2 : \Re(x) < r, |\Im(x)| < c\,\Re(x), |y| < 2^d\,\Re(x)^{d-1}\}$$

So $\mathcal{D}_{c,r}$ is a thin neighborhood of the real interval $(0,r) \times \{0\}$. If $|\Im(x)| < c\,\Re(x)$, we have

$$\Re(x) \leq |x| \leq \Re(x)(1 + c^2/2)$$

$$\Re(x^{d-1}) = \Re(x)^{d-1} - \Im(x)^2\left(\binom{d-1}{2}\Re(x)^{d-3} + \cdots\right) = \Re(x)^{d-1}\left(1 + O(c^2)\right)$$

$$\Im(x^{d-1}) = (d-1)\Im(x)\Re(x)^{d-2} - \Im(x)^3\left(\binom{d-1}{3}\Re(x)^{d-4} + \cdots\right) = (d-1)\Im(x)\Re(x)^{d-2}(1 + O(c^2)).$$

Let us first show that $\mathcal{D}_{c,r}$ is invariant under the map $F_0$ given in (12). For this we write $(x_1, y_1) = F_0(x,y)$. Thus $x_1 = x - 2(2x+y)^{d-1}$, so if $(x,y) \in \mathcal{D}_{c,r}$,

$$\Re(x_1) = \Re(x) - 2^d\,\Re(x^{d-1}) + O(|x|^{d-2}|y|) = \Re(x)\left(1 - 2^d\,\Re(x)^{d-2}(1 + O(c^2)) + O(\Re(x)^{2d-3})\right)$$

and

$$\Im(x_1) = \Im(x) - 2^d\,\Im(x^{d-1}) + O(|x|^{d-2}|y|) = \Im(x) - (d-1)2^d\,\Im(x)\Re(x)^{d-2}(1 + O(c^2)) + O(\Re(x)^{2d-3}).$$

Thus

$$\frac{\Im(x_1)}{\Re(x_1)} = \frac{\Im(x)}{\Re(x)}\frac{1 - (d-1)2^d(1 + O(c^2))\,\Re(x)^{d-2}}{1 - 2^d(1 + O(c^2))\Re(x)^{d-2} + O(\Re(x)^{2d-3})} + \frac{O(\Re(x)^{2d-3})}{\Re(x)(1 + o(1))}$$

$$< c\,\frac{1 - (d-1)2^d(1 + O(c^2))\,\Re(x)^{d-2}}{1 - 2^d(1 + O(c^2))\Re(x)^{d-2} + O(\Re(x)^{2d-3})} + \frac{O(\Re(x)^{2d-3})}{\Re(x)(1 + o(1))}.$$

Let us choose $c$ sufficiently small that the two (independent) expressions of $O(c^2)$ satisfy

$$(d-1)(1 + O(c^2)) > 1 + O(c^2)$$

Now as $r \to 0$, the right hand side approaches

$$\frac{|\Im(x_1)|}{\Re(x_1)} < c\,\frac{1 - (d-1)2^d\,\Re(x)^{d-2}(1 + O(c^2))}{1 - 2^d\,\Re(x)^{d-2}(1 + O(c^2))} < c$$

so that for $r$ and $c$ small, we have $|\mathfrak{I}(x_1)|/\mathfrak{R}(x_1) < c$. Since $y_1 = (2x + y)^{d-1}$, it follows (after possibly further shrinking $r$) that $(x_1, y_1) \in \mathcal{D}_{c,r}$.

Since the dominant behavior of the first coordinate at the origin is parabolic: $x \mapsto x - 2^d x^{d-1}$, it follows that $\mathcal{D}_{c,r}$ is in the basin of $(0,0)$ for $F_0$.

Finally, it is clear that the arguments above remain valid if we add terms $O((|x| + |y|)^d)$ to $F_0$. Thus $\mathcal{D}_{c,r}$ is in the basin of $(0,0)$ for $S_f^3$. □

None of the coordinates in the 3-cycle (10) is a root of $f$, so we have:

**Corollary 3.3.** *If $f$ has degree at least 3 and has a simple critical point which is not a zero, then there is an open set of points whose orbits are not attracted to roots of $f$.*

## 4 Special cases: the Fibonacci monomial map

Given a $2 \times 2$ integer matrix $A$, we define the monomial map

$$M_A(z_1, z_2) = z^A = (z_1^{a_{1,1}} z_2^{a_{1,2}}, z_1^{a_{2,1}} z_2^{a_{2,2}})$$

This iterates easily: $(M_A)^n = M_{A^n}$. We define $A_{\text{Fib}} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, which satisfies

$$A_{\text{Fib}}{}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$$

where $F_n$ is the $n$th Fibonacci number. The eigenvalues of $A_{\text{Fib}}$ are $\phi = (1 + \sqrt{5})/2 \sim 1.61803$ and $-1/\phi = (1 - \sqrt{5})/2$. Taking the trace of $A_{\text{Fib}}{}^n$, we have

$$F_{n-1} + F_{n+1} = \phi^n + (-\phi)^{-n}$$

We refer to

$$M_{\text{Fib}}(x, y) := M_{A_{\text{Fib}}}(x, y) = (y, x y)$$

as the Fibonacci monomial map.

The set $\mathbb{T}^2 = \{|x| = |y| = 1\}$ is invariant under $M_{\text{Fib}}$, and the restriction of $M_{\text{Fib}}$ to $\mathbb{T}^2$ is a hyperbolic toral automorphism of entropy $\log(\phi)$. The set $\{|x| |y|^\phi = 1\}$ is foliated by the curves $\{x y^\phi = e^{i\theta}\}$, where $y^\phi$ is defined locally, which are the complexifications of the stable manifolds of this toral automorphism. Similarly, the complexifications of the unstable manifolds give a foliation of $\{|y| = |x|^\phi\}$.

Every monomial map commutes with the Cremona involution $j(x, y) = (1/x, 1/y)$, so it follows that the behavior of the Fibonacci monomial map at $(0,0)$ is the same as the behavior at $(\infty, \infty) \in \mathbb{P}^1 \times \mathbb{P}^1$. Thus $j$ gives a symmetry between the basins $\mathcal{B}_{M_{\text{Fib}}}(0,0)$ and $\mathcal{B}_{M_{\text{Fib}}}(\infty, \infty)$.

**Theorem 4.1.** *The set $\{|x| |y|^\phi < 1\}$ is the attracting basin of $(0,0)$ of $M_{\text{Fib}}$, and $\{|x| |y|^\phi > 1\}$ is the basin for $(\infty, \infty) \in \mathbb{P}^1 \times \mathbb{P}^1$. In fact, the rate of convergence of the nth iterate to $(0,0)$ is at the superexponential rate $O(r^{\phi^n})$ with $r < 1$.*

*Proof.* If $(x, y)$ is on one of the coordinate axes, i.e., if $x y = 0$, then $M_{\text{Fib}}^2(x, y) = (0,0)$. Thus to discuss the basin of $(0,0)$, we may assume $x y \neq 0$, and let $\xi = (\log|x|, \log|y|)$ denote logarithmic coordinates. Let $v_+ := (1, \phi)$ (resp. $v_- := (1, -1/\phi)$) denote the $\phi$ (resp. $(-1/\phi)$) eigenvector for $A_{\text{Fib}}$. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, there are $c_+, c_- \in \mathbb{R}$ such that $\xi = c_+ v_+ + c_- v_-$. In logarithmic coordinates, the map $z \mapsto M_{\text{Fib}}^n(z)$ corresponds to

$$\xi \mapsto A_{\text{Fib}}^n \xi = c_+ \phi^n v_+ + c_-(-\phi)^{-n} v_-. \tag{13}$$

Thus $\{\xi : c_+ < 0\}$ is the basin of $(-\infty, -\infty)$ as $n \to +\infty$, and $\{\xi : c_+ > 0\}$ is the basin of $(\infty, \infty)$ as $n \to -\infty$. Exponentiating, we see that if $c_+ < 0$, then $M_{\text{Fib}}^n(z) \to (0,0)$, and the exponential of $\{c_+ < 0\}$ is $\{0 < |x| |y|^\phi < 1\}$. The rate of convergence also is clear from (13) when we exponentiate. □

The following is well known, at least from the point of view of numerical computation.

**Theorem 4.2.** *If $\alpha$ is a simple zero of $f$, then for $(x, y)$ near $(\alpha, \alpha)$, the iterates $S_f^n(x, y)$ converge to $(\alpha, \alpha)$ at the superexponential rate $O(r^{\phi^n})$ with $r < 1$.*

*Proof.* Using (4) we may assume that $\alpha = 0$, so $f(x) = x + a_2 x^2 + O(x^3)$, and up to conjugation, we may assume that $a_2 = 1$. Thus $S_f$ is regular at $(0,0)$, and $S_f(x, y) - a_2 M_{\text{Fib}}(x, y)$ is bounded by a multiple of $(|x| + |y|)^3$, so this result is a consequence of the rate of convergence seen in Theorem 4.1. □

**Example 4.1** (Möbius transformation.). If $f(x)$ is a Möbius transformation, we may assume it is of the form $\frac{x-\alpha}{x-\beta}$, and we introduce an affine change of coordinate in $x$ to make $\alpha = 0$ and $\beta = 1$. Thus $f(x) = x/(x-1)$. It follows that $S_f(x, y) = (y, x y)$ is the Fibonacci monomial map. The basin of $(0,0)$ for $S_f$ corresponds to the basin of the unique root of $f(x)$. The other basin of $S_f$ corresponds to the (degenerate) critical point $x = \infty$ for $f$.

**Example 4.2** (Quadratic polynomial.)**.** If $f(x)$ is a quadratic polynomial with a double root, we may replace $f$ by $\lambda f$ and conjugate with a translation so that $f(x) = x^2$. In this case, $S_f(x, y) = (y, xy/(x + y))$. If we now conjugate with the Cremona involution $j(x, y) = (1/x, 1/y)$, we find that $S_f(x, y)$ is conjugate to the linear map

$$z \mapsto A_{\text{Fib}} z$$

Similarly, if $f(x)$ is a quadratic polynomial with distinct roots, we may replace $f$ by $\lambda f$ and compose with an affine transformation in the $x$-coordinate so that $f(x) = x^2 - 1$. Thus we have $S_f(x, y) = (y, \frac{1+xy}{x+y})$. We now move the roots of $f$ to 0 and $\infty$ via the Möbius transformation $\varphi(z) = \frac{z+1}{z-1}$, and the map $\check{\varphi}(x, y) = (\varphi(x), \varphi(y))$ conjugates $S_f(x, y)$ to the Fibonacci monomial map.

# 5 Dynamical degree

A rational map $F = (F_1, F_2) : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ is defined by a pair of rational functions $F_j = P_j/Q_j$, $j = 1, 2$, where $(P_j, Q_j) = 1$, i.e., they have no common factor. We can define the degree of $F$ to be the maximum of the degrees of each of the polynomials $P_j$ and $Q_j$. However, this is not a conjugacy invariant and does not behave well under composition because the cancellation of factors is "unpredictable", meaning that the degree can increase or decrease. Thus we define the *dynamical degree* $\delta(F)$ to be

$$\delta(F) := \lim_{n \to \infty} (\text{degree}(F^n))^{1/n}$$

This limit exists and is an invariant of birational conjugacy (see [5]), and is an important characteristic of $F$ as a dynamical system.

We will compute the dynamical degree $\delta(S_f)$, following the outline given in [5]. The reader who is not familiar with this approach might look at the expository article [1].

Let us start with the case $a_\infty < 0$, so by Proposition 1.3, all the exceptional curves are of the form $\{y = \alpha\}$ with $\alpha \in \mathbb{C}$, and $\{y = \alpha\}$ is mapped to $(\alpha, \alpha)$, where $\alpha$ is a root of $f$. If $(\alpha, \alpha)$ is a point of indeterminacy, then by Proposition 1.4, it must be a root with multiplicity $\geq 2$. By (4) we may suppose that $\alpha = 0$.

Let $\pi(s, t) = (s, st) = (x, y)$, $\pi : M \to \mathbb{C}^2$ be the blowup of the point $(0, 0)$, as was discussed in Section 2. We assume that 0 is a root of $f$ with multiplicity $a \geq 2$, so we use (7) to obtain

$$\hat{S}|_E(0, t) = \frac{1 + t + \cdots t^{a-2}}{1 + t + \cdots t^{a-2} + t^{a-1}}$$

and thus $\hat{S}$ is regular on $E$ in the $(s, t)$ coordinate system.[1] We let $X = \{t = 0\} \subset M$ denote the strict transform of the $x$-axis under $\pi^{-1}$. The $x$-axis is exceptional for $S_f$, and $\hat{S}(X) = (0, 1) \in E$, so that $X$ is exceptional and is mapped to a regular point for $\hat{S}$. By (8) and (9), we see that the further iterates $\hat{S}^n(0, 1)$ all lie in the interval $\{(0, t) : 0 \leq t \leq 1\}$ and thus the forward orbit of $X$ under $\hat{S}$ never encounters a point of $\mathcal{I}(\hat{S})$.

Now let us proceed to blow up all the points $(\alpha_j, \alpha_j) \in \mathbb{C}^2$, $1 \leq j \leq s$, where $f$ has a zero of order $a_j \geq 2$. Again we will refer to the resulting space as $M$. Every dominant rational map has a well-defined pullback on cohomology (see, for instance [5] or [2]). Thus we have a well-defined linear map $\hat{S}^* : H^{1,1}(M; \mathbb{Z}) \to H^{1,1}(M; \mathbb{Z})$. In the previous paragraph, we showed that for every exceptional curve $C \subset M$, the iterates $\hat{S}^n(C)$, $n \geq 0$, are disjoint from the indeterminacy locus of $\hat{S}$. It follows that the pullback operation satisfies

$$(\hat{S}^n)^* = (\hat{S}^*)^n$$

(see, for instance [5] or [2]), and thus the dynamical degree of $\delta(S_f) = \delta(\hat{S})$ is equal to the spectral radius of $\hat{S}^*$.

The cohomology group $H^{1,1}(M; \mathbb{Z})$ is generated by the classes $V = \{x = const\}$, $H = \{y = const\}$, and $E_1, \ldots, E_s$, where $E_j$ denotes the blowup divisor over $(\alpha_j, \alpha_j)$. To find the dynamical degree, we will find the matrix that represents $\hat{S}^*$ with respect to this ordered basis. We have seen in Proposition 1.3 that a vertical line $\{x = c\}$ is pulled back to a horizontal line $\{y = c\}$. This gives us the first column in the matrix (14) below. Now look at the preimage of a general horizontal line $\{y = c\}$. This is pulled back to the curve $\{R(x, y) = c\}$. By Proposition (1.1), we see that $R(x, y)$ has degree $d_t$ in $x$ when $y$ is fixed, and vice versa. Further, $R(x, y)$ vanishes at all the centers of blowup $(\alpha_j, \alpha_j)$. Looking at (6) we see that $\{R(x, y) = c\} = \{c D_1(x, y) = y N_1(x, y)\}$. Thus we see that $(0, 0)$ is a point of multiplicity $a_j - 1$. It follows that the class of $\{R(x, y) = c\}$ is $d_t H + d_t V + \sum(1 - a_j)E_j$. Finally, we have $\hat{S}(s, t) = (st, \star)$, so the pullback of $E = \{s = 0\}$ is $\{st = 0\}$ which is the class of a horizontal line, so we obtain:

$$\hat{S}^* = \begin{pmatrix} 0 & d_t & 0 & \ldots & 0 \\ 1 & d_t & 1 & \ldots & 1 \\ 0 & 1 - a_1 & 0 & \ldots & 0 \\ \vdots & \vdots & & & \\ 0 & 1 - a_s & 0 & \ldots & 0 \end{pmatrix} \tag{14}$$

The characteristic polynomial of $\hat{S}^*$ is

$$x^s(x^2 + d_t x + ((a_1 - 1) + \cdots + (a_s - 1) - d_t))$$

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[1] We comment that the other coordinate chart at $E$ is $\pi'(u, v) = (uv, v)$, and in this chart we have $E = \{v = 0\}$. The induced map $\hat{S} : M \dashrightarrow M$ has a point of indeterminacy at $(u, v) = (0, 0) \in E$, which corresponds to the point $(s, t) = (0, \infty) \in E$, which is outside of the $(s, t)$ coordinate system.

The largest root of this polynomial is $\delta = (d_t + \sqrt{d_t^2 + 4(d_t + \sum(1-a_j))})/2$. Since $a_\infty < 0$, we have $\deg(p) > \deg(q)$, so $d_t = \deg(p) - 1$. Since $\deg(p) = \sum a_j$, it follows that $\delta > d_t$ unless $p$ has exactly one zero, in which case $p(t) = c(t-a)^d$.

There remains the case $a_\infty \geq 0$ to consider. By Proposition 1.3, $\mathbb{P}^1 \times \{\infty\}$ is exceptional and maps to $(\infty, \infty)$, which by Proposition 1.4 is indeterminate exactly when $a_\infty \geq 1$. Thus if $a_\infty = 0$, then $\delta$ is given by the formula above. If $a_\infty \geq 1$, then we must look at the exceptional curve $\mathbb{P}^1 \times \{\infty\}$. The auxiliary function $g(t) = t f(1/t)$ vanishes to order $a_\infty + 1$ at $t = 0$, so by (4) it suffices to determine the dynamical degree of $S_g$. For this, we apply the previous argument. For the new exceptional curve, we augment $\hat{S}^*$ with an extra column as before, and in the extra row, we have the coefficient $1 - (a_\infty + 1) = -a_\infty$ which gives:

$$\hat{S}^* = \begin{pmatrix} 0 & d_t & 0 & \dots & 0 & 0 \\ 1 & d_t & 1 & \dots & 1 & 1 \\ 0 & 1-a_1 & 0 & \dots & 0 & 0 \\ & & \vdots & & & \\ 0 & 1-a_s & 0 & \dots & 0 & 0 \\ 0 & -a_\infty & 0 & \dots & 0 & 0 \end{pmatrix} \tag{15}$$

The largest root of the characteristic polynomial of (15) is

$$\delta = \left(d_t + \sqrt{d_t^2 + 4\left(d_t - a_\infty + \sum(1-a_j)\right)}\right)/2.$$

In the case where $a_\infty \geq 0$, we have $d_t = a_\infty + \deg(p) = \deg(q)$. Thus $a_\infty + \sum a_j = d_t$, from which we conclude that $\delta > d_t$. Combining the cases $a_\infty < 0$ and $a_\infty \geq 0$, we have:

**Theorem 5.1.** *If $p(t) = c(t-a)^d$, and $\deg(p) > \deg(q)$, then $\delta = d_t = d - 1$. Otherwise, $S_f$ is of small topological degree, meaning that the dynamical degree is larger than the topological degree.*

If $\delta > d_t$, there are invariant currents $T^\pm$ with the properties $S^* T^+ = \delta T^+$ and $S_* T^- = \delta T^-$ (see [2]). In our argument discussing the forward iterates of the exceptional curves, we in fact showed that the forward orbits of exceptional curves remain at bounded distance from the indeterminacy locus. Thus by [4], it follows that there is an invariant measure $\mu = T^+ \wedge T^-$. A number of the dynamical properties of $T^\pm$ and $\mu$ are presented in [3]. It will be interesting to apply them to the Secant Method.

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