# Fatou Components for Conservative Holomorphic Surface Automorphisms

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§**0. Introduction.** The dynamics of automorphisms of complex surfaces is an active area where there has been a productive interaction between questions of dynamics and complex analysis. We will describe some questions and problems which arise out of complex dynamics but which have a strong component of complex analysis. We will discuss two general cases. The first is polynomial automorphisms of $\mathbb{C}^2$. By Friedland-Milnor [FM], it suffices to restrict our attention to compositions of *generalized Hénon mappings*, which have the form

$$f(x,y) = (y, p(y) - \delta x) \tag{0.1}$$

where $p(y)$ is a monic polynomial of degree at least 2, and $\delta \in \mathbb{C}$ is a nonzero scalar. Some basic dynamical properties of a general Hénon map are given in [H], [HO], [FS] and [BS]. An intriguing aspect here is that the mappings themselves are simple to write down, but the dynamical questions have led to a rather elaborate theory. The results obtained have been most successful/complete in the case of dissipative maps. This paper is devoted to an outline of the conservative (volume-preserving) case, which corresponds to $|\delta| = 1$. Our theme will be that in the conservative case there seem to be more open questions than proved results.

The second case we will discuss is rational surface automorphisms.* The dynamics of a positive entropy automorphism of a compact, complex surface may be studied along the same lines as the complex Hénon maps (see Cantat [C]). Here we consider the case where $X$ is a rational surface (that is, $X$ is birationally equivalent to $\mathbb{P}^2$). In the Appendix, we explain why we do not consider other complex surfaces. Consider the family of birational maps of $\mathbb{P}^2$ given by:

$$f_{a,b}(x,y) = \left(y, \frac{y+a}{x+b}\right) \tag{0.2}$$

The map $f_{a,b}$ may not look like an automorphism because the line $\{y+a=0\}$ is exceptional: it is mapped to the point $(-a, 0)$. However, let $\pi : X \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at the point $(-a, 0)$, and let $L$ denote the lift of the line $\{y+a=0\}$ to $X$. If we lift $f_{a,b}$ to a map of $X$, then the line $L$ is no longer exceptional. Although this new map is not yet an automorphism, there is an infinite family of cases (see Theorem 8.1) where $f_{a,b}$ induces an automorphism after suitable further blowups.

All Hénon maps have constant Jacobian, which means that the holomorphic 2-form $\eta = dx \wedge dy$ satisfies $f^*(\eta) = \delta\eta$. If we extend $f$ to a birational map of $\mathbb{P}^2$, then $\eta$ becomes a meromorphic 2-form, with a pole of order 3 at infinity. When we consider an automorphism

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\* A Hénon map cannot be expressed as an automorphism of a compact surface. This is because by [FM], the dynamical degree of a Hénon map is an integer greater than 1, whereas the dynamical degree of an automorphism of a compact surface cannot be rational, unless it is 1 (see [BC]).

$f$ of a compact, complex surface $X$, we require that there is a meromorphic 2-form $\eta$ on $X$ such that $f^*\eta = \delta\eta$ for some $|\delta| = 1$.

**Acknowledgment.** I wish to thank Prof. Shigehiro Ushiki for generously explaining his computer work and sharing his ideas with me. His work and vision have been an inspiration and motivation for my work.

**§1. Fatou set of a conservative Hénon map.** A traditional starting place for complex dynamics is the dichotomy between the Fatou and Julia sets: the regions of regular and chaotic dynamics. If $f : X \to X$ is a holomorphic map, we define the *(forward) Fatou set* $\mathcal{F}^+$ to be the largest open subset of $X$ such that the iterates form a *normal family*. This means that any sequence $\{f^{n_j}\}$ has a subsequence which converges uniformly on compact subsets of $\mathcal{F}$. This is equivalent to pre-compactness in the compact-open topology (which is the topology of uniform convergence on compact subsets). The Fatou set is also the set of points which are Lyapunov stable. If $f$ is invertible, then the *backward Fatou set* $\mathcal{F}^-$ is defined similarly, with $f$ replaced by $f^{-1}$.

In analogy with the case of a one-dimensional polynomial map, we define the *escape locus* $U^+$ and *boundedness locus* $K^+$ by

$$K^+ = \{q \in \mathbb{C}^2 : \{f^n(q) : n \geq 0\} \text{ is bounded}\}, \quad U^+ := \mathbb{C}^2 - K^+$$

as well as the set $J^+ := \partial K^+$. It follows that the forward Fatou set is $\mathcal{F}^+ = \mathbb{C}^2 - J^+$. Similarly, we define $K^-$, $U^-$ and $J^-$ using $f^{-1}$ in place of $f$, and we set

$$J := J^+ \cap J^-, \quad K := K^+ \cap K^-$$

Let us recall that the *Shilov boundary* of a compact set $X \subset \mathbb{C}^2$, written $\partial_s X$, is the smallest closed set such that

$$\max_{x \in X} |P(x)| = \max_{x \in \partial_s X} |P(x)|$$

By the Maximum Principle, $\partial_s X \subset \partial X$. We define $J^* := \partial_s K$.

In dimension 1, the set $K$ is called the *filled Julia set*, and the *Julia set* itself is $J := \partial K$. For complex Hénon maps, the sets $K^+$, $K^-$ and $K$ are analogues of the filled Julia set, and the sets $J^+$, $J^-$, $J$ and $J^*$ are all analogues of the Julia set. While $J^*$ may seem to be the most exotic of these sets, it has a number of natural characterizations (see [BT] for (1) and [BLS] for (2)): (1) it is the support of the invariant (equilibrium) measure $\mu$, and (2) it is the closure of the set of periodic points of saddle type. It is easily seen that $J^* \subset J$, and equality $J^* = J$ holds in a number of cases, but the problem of determining whether these two sets are always equal has remained elusive.

We recall the following basic result:

**Theorem 1.1 ([FM]).** *If $f$ is a volume-preserving (composition of) complex Hénon maps, then*

$$\text{int}(K^+) = \text{int}(K^-) = \text{int}(K)$$

This set is bounded, and if $\Omega$ is any connected component of $\text{int}(K)$, then there is an $N$ such that $f^N(\Omega) = \Omega$.*

A consequence of Theorem 1.1 is that $\mathcal{F}^\pm = U^\pm \cup \text{int}(K)$. A *Fatou component* is a connected component of $\mathcal{F}^+$. We observe that since $\Omega$ is the set of normality of a sequence of polynomial mappings, it is a Runge domain, which means in particular that it is polynomially convex.

Although the unbounded set $U^+$ is a Fatou component, it is different from the others because it is the basin of attraction of a point at infinity, whereas, as we will see in the next section, all other Fatou components are rotation domains. Henceforth, we will consider only bounded Fatou components.

One of the most basic quadratic maps $p_c(z) = z^2 + c$ corresponds to $c$ in the interior of the main cardioid of the Mandelbrot set. In this case, $p_c$ has an attracting fixed point $z_c$, and the interior of $K(p_c)$ is the immediate basin of $z_c$. The interior of $K(p_c)$ is a Fatou component, and $K(p_c)$ itself is the closure of this component. In fact, $K(p_c)$ is topologically equivalent to a closed disk, and its boundary $J(p_c)$ is a Jordan curve.

Let $p_c$ be as above, and let $f_\delta(x, y) = (y, p_c(y) - \delta x)$ be the Hénon map associated with $p_c$. It was shown in [HO] and [FS] that if $|\delta| > 0$ is sufficiently small, then $f_\delta$ has an attracting fixed point, and the interior of $K^+$ is the basin of attraction. Thus $K^+$ is the closure of a Fatou component. In fact, $J^+$ is a topological 3-manifold, and $(K^+, J^+)$ is a topological manifold-with-boundary. Our first question is whether anything similar can happen in the conservative case:

**Question 1.** *If $\Omega$ is a Fatou component of a conservative Hénon map, is it possible that $\overline{\Omega} = K$? Or is $\overline{\Omega}$ always a strict subset of $K$?*

§**2. Rotation domains.** We suppose that $\Omega$ is a conservative Fatou component with $f(\Omega) = \Omega$, and we define the set of all limits of convergent subsequences

$$\mathcal{G} := \{g = \lim_{n_j \to \infty} f^{n_j} : \Omega \to \overline{\Omega}\}$$

If $g = \lim_{n_j \to \infty} f^{n_j}$ is such a limit, then $g$ must preserve volume, and thus it is locally invertible. It follows that $g : \Omega \to \Omega$. Further, we may pass to a subsequence such that $m_j := n_{j+1} - 2n_j \to \infty$ and $h := \lim_{j \to \infty} f^{m_j}$ exists. It follows that

$$g \circ h = \lim_{j \to \infty} f^{n_j} \circ f^{m_j} = \lim_{j \to \infty} f^{n_{j+1} - n_j} = \text{id}$$

and thus $h = g^{-1}$, so $\mathcal{G}$ is a group. Since $\Omega$ is a Fatou component, it follows that $\mathcal{G}$ is compact in the compact-open topology. By a Theorem of H. Cartan (see Narasimhan [N]), it follows that $\mathcal{G}$ is a Lie group. Thus the connected component $\mathcal{G}_0$ of the identity must be a (real) torus.

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* A component $\Omega$ is said to be *wandering* if $f^n(\Omega) \cap \Omega = \emptyset$ for all nonzero $n \in \mathbb{Z}$. It is an open question, in the dissipative case $|\delta| < 1$, whether polynomial automorphisms can have wandering Fatou components.

**Theorem 2.1 ([BS2]).** $\mathcal{G}_0$ *is isomorphic to* $\mathbb{T}^\rho$ *with* $\rho = 1$ *or* $2$.

We conclude that $\Omega$ is invariant under a nontrivial torus of rotations, so we call it a *rotation domain*, and we refer to $\rho$ as the *rank* of the rotation domain.

**Question 2.** *Does a rotation domain necessarily contain a fixed point?*

**Rank 1.** We first discuss the case of rank 1. In this case, $\mathcal{G}_0 \cong \mathbb{T}^1$, and $\mathcal{G}_0$ is generated by the real part of a holomorphic vector field $\mathcal{V}$ on $\Omega$. It follows that the restriction of $f$ to $\Omega$ is part of the flow generated by $\mathcal{V}$, so $f|_\Omega = \exp(t_1 \Re(\mathcal{V}))$ for some $t_1 > 0$. The zeros of $\mathcal{V}$ correspond to the fixed points of $f$ in $\Omega$; by [FM], there are a total of $d$ fixed points (counted with multiplicity) in $\mathbb{C}$. Suppose that $\omega \in \Omega$, and $\mathcal{V}(\omega) \neq 0$. Since $\mathcal{G}$ is compact, the orbit $\mathcal{G} \cdot \omega$ is a closed curve. It follows that the orbit $\mathcal{G} \cdot \omega$ is contained in a Riemann surface $\mathcal{R}_\omega$ and a proper map $\varphi_\omega : \mathcal{R}_\omega \to \Omega$. Now $\mathcal{R}_\omega$ is a Riemann surface which carries a $\mathbb{T}^1$ of automorphisms, so it follows that $\mathcal{R}_\omega$ must either be the disk or an annulus. We may write $\mathcal{R}_\omega$ as $\{|\zeta| < 1\}$ or $r_1 < |\zeta| < r_2$. Then the restriction $f|_{\mathcal{R}_\omega}$ is given by $\zeta \mapsto \alpha\zeta$, where $\alpha = \exp(2\pi i a)$, $a > 0$, and $a$ is the rotation number of $f$ on the curve $\mathcal{G} \cdot \omega$. The fixed points of $f$ are isolated, so $a$ must be irrational. Further, since $a$ depends continuously on $\omega$, we conclude that $a$ must be constant. We call this the *rotation number of* $\Omega$, written $\mathrm{rot}(\Omega)$.

The question arises whether the (abstract) torus action on $\Omega$ is equivalent to a more familiar circle action. Let $D \subset \mathbb{C}^2$ be a connected open set. We say that $D$ is a $(p, q)$ *domain* if $(e^{ip\theta}z, e^{iq\theta}w) \in D$ whenever $(z, w) \in D$ and $\theta \in \mathbb{R}$.

**Question 3.** *Suppose that* $\Omega$ *is a rank 1 rotation domain. Is there a* $(p, q)$*-domain* $D \subset \mathbb{C}^2$ *and a biholomorphic* $\Phi : \Omega \to D$ *satisfying* $L \circ \Phi = \Phi \circ f$, *with* $L = \begin{pmatrix} \alpha^p & 0 \\ 0 & \alpha^q \end{pmatrix}$ *for some* $\alpha \in \mathbb{C}$, $|\alpha| = 1$?

If there is such a domain $D$, then we may take $p$ and $q$ to be relatively prime, and $\alpha = e^{2\pi i r}$, where $r = \mathrm{rot}(\Omega)$.

**Question 4.** *Can the case* $pq < 0$ *occur in Question 3?*

**§3. Reinhardt domains.** Let $D \subset \mathbb{C}^2$ be a connected open set. We say that $D$ is a *Reinhardt domain* if $(e^{i\theta}z, e^{i\phi}w) \in D$ for all $(z, w) \in D$ and all $\theta, \phi \in \mathbb{R}$. A Reinhardt domain is determined by its logarithmic image

$$\log(D) := \{(\xi, \eta) = (\log|z|, \log|w|) \in \mathbb{R}^2 : (z, w) \in D\}$$

A classical result asserts that $D$ is holomorphically convex if and only if $\log(D)$ is convex. If $\Omega$ is a rank 2 rotation domain, then by [BBD] the $\mathcal{G}$-action on $\Omega$ may be conjugated to the standard linear action on $\mathbb{C}^2$:

**Theorem 3.1([BBD]).** *There are a Reinhardt domain* $D \subset \mathbb{C}^2$, *a linear map* $L = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$, $|\alpha_1| = |\alpha_2| = 1$, *and a biholomorphic map* $\Phi : \Omega \to D$ *such that* $\Phi \circ f = L \circ \Phi$.

**Question 5.** *What are the Reinhardt domains that can arise as rank 2 rotation domains?*

Note that Reinhardt $D$ is the biholomorphic model of $\Omega$, whereas $\Omega$ is a subset of dynamical space. Thus the boundary of $D$ is logarithmically convex and rather "tame", whereas we expect that $\Omega$ may have a possibly "wild", fractal boundary.

The torus action on $D$ has no fixed point except the origin $(0,0)$. In our case, $\Omega$ is polynomially convex, and for polynomially convex sets we have $H_2(\Omega; \mathbb{Z}) = 0$. There are two possibilities: a polynomially convex Reinhardt domain is either:
(1) topologically equivalent to a ball (in which case it contains the fixed point $(0,0)$), or
(2) topologically equivalent to disk $\times$ annulus (in which case it contains an invariant annulus inside one of the coordinate axes).
In either case, $\log(D)$ will be an unbounded, convex subset of $\mathbb{R}^2$. It seems hard to imagine that the Reinhardt model can be a domain that is "familiar," so we ask (expecting the answer "no"):

**Question 6.** *Can there be a "Siegel bidisk"? That is, can $\Omega$ be biholomorphic to the bidisk $\Delta^2 := \{(x, y) \in \mathbb{C}^2 : |x|, |y| < 1\}$? More generally, can $\Omega$ be an analytic polyhedron?*

**Question 7.** *Can there be a "Siegel ball"? That is, can $\Omega$ be biholomorphic to the standard ball $\mathbb{B}^2 := \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 < 1\}$? More generally, can the boundary of $\Omega$ (or its Reinhardt model) be smooth?*

## §4. Existence of rotation domains.

Let a linear map $L = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ be given, and consider a map

$$F(x, y) = L \begin{pmatrix} x \\ y \end{pmatrix} + \sum_{j+k \geq 2} F_{jk} x^j y^k, \quad F_{jk} \in \mathbb{C}^2 \tag{4.1}$$

Consider a power series

$$\Phi = (x, y) + \sum_{j+k \geq 2} \Phi_{jk} x^j y^k, \quad \Phi_{jk} \in \mathbb{C}^2 \tag{4.2}$$

and the power series equation

$$\Phi \circ F = L \circ \Phi$$

A *resonance* is a relation of the form $\lambda_j = \lambda_1^{m_1} \lambda_2^{m_2}$, where $j = 1$ or $2$, and $m_1, m_2 \geq 0$, $m_1 + m_2 \geq 2$. Note that if $\lambda_1 = \alpha^p$, $\lambda_2 = \alpha^q$, then there are infinitely many resonances whenever $pq < 0$. If $pq > 0$, there is a resonance if and only if $\alpha$ is a root of unity. If there is no resonance, then it is possible to solve algebraically for the coefficients $\Phi_{jk}$. In fact, the solution $\Phi_{jk}$ is a rational function in the coefficients $F_{pq}$ with $p < j$ and $q < k$. The solution $\Phi$ is a formal power series, and convergence is a question. A classic theorem of C.L. Siegel (see Herman [H] and Pöschel [P] for subsequent developments) asserts:

**Theorem 4.1 (Siegel ...).** *If $\lambda_1$ and $\lambda_2$ are "sufficiently irrational", then the power series $\Phi$ converges in a neighborhood of the origin.*

The convergence of such a series has been much studied and is an example of a "small divisor" problem. Let us restrict ourselves to noting that the condition "sufficiently irrational" holds for almost every choice of $(\lambda_1, \lambda_2)$ with $|\lambda_1| = |\lambda_2| = 1$. We say that $\lambda_1$

and $\lambda_2$ are *multiplicatively independent* if $\lambda_1^{m_1}\lambda_2^{m_2} = 1$ implies $m_1 = m_2 = 0$. Otherwise, there exists $\alpha$ such that $\lambda_1 = \alpha^p$, $\lambda_2 = \alpha^q$. (If $pq < 0$, there is a resonance between $\lambda_1$ and $\lambda_2$, and the "sufficiently irrational" hypothesis of Siegel's Theorem is not met.) When we apply Siegel's Theorem, the rotation domain $\Omega$ will have rank 2 if the $\lambda_j$'s are multiplicatively independent, and rank 1 otherwise.

If a quadratic Hénon map has a fixed point whose differential is diagonalizable with eigenvalues $\lambda_j$, $j = 1, 2$, then it is conjugate to

$$H_{\lambda_1,\lambda_2} : (x, y) \mapsto (\lambda_1 x, \lambda_2 y) + (1, -1)(\lambda_1 x + \lambda_2 y)^2 \tag{4.3}$$

If $H_{\lambda_1,\lambda_2}$ is of the form (4.3), and if the eigenvaues $|\lambda_1| = |\lambda_2| = 1$ are suitable for Siegel's Theorem, then $H_{\lambda_1,\lambda_2}$ is linearizable in a neighborhood $U$ of the origin. It follows that $U$ is contained in a Fatou component (rotation domain) $\Omega = \Omega_{\lambda_1,\lambda_2}$ with $H(\Omega) = \Omega$. Although the domain $\Omega_{\lambda_1,\lambda_2}$ is nonempty for almost every choice of $|\lambda_1| = |\lambda_2| = 1$, it is unstable, because the roots of unity are dense in $|\lambda_1| = |\lambda_2| = 1$, and whenever $\lambda_1$ and $\lambda_2$ are roots of unity, $H_{\lambda_1,\lambda_2}$ cannot be linearized, and thus $\Omega_{\lambda_1,\lambda_2} = \emptyset$. In dimension 1, the dependence of the radius of a Siegel disk on the multiplier $\lambda$ is related to the Brjuno function (see [Ma] and [MMY]).

Let us revisit Question 4 above in the special case where $\Omega$ contains a fixed point. In this case, the eigenvalues have a resonance at the fixed point. Thus the generic polynomial map cannot be linearized. Question 4 asks whether *every* map with this resonance will fail to be linearizable.

We close this section with an easy complement to Siegel's Theorem. This is Proposition 4.2, which allows us to conclude that any Fatou component with a fixed point could have been constructed by linearization. Let us suppose that $\Omega$ is a rotation domain, and suppose that $(0, 0) \in \Omega$ is a fixed point for $F$. Then $DF(0, 0)$ is linearly conjugate to a matrix $L = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $|\lambda_1| = |\lambda_2| = 1$. Let us consider the sequence of maps

$$\Phi_n = n^{-1}\left(Id + L^{-1}F + \cdots L^{-n+1}F^{n-1}\right)$$

Now $F^n$ is bounded on any compact subset of $\Omega$, so the $\Phi_n$ are bounded there, too. Thus $\{\Phi_n\}$ is a normal family of mappings, and we can let $\tilde{\Phi} := \lim_{j\to\infty}\Phi_{n_j}$ denote any sub-sequential limit. It is easy to see that $\tilde{\Phi}$ linearizes $F$ in a neighborhood of $(0, 0)$:

**Proposition 4.2.** $\tilde{\Phi} \circ F = L \circ \tilde{\Phi}$.

**Question 8.** *Is there a construction that gives a Fatou component without starting at a fixed point?*

§**5. Nonexistence of rotation domains.** The simplest resonance is where one of the eigenvalues is 1. Suppose $f$ is a Hénon map with a fixed point $z_0$. If the eigenvalues of $Df(z_0)$ are 1 and $\lambda$, then $f$ cannot be locally linearized at $z_0$ because the linear map with eigenvalues 1 and $\lambda$ has a curve of fixed points, corresponding to the multiplier 1, but the fixed points of $f$ are isolated. The next simplest case is where the eigenvalues are $\alpha$ and $\alpha^{-1} = \bar{\alpha}$, and thus $|\alpha| = 1$. This leads to an infinite number of resonances, each of which is a possible obstruction to linearization.

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Much of the early interest in Hénon maps arose from the real, area-preserving case (see [Hen]). A complex Hénon map preserves $\mathbb{R}^2$ when it has real coefficients. In this case, the jacobian is $\delta = \pm 1$, which means that a periodic point has multipliers $\lambda$ and $\pm\lambda^{-1}$, which forces a resonance. A fixed point for a generic area-preserving map with $|\lambda| = 1$ will be of "twist" type. The classical KAM theory asserts that near such a fixed point, there will be a positive measure set of rotation numbers $\omega$, and for each $\omega$ there will be an invariant "KAM curve" $\gamma_\omega$. For a complex Hénon map, $\gamma_\omega$ has a complexification to an annulus $\tilde{\gamma} \subset K \subset \mathbb{C}^2$. The "standard picture" of a twist map does not allow $\gamma_\omega$ to be inside the Fatou set.

We must ask whether, contrary to the generic picture, there is a map that can have a rotation domain. We expect that the answer to the following question is "no" in each case:

**Question 9.** *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-preserving real Hénon map.*
 *(i) Is it possible for $h$ to have a linearizable fixed (periodic) point?*
 *(ii) Is it possible for the Fatou set $\mathcal{F}(h)$ to be nonempty?*
*(iii) Is it possible for $\mathcal{F}(h) \cap \mathbb{R}^2 \neq \emptyset$?*


**§6. Computer pictures: the Ushiki approach.** Computers have been very useful in illustrating theorems and motivating new results in the study of complex Hénon maps. The most useful computer picture has been the *unstable slice* picture, which was introduced and widely used by J. Hubbard. To make such a picture, you start with a saddle fixed (or periodic) point $q$. The "unstable slice" is $W^u(q) \cap K^+$. The unstable manifold $W^u(q)$ is conformally equivalent to $\mathbb{C}$, and it is not hard to compute the uniformization $\psi_q : \mathbb{C} \to W^u(q) \subset \mathbb{C}^2$. The rate of escape function (Green function of $K^+$) is given by

$$G^+(x,y) := \lim_{n \to \infty} \frac{1}{d^n} \log^+ ||f^n(x,y)||$$

In order view the unstable slice, we simply look at the level sets of $G^+$ and its harmonic conjugate. We refer the reader to [K] for an extended discussion of this and related pictures. Two important features of this picture are:

(6.1) The unstable slice $K^+ \cap W^u(q)$ is invariant under $f$, and the resulting picture in the $\zeta$-plane $\mathbb{C}$ is invariant under a complex scaling $\zeta \mapsto \beta\zeta$.

(6.2) The restriction of $G^+$ to the unstable manifold is subharmonic, and it is is harmonic outside $K^+$. Thus the level sets will be compatible with the maximum principle and the mean value property.

While the unstable slice picture is quite useful in the dissipative case, the unstable slice cannot "see" the Fatou set of a conservative map since $W^u(q) \cap \Omega = \emptyset$ for any bounded Fatou component $\Omega$. (For if a point belongs to $\Omega$, then its backward/forward orbit stays away from the boundary; but if a point belongs to $W^u(q)$, its backward orbit converges to $q \notin \Omega$.) The inability of the stable/unstable manifolds to "penetrate" the Fatou set may be a reason why the conservative case seems more difficult than the dissipative case.

We say that a map $f$ is *reversible by an involution* $\tau$ if $\tau \circ f \circ \tau = f^{-1}$ (see [GM]). For instance, if $\delta = 1$, then $f(x, y) = (y, p(y) - x)$, and $f$ is reversible by the involution $(x, y) \mapsto (y, x)$. If $h$ is a polynomial automorphism, then the (constant) Jacobian determinant of $h^{-1} \circ f \circ h$ is the same as that of $f$. Thus $f$ cannot be reversible by a (holomorphic) polynomial automorphism unless $\delta = \pm 1$.

**Theorem 6.1 (Ushiki).** *A Hénon map is reversible by the (anti-holomorphic) involution $\tau(x, y) = (\bar{y}, \bar{x})$ if and only if it has the form*

$$f(x, y) = (y, \beta p(y) - \beta^2 x)$$

*where $p(y)$ is a real polynomial, and $|\beta| = 1$.*

The fixed point set of $\tau$ is the conjugate diagonal:

$$\mathrm{Fix}(\tau) = \Delta' := \{(\zeta, \bar{\zeta}) : \zeta \in \mathbb{C}\}$$

The Ushiki approach is to look at maps that are reversible under $\tau$ and look at the slice of the interior of $K$ by the conjugate diagonal $\Delta'$. The new pictures will not have the same "feel" as the unstable slice pictures because properties (6.1) and (6.2) above do not hold. However, these slices are well suited to reversible maps, which seem to be a rich source of rotational behavior. Restricting to reversible maps also has the benefit of reducing the dimension of the (real) parameter space to two.
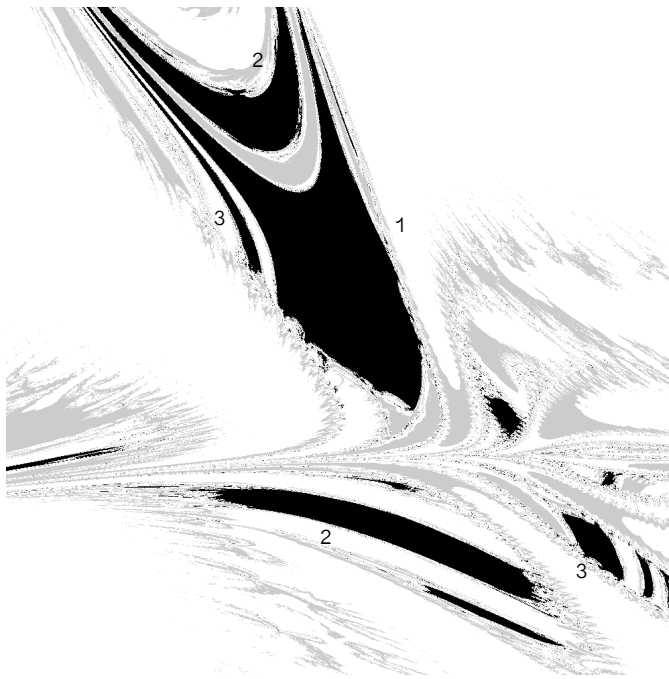


Figure 1. Hénon map $f(x, y) = (y, e^{i\theta}(y^2 + \alpha) - e^{2i\theta})$: $\alpha = 0.269423$, $\theta = 1.02773$.
Slice of $K$ by the conjugate diagonal $\Delta'$.

Figure 2. Same Hénon map as Figure 1. Slice of $K$ by complex line.

Ushiki has made a number of pictures of $K \cap \Delta'$ and has found a number of interesting phenomena. One of Ushiki's parameter values is used to make Figures 1 and 2, which show two slices of $K$ for the reversible map $f(x, y) = (y, e^{i\theta}(y^2 + \alpha) - e^{2i\theta}x)$. Figure 1 is the slice of $K$ by the conjugate diagonal $(\zeta, \bar{\zeta})$ for $.2 \leq \Re(\zeta) \leq .6$, $-.2 \leq \Im(\zeta) \leq .4$; Figure 2 shows the slice of $K$ by the "horizontal" line $(\zeta, .4 - .1\, i)$, $.3 \leq \Re(\zeta) \leq .56$, $-.05 \leq \Im(\zeta) \leq .3$. Points of $K = \{G^+ + G^- = 0\}$ are black, and other points are white/gray according to the value of $G^+ + G^- > 0$.
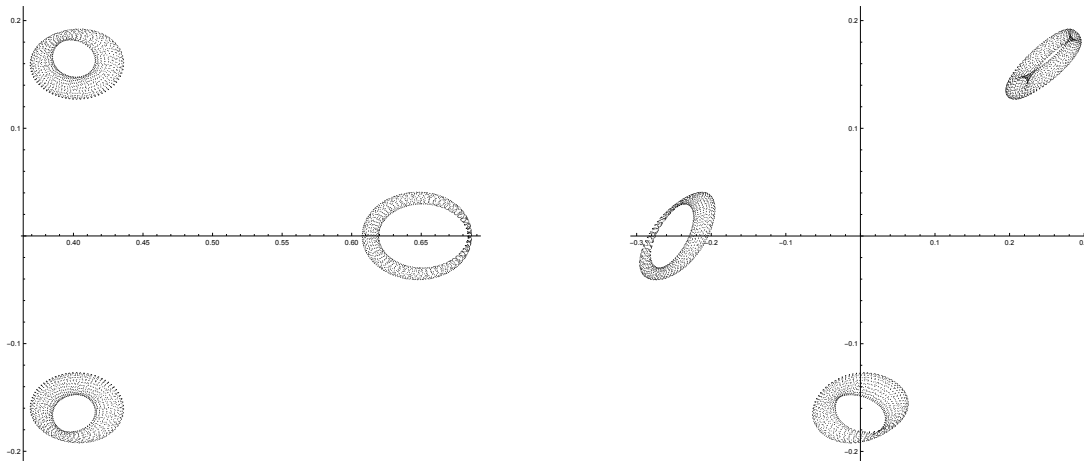


Figure 3. First 5000 iterations of a point from region '1'; two projections.

The black components labeled '1' in Figures 1 and 2 represent the same component of the interior of $K$: points from each of these regions have the same orbits. The point $(\zeta_1, \bar{\zeta}_1)$,

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$\zeta_1 = 0.396+0.19i$ is taken from region '1'. The orbit of this point is shown in Figure 3 under two projections: $\pi_1(x,y) = (Re(x), Re(y))$ on the left, and $\pi_2(x,y) = (Re(x-y), Im(y))$ on the right. The closure of the orbit appears to be union of three 2-tori. Region '1' contains a point of period 3 and appears to be the slice of a 3-cycle of rotation domains of rank 2.
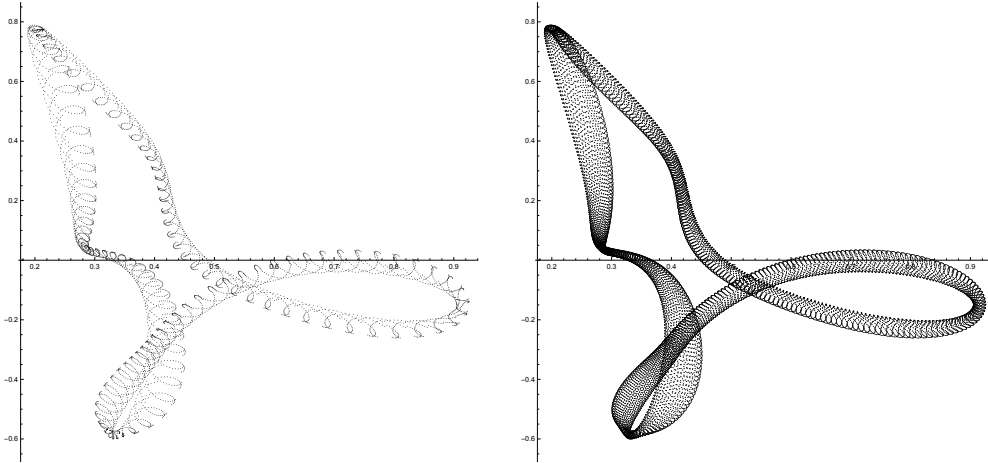


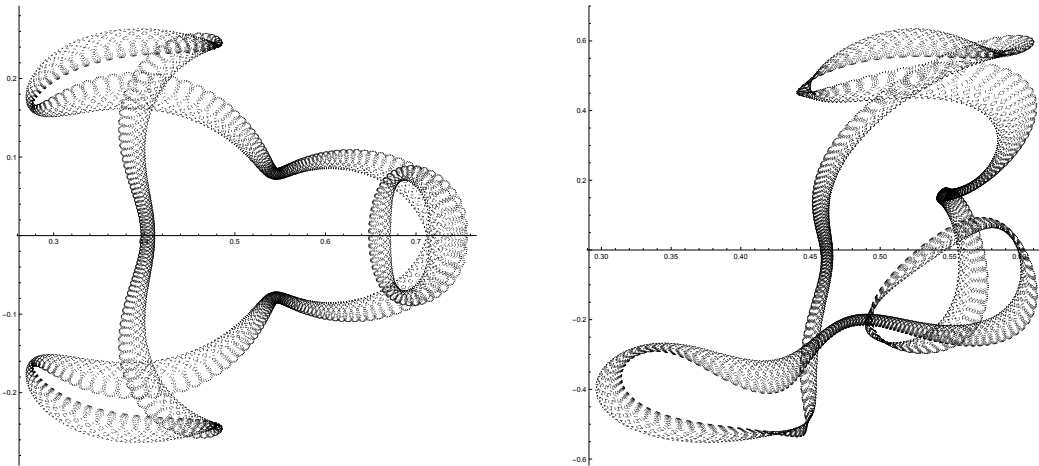Figure 4. 5000 and 50000 iterations of a point from region '2'.



Figure 5. Orbit (40000 points) from region '3': two projections.

The black components labeled '2' in Figures 1 and 2, represent the same connected component of the interior of $K$ in $\mathbb{C}^2$. The point $(\zeta_2, \bar{\zeta}_2)$, $\zeta_2 = 0.36 + 0.298i$, is taken from region '2', and Figure 3 shows the projection of points from the orbit of this point. Recall that by Theorem 2.1, the closure of an orbit is either a closed curve or a 2-torus. Thus it would appear that the orbits in Figure 3 are increasing to something whose closure would be a (connected) 2-torus. If this is in fact the case, then region '2' is contained in an invariant Fatou component $\Omega_2$ which has rank 2. On the other hand, $\Omega_2$ cannot contain a fixed point because the 2 fixed points of $f$ are both of saddle type. Thus $\Omega_2$ would appear to be "exotic", which means that it is a rotation domain without a fixed point. There are also other components, such as '3', which appear to belong to "exotic" rotation domains, which was a motivation for Ushiki to find Hénon maps like this.

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**Problem 10.** *Prove mathematically that there are Hénon maps with exotic rotation domains.*

It may appear surprising that the 2-tori in Figures 3 and 4 are long and thin. If '1', '2', and '3' actually represent rotation domains of rank 2, then as was noted in §2, each of these domains is uniformized by a Reinhardt domain $D$, and each $D$ has nonempty intersection with one or both coordinate axes. The axes are not generic for the torus action: the $f$-orbits of these points are dense in closed curves. So for a point of $D$ which is close to one of the axes, we expect the orbit to be dense in a long, thin torus which looks almost like a closed curve.

**Question 11.** *Can a Hénon map have infinitely many rotation domains? Can it have an infinite number of rotation domains with fixed (periodic) points?*

This question is motivated by the fact that there seem to be so many black components in Figure 1 and any one Fatou component should not create many slice components because of the following rank 2 phenomenon:

**Theorem 6.2.** *Let $f$ be reversible by $\tau$, and let $\Omega = f(\Omega)$ be a rank 2 Fatou component with $\Omega \cap \Delta' \neq \emptyset$. If $\Omega$ contains a fixed point, then $\Omega \cap \Delta'$ is connected; otherwise it has exactly two connected components.*

*Proof.* Since $f$ is $\tau$-reversible, $\tau(\Omega)$ is a Fatou component. Further, since $\Omega \cap \Delta' \neq \emptyset$, it follows that $\Omega \cup \tau(\Omega)$ is connected. Thus $\Omega = \tau(\Omega)$ is invariant under $\tau$. Now we let $D \subset \mathbb{C}^2_{z_1, z_2}$ be the Reinhardt model for $\Omega$, and the conjugacy takes $f$ to a diagonal map $L = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ with $|\mu_1| = |\mu_2| = 1$. Let $\hat{\tau}$ denote the map of $D$ induced by this conjugacy. Thus $\hat{\tau}$ is an anti-holomorphic and reverses $L$. Since $L$ generates the standard torus action on $\mathbb{C}^2$, it follows that $\hat{\tau}$ reverses the torus action. Now we may write $\hat{\tau}(z) = (\hat{\tau}_1, \hat{\tau}_2)$, where each $\hat{\tau}_j = \sum a_\alpha \bar{z}^\alpha$ is a Laurent series in the anti-holomorphic variables $(\bar{z}_1, \bar{z}_2)$. The property of reversing the torus action means that for all $\theta_1, \theta_2 \in \mathbb{R}$, we have

$$\hat{\tau}(e^{i\theta_1} z_1, e^{i\theta_2} z_2) = (e^{-i\theta_1} \hat{\tau}_1, e^{-i\theta_2} \hat{\tau}_2)$$

By checking the coefficients of the Laurent series, we conclude that $\hat{\tau}(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$.

The fixed point set of $\hat{\tau}$ is $\mathbb{R}^2$, so $\Omega \cap \Delta'$ corresponds to $D \cap \mathbb{R}^2$. If $\Omega$ has a fixed point, then $D$ must contain the origin, and thus $D \cap \mathbb{R}^2$ is a connected set (which is logarithmically convex). On the other hand, if $\Omega$ does not have a fixed point, then $D$ must be disjoint from one of the coordinate axes, say $z_1$. Thus $D \cap \mathbb{R}^2$ consists of two (logarithmically convex) open sets, which are symmetric under the map $(x_1, x_2) \mapsto (x_1, -x_2)$. This completes the proof.

**Question 12.** *Can a Hénon map have exactly one rotation domain? Or can the existence of one rotation domain cause the existence of another?*

The essence of this question is whether there might be some phenomenon for rotation domains in the complex domain which is reminiscent of the island chains for twist maps: one twist causes the existence of others.

§**7. Herman rings for dissipative maps?** For dissipative maps, $\text{int}(K^+)$ can be unbounded, and it is not known whether every Fatou component $\Omega \subset \text{int}(K^+)$ is necessarily periodic. Let us suppose that $f(\Omega) = \Omega$. We say that $\Omega$ is *recurrent* if there is a point $z_0 \in \Omega$ whose $\omega$-limit set contains a point $z_1 \in \Omega$. In other words, there is a point $z_0 \in \Omega$ so that all the forward iterates do not converge to $\partial\Omega$. In the dissipative case, it is shown in [BS2] that every recurrent, periodic Fatou component $\Omega$ is a basin; there are 3 possibilities:
  (1) $\Omega$ is the basin of an attracting fixed point;
  (2) $\Omega = \mathcal{B}(\mathcal{S})$ is the basin of a Siegel disk; that is, $\varphi : \{|\zeta| < 1\} \to \mathcal{S} \subset \mathbb{C}^2$ is a holomorphic imbedding, and $f|_{\mathcal{S}}$ is conjugate to an irrational rotation;
  (3) $\Omega = \mathcal{B}(\mathcal{A})$ is the basin of an annulus $\mathcal{A} \cong \{r_1 < |\zeta| < r_2\}$, and $f|_{\mathcal{A}}$ is conjugate to an irrational rotation.
We may linearize maps (4.3) at the origin with suitable eigenvalues $\lambda_j$, $j = 1, 2$, and show that cases (1) and (2) can occur. However the possibility of case (3) remains an unanswered question. In case (3), we have the intriguing situation that $\mathcal{B}(\mathcal{A})$ is biholomorphically equivalent to the product $\{r_1 < |\zeta| < r_2\} \times \mathbb{C}$, yet it must also be polynomially convex.

Let us suppose that $\Omega$ is a rank 2 rotation domain for the case $|\delta| = 1$, and let $D$ be its Reinhardt model. If $\Omega$ has no fixed point, then $D \subset \mathbb{C}^2_{z,w}$ must intersect one of the axes, say $\{w = 0\}$, so $A := D \cap \{w = 0\}$ must be an annulus, which corresponds (biholomorphically) to an invariant annulus $\mathcal{A} \subset \Omega$.

**Question 13.** *What happens to the invariant annulus $\mathcal{A}$ inside an exotic rotation domain if we perturb the map slightly to become dissipative? Can it "persist", or does it always "disappear"?*

The point here is that if $\mathcal{A}$ does not disappear, then we would have an example of case (3).

§**8. Rational surface automorphisms preserving a 2-form.** The rational surface maps with invariant 2-form have been classified by Diller and Lin [DL]. Rational surface automorphisms with invariant 2-forms have been given by [M2], [BK2], [D], and [Ue]. The polynomial

$$\chi_n(t) = t^n(t^3 - t - 1) + t^3 + t^2 - 1 \tag{8.1}$$

is related to the family of rational surface automorphisms (0.2). It may be factored as $\chi_n = C_n S_n$, where $C_n$ is a product of cyclotomic factors (all of whose zeroes are roots of unity), and $S_n$ is a Salem polynomial, which means that it has two real roots $\lambda$ and $\lambda^{-1}$, with $\lambda > 1$, and all other roots have modulus 1.

**Theorem 8.1 ([BK1]).** *There is a blowup $\pi : X \to \mathbb{P}^2$ at $n+3$ points such that $f_{a,b}$ lifts to an automorphism of $X$ if and only if $f_{a,b}^n(-a, 0) = (-b, -a)$. If in addition $n \geq 7$, then the entropy of the automorphism $f_{a,b}$ is $\log\lambda$, where $\lambda > 1$ is the largest real root of $S_n$.*

Some of the automorphisms given in this Theorem have invariant curves and some do not. The curve $\{y = x^3\}$ is a cubic with a cusp at infinity. We will let $\mathcal{C}$ denote the image of this cubic under a linear automorphism of $\mathbb{P}^2$, and we let $\eta = dx \wedge dy/p(x, y)$ denote a 2-form with a simple pole along $\mathcal{C}$. McMullen [M2] (see also [BK2] for a different approach) shows that among the maps $f_{a,b}$, there is a map with invariant curve for every root of $S_n$.

**Theorem 8.2 ([M2]).** *If $\delta$ is a root of $S_n$, then there is an automorphism $f_{a,b}$ which leaves the cubic curve $\mathcal{C}$ invariant and which satisfies $f_{a,b}^*\eta = \delta\eta$.*

Let us change notation and write $f_\delta$ for the map in Theorem 8.2. We note that for each $n \geq 7$, all but two of the roots of $S_n$ have modulus 1, which means that all but two of the $f_\delta$ are conservative automorphisms.
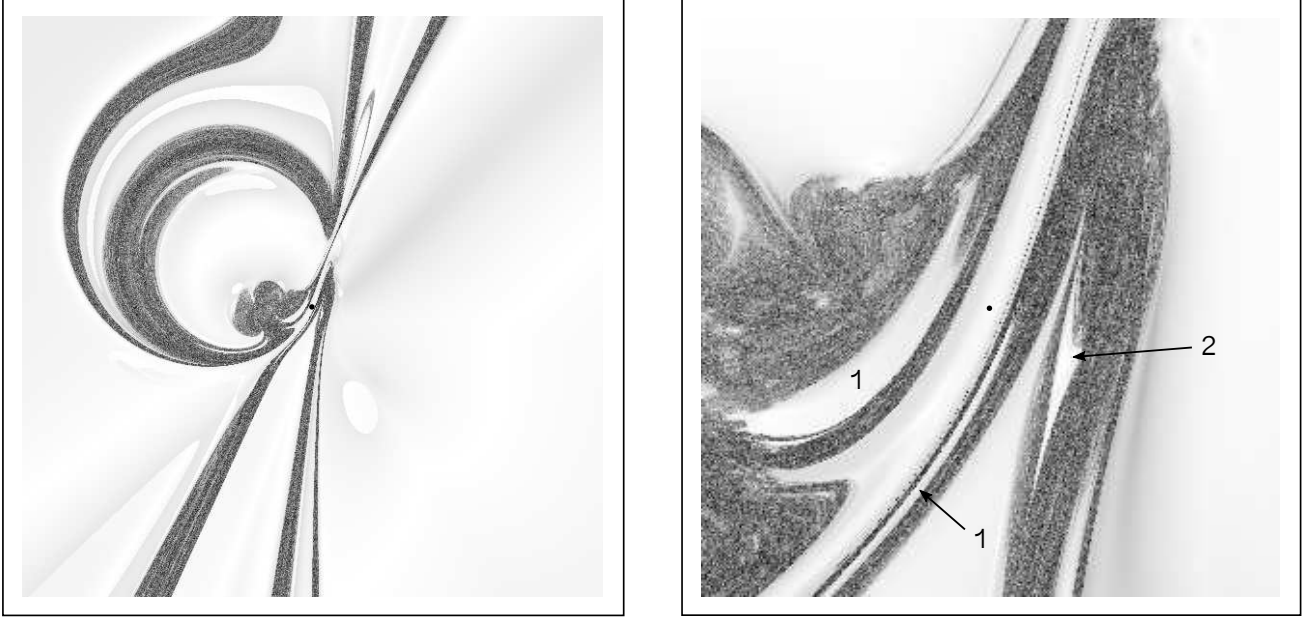


Figure 6. Lyapunov exponent of map (8.2) with $\alpha \sim .5695$, $\beta \sim .3977$.
Slice by conjugate diagonal $\Delta'$ (left); 'dot' is fixed point $p_2$. $8\times$ zoom about $p_2$ (right).

**Theorem 8.3 (Ushiki).** *The maps $f_\delta$ with $|\delta| = 1$ are conjugate to*

$$(x, y) \mapsto \left(y, \frac{y + \alpha}{x + i\beta} + i\beta\right) \tag{8.2}$$

*with $\alpha, \beta \in \mathbb{R}$. This map is reversible by the involution $\tau(x, y) = (\bar{y}, \bar{x})$, and $\mathcal{C} \cap \mathrm{Fix}(\tau)$ is a real curve.*

Now let $\mathcal{F}^+$ denote the Fatou set, and let $\Omega$ be a Fatou component such that $f(\Omega) = \Omega$. We again consider the set of limits, $\mathcal{G}$, as in §1, and we find that the connected component of the identity $\mathcal{G}_0$ is a torus $\mathbb{T}^\rho$ with $\rho = 1$ or 2 (see [BK3] for details). Thus all periodic Fatou components are rotation domains of rank either 1 or 2. In particular, we see that $\mathcal{F}^- = \mathcal{F}^+$, so in the case of a conservative automorphism we will denote it simply as $\mathcal{F}$.

The multipliers of $Df_\delta$ at the cusp point of $\mathcal{C}$ are $\delta^{-2}$, $\delta^{-3}$, so when $|\delta| = 1$ each $f_\delta$ has a rotation domain of rank 1. It was shown in [M2] and in [BK2] that in many cases the maps $f_\delta$ also have rank 2 rotation domains. We ask whether this always happens:

**Question 14.** *Does $f_\delta$ always have a rotation domain in addition to the one centered at the cusp point? In other words, is the Fatou set for $f_\delta$ always disconnected?*
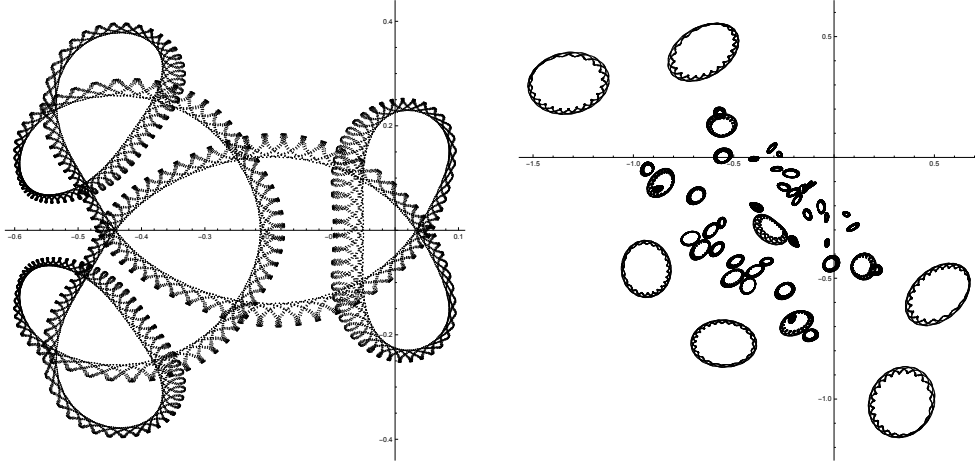
Figure 7. Two orbits from Figure 6. Region '1' (left); region '2' (right).

Let $p_2 \in \mathcal{C}$ denote the other fixed point (not the cusp) of $f$. The eigenvalues of $Df$ at $p_2$ are $\delta$ and $\delta^{-n}$, so there is a resonance. By Pöschel [P] there are two invariant complex disks passing through $p_2$. Ushiki's computer work suggests more: an answer of "yes" the following:

**Question 15.** *Does the rotation domain containing the cusp contain the whole curve $\mathcal{C}$? In particular, is $f_\delta$ linearizable at $p_2$?*

An immediate consequence of "yes" would be that $f_\delta$ can have no wandering Fatou component, since the invariant volume form $\eta \wedge \bar{\eta}$ is bounded outside a neighborhood of $\mathcal{C}$. Some analogous resonant fixed points for (other) rational surface automorphisms were shown in [BK3] to be linearizable.

**Further Questions.**    All of the questions that we have asked about Fatou components of Hénon maps apply equally to surface automorphisms. In order to avoid duplication, we do not re-state them here. We note moreover that these same questions are interesting also in the case where $f$ is merely birational (invertible but not required to be everywhere regular) and conservative.

*How can we draw a computer picture of the Fatou set of a conservative surface automorphism?* In the case of Hénon maps, it suffices to draw the set $K^+$ or $K$ where orbits are bounded, but there seems to be nothing analogous for other automorphisms. The other useful object, the Green function $G^+$, exists only because there is a super-attracting invariant curve (the line at infinity). One alternate approach is to consider the maximal Lyapunov exponent of a point $p \in X$:

$$\Lambda(p) = \limsup_{n \to \pm\infty} \frac{1}{|n|} \log ||Df^n(p)||.$$

Here $||\cdot||$ denotes the operator norm of $Df$ with respect to any norm on the tangent space of $X$; the limit is independent of choice of norm.

If $p$ belongs to the Fatou set, then $Df^n$ is bounded in a neighborhood of $p$ for all $n \geq 0$, and thus $\Lambda(p) = 0$. The converse, however, is not always true. For example, consider the automorphisms $f_{a,b}$ as in Theorem 8.1 for $n = 6$ (that is, $X$ is obtained by blowing up $\mathbb{P}^2$ at 9 points). These automorphisms have the property that $\deg(f^n) \sim n^2$, and thus the derivative of $f^n$ grows quadratically. Thus $\Lambda = 0$, and $f_{a,b}$ has zero entropy. On the other hand, these examples have an invariant fibration on which $f$ acts as a "twist", so $\mathcal{F} = \emptyset$.
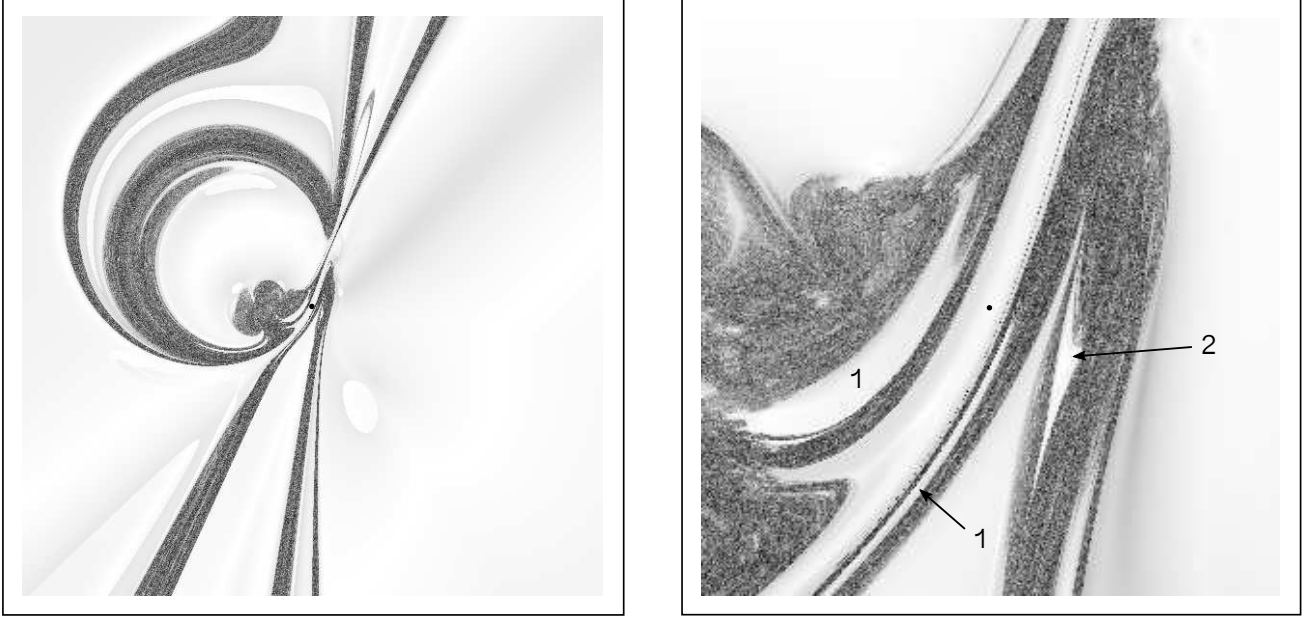


Figure 6. Lyapunov exponent of map (8.2) with $\alpha \sim .5695$, $\beta \sim .3977$.
Slice by conjugate diagonal $\Delta'$ (left); 'dot' is fixed point $p_2$. $8\times$ zoom about $p_2$ (right).

Let us restrict our attention now to the case of an automorphism with positive entropy $\log \lambda > 0$. In this case, there are positive closed currents $T^{\pm}$ which are invariant in the sense that $f^* T^{\pm} = \lambda^{\pm 1} T^{\pm}$. Further, the wedge product of these currents defines a measure $\mu := T^+ \wedge T^-$, which is the unique measure of maximal entropy. The reader is referred to Cantat [C] for a presentation of this material. In particular, (see [C, Theorem 7.5]), we have:

**Theorem 8.4 [Dinh-Sibony, Moncet, Ueda].** *For an automorphism of positive entropy, $\mathcal{F} := \mathcal{F}^+ \cap \mathcal{F}^-$ is the complement of the support of $T^+ + T^-$, modulo periodic curves.*

We would now like to give another description of the Fatou set in terms of the support of $T^+ + T^-$. There are two Lyapunov exponents $\lambda_1 \geq \lambda_2$ with respect to $\mu$. The larger one is given by

$$\lambda_1 = \lim_{n \to \infty} \frac{1}{n} \int \log ||Df^n(p)|| \, \mu(p)$$

In the conservative case, we have $\lambda_1 + \lambda_2 = 0$, so the larger exponent determines the smaller one. In general (see [Duj2]) we have

$$\lambda_1 \geq \frac{\log \lambda}{2} > 0 > -\frac{\log \lambda}{2} \geq \lambda_2$$

Thus $\mu$ is a hyperbolic measure, and by Pesin Theory, there are stable/unstable manifolds $W^{s/u}(x)$ through $\mu$ a.e. point.

**Theorem 8.5 [Dujardin].** *For $\mu$ a.e. $x$, the Pesin manifold $W^{s/u}(x)$ is dense in the support of $T^{+/-}$.*

*Proof.* It suffices to consider $T^+$. This is generated by submanifolds in the "strongly laminar" sense of [Duj1], which also gives a definition of "being in the same leaf", and shows that this is an equivalence relation on the submanifolds of $T^+$. By [Duj1, Corollary 5.8] the space of leaves is ergodic with respect to the trace measure of $T^+$: for each measurable subset of leaves, either it or its complement has measure zero. Now let $\{U_i\}$ be a neighborhood basis for the support of $T^+$. It follows that the set of leaves that do not intersect $U_i$ has trace measure zero. Taking a countable union of sets of zero measure, we see that the set of leaves which intersect every $U_i$ has full measure. Now consider the leaves which are Pesin manifolds. This is a measurable, invariant set, which must have full trace measure, by the product structure of $\mu$. Thus almost every Pesin manifold intersects every $U_i$.

**Theorem 8.6.** *Let $U$ be an open set such that $\Lambda(p) < \frac{\log \lambda}{2}$ for all $p \in U$. Then $U$ is disjoint from the support of $T^+ + T^-$.*

*Proof.* It suffices to consider the case where $U$ intersects the support of $T^+$. By Theorem 8.5, there is a Pesin manifold $W^s(x)$ and a point $p \in W^s(x) \cap U$. If $t \in T_p W^s(x)$ is a nonzero tangent vector, then $\log |Df^n(p)t|$ decreases like $n\lambda_2(x)$. Since $f$ is conservative, it follows that $\log ||Df^n(p)||$ grows like $n\lambda_1(x)$. We conclude that $\Lambda(p) \geq \frac{\log \lambda}{2}$.

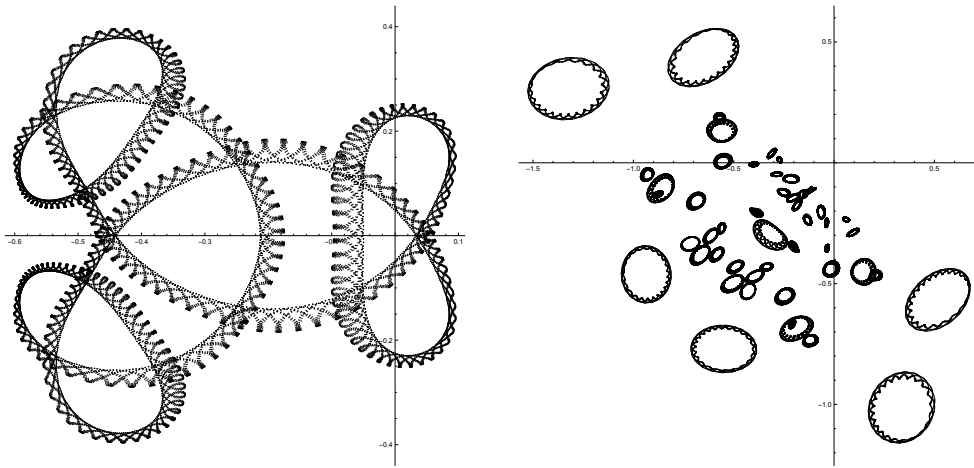Theorems 8.4 and 8.6 justify computing the set $\{\Lambda(p) > 0\}$ as the complement of $\mathcal{F}$.



Figure 7. Two orbits from Figure 6. Region '1' (left); region '2' (right).

**Figures 6 and 7.** For Figures 6 and 7, we have used a map corresponding to $n = 7$ in Theorem 8.1. The Salem polynomial $S_7$ has degree 10. Eight of the roots of $S_7$ have modulus 1, and we may pair them up $\{\delta_j, \bar{\delta}_j\}$, $1 \leq j \leq 4$, where $f_{\bar{\delta}_j}$ is conjugate to the inverse of $f_{\delta_j}$. This gives four essentially distinct conservative maps. When $n = 7$,

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the automorphism $f_\delta$ is obtained from $\mathbb{P}^2$ by blowing up 10 points, and the entropy is $\log 1.17628$, which was shown in [M2] to be the minimum possible for an automorphism of a compact, complex surface. Figure 6 shows the map $f_\delta$ with $\delta \sim -0.2344 + 0.9721i$. The left hand image shows the slice by conjugate diagonal points $p_2 + (\zeta, \bar{\zeta})$, where $p_2$ is the resonant fixed point of Question 15, and $-2.4 \le \Re(\zeta), \Im(\zeta) \le 2.4$. The right hand frame of Figure 6 is a detail, centered at $p_2$.

The coloring of Figures 6 and 8 is opposite from Figures 1 and 2, where the Fatou set was the interior of the black; here the white regions correspond to the Fatou set. Or perhaps more precisely, Figures 1 and 2 show the basin of infinity in shades of white/gray, while Figures 6 and 8 use shades of gray to show the Julia set. The region '1' (whose slice by the conjugate diagonal appears to contain two components) was brought to our attention by Ushiki. An orbit from '1' is given on the left hand frame of Figure 7. It suggests that '1' is part of a connected Fatou component $\Omega_1$ of rank 2. If this is so, then $\Omega_1$ must be "exotic" since it cannot contain a fixed point. The reason for this is the fixed points of $f_\delta$ are $p_1$ and $p_2$ which have been discussed earlier. Their eigenvalues are not multiplicatively independent, and thus they cannot be contained in a rotation domain of rank 2. The right hand frame of Figure 7 indicates that '2' is contained in a rank 2 Fatou component of period 50. The number 50 is seen because an orbit of $f^2$ has visibly fewer components, and an orbit of $f^{10}$ appears to consist of 5 components.
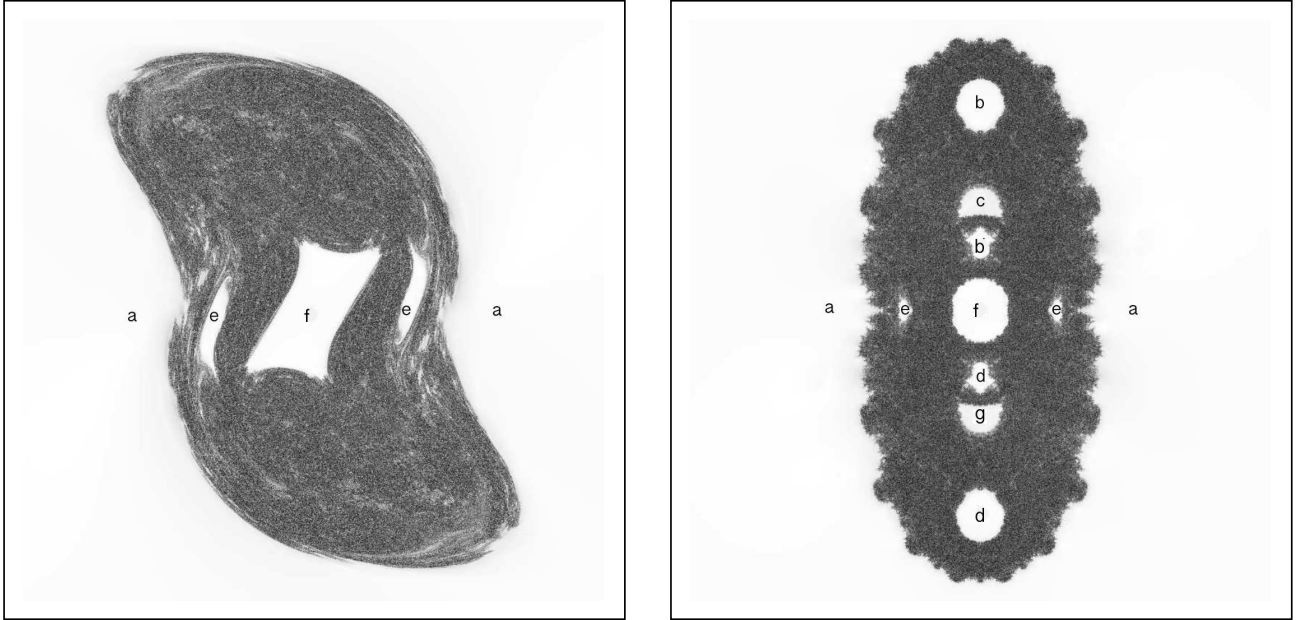


Figure 8. Lyapunov exponent for map $(x, y) \mapsto (y, \beta(\sqrt{2}y + 1/y) - \beta^2 x)$
$\beta \sim 0.4174 + .9086i$ satisfies $\chi_{4,1}(\beta^2) = 0$
Slice by conjugate diagonal $\Delta'$ (left) and by complex diagonal (right).

Another family of birational maps is given by

$$g_{c,\beta}(x, y) = \left(y, \beta(c\,y + 1/y) - \beta^2 x\right), \quad \beta^2 = \delta \qquad (8.3)$$

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This map is reversible under $\tau(x, y) = (\bar{y}, \bar{x})$ if $c \in \mathbb{R}$ and $|\delta| = 1$ and preserves the 2-form $dx \wedge dy$. Associated with (8.3) is the family of polynomials

$$\chi_{n,m}(t) = 2 \sum_{k=0}^{m} t^{kn} - \sum_{k=0}^{mn} t^k \tag{8.4}$$

As was the case with (8.1), we may factor $\chi_{n,m} = C_{n,m} S_{n,m}$, where $C_{n,m}$ is a product of cyclic polynomials, and $S_{n,m}$ is a Salem polynomial. The roots of this Salem polynomial give us automorphisms of the form (8.3).

**Theorem 8.7 ([BK3]).** *Suppose that $n \geq 4$, $m \geq 1$ or $n = 3$, $m \geq 2$. Let $\delta = \beta^2$ be a root of $S_{n,m}$, and let $c = 2\cos(j\pi/n)$, where $(j, n) = 1$. Then there is a blowup $\pi : X \to \mathbb{P}^2$ such that $g_{c,\beta}$ induces an automorphism $g_X$ of $X$, and the entropy of $g_X$ is $\log(\lambda_{n,m}) > 0$, where $\lambda_{n,m}$ is the largest root of $S_{n,m}$.*

The construction of $g_X$ differs from the construction for the maps (0.2) and (8.2) because the blowups to make the space $X$ are iterated to height 3. This is because $dx \wedge dy$ has a pole of order 3 along the line at infinity. However, by a general result of [DL], there is a birational conjugacy under which the invariant 2-form of (8.3) will be transformed to another 2-form with only simple poles.
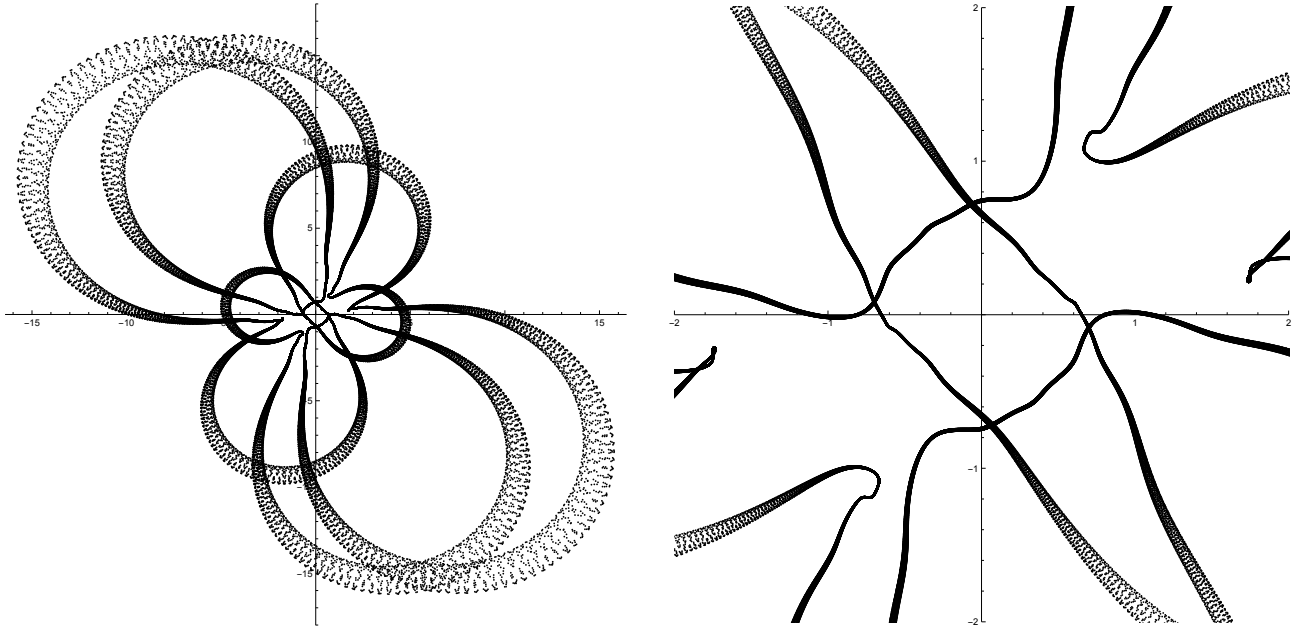


Figure 9. Projection of an orbit of a point from region 'e' in Figure 8 (left); detail (right).

**Figures 8 and 9.** Figure 8 gives a map from Theorem 8.7 with $n = 4$, $m = 1$, $j = 1$. If $L \subset X$ denotes the strict transform of the line at infinity, then $L$ is invariant under $g_X$, and $g_X$ acts as a rotation of period 4 on $L$. There is a rank 1 rotation domain $\Omega_L \supset L$, and the induced group is $\mathcal{G}(\Omega_L) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}$. The global nature of $\Omega_L$ is discussed in [BK3]. Figure 8 shows two slices of the Fatou set (white). On the left are the points for

a conjugate diagonal slice $\{(\zeta, \bar{\zeta}), -1.4 < \Re(\zeta), \Im(\zeta) < 1.4\}$, and on the right a complex diagonal slice $\{(\zeta, \zeta), -1.8 < \Re(\zeta), \Im(\zeta) < 1.8\}$. The regions marked 'a' correspond to $\Omega_L$. The two fixed points in $\mathbb{C}^2$ were shown in [BK3] to be rotational of rank 2, and these are contained in the components 'b' and 'd'. The computer evidence suggests that the two regions 'b' are connected inside $\mathbb{C}^2$ and are disjoint from the two components 'd'. The components 'c' and 'g' appear to be rank 2 rotation domains of period 6, and component 'f' appears to be rank 2 with period 5. The projection of 220000 points of an orbit from 'e' is shown on the left half of Figure 9. The detail on the right hand side lends evidence that the closure of the orbit is connected. The component 'e' was shown to us by Ushiki as a possible exotic rank 2 rotation domain. The reason why it might be exotic (why it cannot contain a fixed point) is that there are only four fixed points. Two of them are contained in $\Omega_L$ and one in each domain 'b' and 'd'. However, $\Omega_L$ has rank 1 and cannot intersect 'e'. Further 'e' is invariant under complex conjugation, so if 'e' intersects component 'b', then it must intersect 'd', which contradicts the apparent disjointness of 'b' and 'd'.

**Appendix. Compact surface automorphisms.** We have considered only rational surface automorphisms in our discussions above. We have not considered other surfaces because of Theorem 10.1 of [C], which we summarize as follows:

**Theorem ([C], [Nag]).** *Suppose that $X$ is a compact complex surface and that $F \in Aut(X)$ has positive entropy. Then there are three possibilities for $X$:*
 *(i) $X = \mathbb{C}^2/\mathcal{L}$ is a complex torus.*
 *(ii) $X$ is a $K3$ surface (or certain quotients).*
*(iii) $X$ is a rational surface. In this case, $\pi : X \to \mathbb{P}^2$ is obtained from $\mathbb{P}^2$ by blowing up.*

In case of the torus, every automorphism $F$ lifts to an affine map. It must preserve the lattice, so its determinant must be $\pm 1$. If $F$ has positive entropy, then the eigenvalues must be $|\lambda_1| < 1 < |\lambda_2|$. Thus it is hyperbolic, and it follows in this case that $\mathcal{F} = \emptyset$.

In the case of a $K3$ surface, McMullen [M1] has shown that rotation domains can exist. He constructs lattices with a lattice automorphism which satisfies the conditions of the Torelli Theorem. The Torelli Theorem gives the existence of a $K3$ surface $X$ with an automorphism $F$ which will have the given behavior $F^*$ on the cohomology lattice. Knowledge of $F^*$ and the holomorphic Lefschetz Index Formula give the existence of a unique fixed point as well as the values of the eigenvalues $\lambda_1$, $\lambda_2$ of $DF$. He shows that the eigenvalues have modulus 1 and are multiplicatively independent and thus are "suitably irrational" for Siegel's Theorem. This automorphism $F$ thus has a rank 2 rotation domain with a fixed point.

As was noted in [M1], this $K3$ surface $X$ is necessarily non-algebraic and cannot be exhibited explicitly. For if $X$ is an algebraic $K3$ surface, then an automorphism $F \in Aut(X)$ cannot have a rank 2 rotation domain with a fixed point since the jacobian $\delta = \lambda_1 \lambda_2$ would be a root of unity.

**Question 16.** *Can an automorphism of an algebraic $K3$ surface have a rotation domain? Equivalently, can it have a nonempty Fatou set?*

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