J. Korean Math. Soc. **40** (2003), No. 3, pp. 373–390

# ON THE DYNAMICS OF BIRATIONAL MAPPINGS OF THE PLANE

Eric Bedford

Abstract. In this paper we discuss how the dynamics of certain birational maps of the real plane may be studied using complex methods.

## 1. Introduction

A map of the plane $f : \mathbf{R}^2 \to \mathbf{R}^2$ is rational if its coordinate functions are given by rational functions. And it is birational if there is a rational mapping $g : \mathbf{R}^2 \to \mathbf{R}^2$ such that the compositions $f \circ g$ and $g \circ f$ are the identity transformation on a dense open subset of $\mathbf{R}^2$. While $f$ is invertible as a meromorphic map, it can fail to be a homeomorphism for two reasons. First, there can be exceptional curves that map to points under $f$, i.e. curves on which $f$ is constant, wherever it is defined. We write the set of exceptional curves as $\mathcal{E}(f)$. Second, there can be points where $f$ is indeterminate. We write the indeterminacy set as $\mathcal{I}(f)$. A point $p$ belongs to $\mathcal{I}(f)$ if the fraction representing one of the coordinates of $f$ has the form $\frac{0}{0}$ at $p$, even after common factors have been removed from numerator and denominator. Geometrically, the point $p \in \mathcal{I}(f)$ is blown up to a curve, which is an exceptional curve for $f^{-1}$.

Let $X$ be a compact, complex surface which is a complexification of $\mathbf{R}^2$. Then $f : \mathbf{R}^2 \to \mathbf{R}^2$ extends to a birational map $F : X \to X$. In Section 2 we discuss the paper of Diller-Favre [11] which develops the approach of studying $F : X \to X$ in terms of invariant currents. Positive, closed (1,1)-currents are a generalization of complex curves (which have real dimension 2). Thus the approach of studying the action of $F^{n*}$ on (1,1)-currents may be viewed as a generalization of the action of $F$ on complex curves.

Received August 30, 2002.
2000 Mathematics Subject Classification: 32H50.
Key words and phrases: dynamics, birational mappings.

Why would it be natural to start by mapping complex curves? For one thing, whenever the orbit of a point encounters a point of indeterminacy, it gets blown up to a curve, so we necessarily find ourselves iterating curves. For another thing, it is well known in the case of diffeomorphisms that volume growth, cohomology growth, and entropy are closely related. For instance, if $\rho$ denotes the spectral radius of $F^*$ on $H^{1,1}$, then by Friedland [13], $\log \rho$ is an upper bound for the entropy of $F$, which is in turn an upper bound for the entropy of $f$.

In Section 3, we discuss a joint paper with Diller (see [4]) which gives a sufficient condition for the existence of an invariant measure under $F$. The main result is the following:

Theorem 1.1 [4]. *Let $F : X \to X$ be a bimeromorphic mapping which satisfies conditions* (3), (6) *and* (7) *below. Then there is an invariant measure $\mu$ which is mixing. Further, one of the Lyapunov exponents of $\mu$ is strictly positive, and the other is strictly negative.*

In Section 4 we discuss two families of real, birational maps. The first of them is:

$$f_a(x,y) = (y\frac{x+a}{x-1}, x+a-1), \tag{1}$$

which have been studied in a series of papers by Abarenkova et al.; see [1]-[3] and the references therein. If $f_a$ is given as in (1), then the exceptional curves are $\mathcal{E} = \{x=1\} \cup \{x=-a\}$, a pair of vertical lines which are mapped to points: $f(x=1) = (\infty, a)$ and $f(x=-a) = (0,-1)$. The indeterminacy locus is $I(f) = \{(-a,\infty),(1,0)\}$, and $f$ blows them up to a pair of horizontal lines: $f(-a,\infty) = \{y=-1\}$ and $f(1,0) = \{y=a\}$.

In the complex case, the current of integration over a compact, complex curve defines a positive, closed current. The real analog to the approach discussed in Section 2 is to consider the action of $f$ on curves rather than points. This is the approach taken in the paper [5], where the main result is:

Theorem 1.2 [5]. *If $a < 0$, $a \neq -1$, then* (3), (6) *and* (7) *hold. The support of the measure $\mu$ from Theorem 1.1 is contained in $\mathbf{R}^2$ and coincides with the nonwandering set of $f$. Further, the restriction of $f$ to the nonwandering set is* (*essentially*) *conjugate to the golden mean shift.*

We also study the family of birational maps given by the formula:

$$g_{\alpha,\beta}(x,y) = (1-x+\frac{x}{y}, \alpha+y-\frac{y}{x}+\beta(1-x+\frac{x}{y})). \tag{2}$$

The behavior of this map is more complicated. However, we are able to show:

THEOREM 1.3 [6]. *If $\beta > 0$ and $\alpha < -1$, then the support of the measure given by Theorem 1.1 is contained in* $\mathbf{R}^2$. *Further, the entropy of $g_{\alpha,\beta}$ is equal to* $\log \rho$, *where $\rho \sim 2.1479$ is the largest root of the equation*

$$\rho^3 - \rho^2 - 2\rho - 1 = 0.$$

We might compare families (1) and (2) with the Hénon family of diffeomorphisms of $\mathbf{R}^2$:

$$h_{a,b}(x, y) = (x^2 + a - by, x).$$

It is known (see [9], [15], [16]) that for certain values of the parameters $(a, b)$, the mapping $h_{a,b}$ generates a horseshoe. The horseshoe mappings are those for which the entropy of $h_{a,b}$ is maximal (and equal to $\log 2$), and these maps seem to play a role of special importance within the family $\{h_{a,b}\}$. In joint work with John Smillie (see [7], [8]), we have studied the real mappings of maximal entropy, which include both the horseshoe mappings and the maps that are limits of horseshoes. The fact that these mappings of maximal entropy have such rich structure indicates that the maximal entropy mappings may be a good starting place for understanding the families $\{f_a\}$ and $\{g_{\alpha,\beta}\}$.

## 2. Basic properties: invariant currents

A densely defined map $F : X \to X$ is *meromorphic* if there is a 2-dimensional subvariety $\Gamma \subset X \times X$ (the graph of $F$) and a pair of holomorphic projections $\pi_1 : \Gamma \to X$ and $\pi_2 : \Gamma \to X$ such that $\pi_1$ is proper and generically one-to-one, and if $F = \pi_2 \circ \pi_1^{-1}$ holds on a dense subset of $X$. $F$ is bimeromorphic if both $\pi_1$ and $\pi_2$ may be taken to be generically one-to-one. We let $\mathcal{E}_j$ denote the exceptional set for $\pi_j$, i.e. $\mathcal{E}_j$ is the union of all pure one-dimensional varieties which are mapped to points under $\pi_j$. Without loss of generality, we may assume that $\mathcal{E}_1 \cap \mathcal{E}_2$ is a finite set. It follows that $\mathcal{I}(f) = \pi_1(\mathcal{E}_1)$ is the set of indeterminacy, and $\mathcal{E}(F) = \pi_1(\mathcal{E}_2)$ is the exceptional set for $F$. We may consider $F^{-1}$ by interchanging the roles of $\pi_1$ and $\pi_2$.

If $\alpha$ is a smooth $(p, q)$ form on $X$, then $\pi_2^*\alpha$ is a smooth form on $\Gamma$. By duality, we may consider this to be a current on $\Gamma$, and we may push it forward under $\pi_1$. Thus we have a mapping $F^*\alpha = \pi_{1*} \circ \pi_2^*(\alpha)$, which induces a mapping on cohomology:

$$F^* : H^{p,q}(X) \to H^{p,q}(X).$$

For the study of $F$, we work only with the case $p = q = 1$. In general, it can be shown that if $\alpha$ is a positive, closed current on $X$, then there is a well-defined pull-back $f^*\alpha$, which is again a positive, closed current.

We consider the condition

$$\mathcal{I}(f^{-1}) \cap f^{-n}(\mathcal{I}(f)) = \emptyset \quad \text{for all } n \geq 0. \tag{3}$$

If (3) holds, then $\mathcal{I}(f^n) = \bigcup_{n\geq 0} f^{-n}\mathcal{I}(f)$. Further, (3) is equivalent to the condition that

$$\mathcal{I}(f) \cap f^n(\mathcal{I}(f^{-1})) = \emptyset \quad \text{for all } n \geq 0.$$

Thus we have

$$\bigcup_{n\geq 0} f^{-n}(\mathcal{I}(f)) \cap \bigcup_{n\geq 0} f^n(\mathcal{I}(f^{-1})) = \emptyset.$$

One consequence of (3) is that for all $x \in X$, either the forward orbit $\{x, fx, f^2x, \dots\}$ or the backward orbit $\{x, f^{-1}x, f^{-2}x, \dots\}$ is defined in the "classical" sense, i.e. the orbit does not encounter a point of indeterminacy.

The interest in this condition is that if $F : X \to X$ be a bimeromorphic map which satisfies (3), then the passage to the map on cohomology is natural: $(F^*)^n = (F^n)^*$.

Theorem 2.1 [11]. *If $F : X \to X$ is a bimeromorphic map, then there are a compact, complex surface $\hat{X}$ and a bimeromorphic map $h : \hat{X} \to X$ such that $\hat{F} = h^{-1} \circ F \circ h : \hat{X} \to \hat{X}$ satisfies* (3).

Replacing $X$ by $\hat{X}$, we may assume that (3) holds. Let $\rho$ denote the spectral radius of $F^*$ on $H^{1,1}$. The spectral radius is the same for any $\hat{X}$ satisfying (3).

Theorem 2.2 [11]. *If $\rho > 1$, then there is a cohomology class $\omega^+$ (unique up to scalar multiple) such that $F^*\omega^+ = \rho\omega^+$. Further, there is an essentially unique, positive, closed current $\mu^+$ such that $F^*\mu^+ = \rho\mu^+$.*

The invariant cohomology class $\omega^+$ is easy to find. How do we get from $\omega^+$ to $\mu^+$? Since $\rho^{-1}F^*\omega^+$ is the same cohomology class as $\omega^+$, it follows that there is a function $\gamma^+$ on $X$ such that

$$\rho^{-1}F^*\omega^+ = \omega^+ + dd^c\gamma^+. \tag{4}$$

In fact, $\rho^{-n}F^{*n}\omega^+$ converges to $\mu^+$. Thus if we apply higher powers of $\rho^{-1}F^*$ to (4), we end up with $\mu^+ = \omega^+ + dd^c g^+$, where

$$g^+ = \sum_{n\geq 0} \frac{\gamma^+ \circ F^n}{\rho^n}. \tag{5}$$

## 3. Invariant measure

We consider $F : X \to X$ such that condition (3) and

$$\rho(F^*) > 1 \tag{6}$$

hold. Since $F_* = (F^{-1})^*$ is dual to $F^*$ under the intersection pairing on $H^{1,1}(X)$, we have that $\rho(F^*) = \rho(F^{-1*})$. Thus we have an invariant cohomology class $\omega^-$ and an invariant (1,1)-current $\mu^-$ for $F^{-1}$. We would like to obtain an invariant measure $\mu$. Note that if $\mu$ is to be an invariant measure, it must put no mass on any point of indeterminacy, and it must put no mass on the exceptional set.

If we could define a wedge product $\mu := \mu^+ \wedge \mu^-$, there is a very tempting (formal) identity:

$$F^*(\mu) = F^*(\mu^+ \wedge \mu^-) = F^*(\mu^+) \wedge F^*(\mu^-) = \rho\mu^+ \wedge \rho^{-1}\mu^- = \mu.$$

We will use the following condition which is stronger than (3):

$$\sum_{n\geq 0} \frac{\log(\operatorname{dist}(F^{-n}\mathcal{I}(F), \mathcal{I}(F^{-1})))}{\rho^n} > -\infty. \tag{7}$$

This has an important connection with the potential $g^+$ of the invariant current $\mu^+$:

Theorem 2.1 [10]. *Condition* (7) *holds if and only if* $g^+(p) > -\infty$ *for all* $p \in \mathcal{I}(F^{-1})$.

Further, (7) is equivalent to the corresponding statement with $F$ replaced by $F^{-1}$ (see [10]). Favre [12] has given an example of a mapping which does not satisfy (7); we give some variants of this in Example 5 of Section 5.

Theorem 2.2 [10]. *If* (7) *holds, then* $g^+$ *is continuous on* $X - \overline{\bigcup_{n\geq 0} F^{-n}\mathcal{I}(F)}$.

It is instructive for us to consider also the condition

$$\overline{\bigcup_{n\geq 0} F^{-n}\mathcal{I}(F)} \cap \overline{\bigcup_{n\geq 0} F^{n}\mathcal{I}(F^{-1})} = \emptyset. \tag{8}$$

Since $\rho > 1$, and since the distance function is bounded on $X$, it follows that (8) $\Rightarrow$ (7). Applying Theorem 2.2 in the case where (8) holds, we see that every $p \in X$ is contained in a neighborhood where either $g^+$ or $g^-$ is continuous.

If $g^+$ is a continuous function for which $dd^c g^+$ is essentially positive, then we may define the wedge product with a positive, closed current $T$, which we write as $\mu = dd^c g^+ \wedge T$. This is done as follows. If $\varphi$ is a test function, we define

$$\langle \mu, \varphi \rangle = \langle (dd^c g^+ \wedge T), \varphi \rangle = \int g^+ \, dd^c \varphi \wedge T,$$

where the far right hand side defines the left. Note that if $g^+$ is smooth, this just corresponds to an integration by parts. This defines $\mu$ as a distribution. Now if $dd^c g^+$ is essentially positive, then $\mu$ is an essentially positive distribution and is thus represented by a signed Borel measure.

Since (8) holds, we may assume (locally) that $g^+$ is continuous and we may set $T := \mu^-$. So we may define $dd^c g^+ \wedge T$ and thus the measure $\mu = (\omega^+ + dd^c g^+) \wedge T$.

For the sake of discussion, let us assume that $\omega^+$ is smooth and positive. Thus $F^*\omega^+ \geq 0$, and from (4) we see that $dd^c \gamma^+ \geq -\omega^+$. This means that $dd^c \gamma^+$ is essentially positive. Further, there are constants $A, B, C$ such that

$$A \log \mathrm{dist}(z, \mathcal{I}(F)) - C \leq \gamma^+(z) \leq B \log \mathrm{dist}(z, \mathcal{I}(F)) + C.$$

In [4] we show that if (7) holds, then the potentials $g^\pm$ given by formula (5) are sufficiently tame that we may define $dd^c g^+ \wedge dd^c g^-$, which in turn allows us to define $\mu := \mu^+ \wedge \mu^-$. This gives the analytic definition of the measure $\mu$ that appears in Theorem 1.1. We also have another, perhaps more dynamical, characterization of $\mu$:

Theorem 2.3 [4]. *Let $\eta_1$ and $\eta_2$ be smooth, closed, real $(1,1)$-forms on $X$. Then there is a constant $c$ such that*

$$c\mu = \lim_{m,n\to\infty} \frac{1}{\rho^{m+n}} F^{m*}\eta_1 \wedge (F^{-n})^*\eta_2.$$

## 4. Families of real, birational maps

Let $\mathbf{P}^1 = \mathbf{C} \cup \infty$ denote the Riemann sphere, which is a compact complexification of $\mathbf{R}$. If $f_a$ is defined as in (1), we let $F_a$ denote the extension of $f_a$ to a birational map $F_a : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$. It follows that $(\infty, \infty)$ is a parabolic fixed point. Condition (3) holds unless

$$a = \frac{1}{n} \text{ for some } n \geq 1, \text{ or } a = \frac{n}{n+2} \text{ for some } n \geq 0. \tag{9}$$

We also require $a \neq -1$ because $f_a$ is affine in this case. The cases where (3) fails have been analyzed in [11] and are found to be more tame than when (3) holds. In case (3) holds,

$$\overline{\bigcup_{n\geq 0} F_a^{-n}(\mathcal{I}(F_a))} \cap \overline{\bigcup_{n\geq 0} F_a^{n}(\mathcal{I}(F_a^{-1}))} = (\infty, \infty),$$

and so (8) does not hold. On the other hand, (7) does hold (since $\mathrm{dist}(F^{-n}\mathcal{I}(F), \mathcal{I}(F^{-1}))$ is bounded below for $n \geq 0$). The cohomology group $H^{1,1}(\mathbf{P}^1 \times \mathbf{P}^1)$ has dimension two, and $F_a^*$ may be represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Thus the spectral radius of $F_a^*$ is given by $\rho = \frac{\sqrt{5}+1}{2}$. It follows that the entropy of $f_a$ is no greater than $\log \frac{\sqrt{5}+1}{2}$.

In fact, the entropy of $f_a$ is equal to $\log \frac{\sqrt{5}+1}{2}$ when $a < 0$, $a \neq -1$. One approach to this is to consider the real horizontal $h = \{y = y_0\}$ for fixed $y_0 \leq -1$ and vertical lines $v = \{x = x_0\}$ for $x_0 \leq -1$. The measure $\mu$, described in Theorem 1.1 is also generated by pushing $h$ forward and pulling $v$ back:

Theorem 3.1 [5].

$$\mu = \lim_{m,n\to\infty} \frac{1}{\rho^{n+m}} \sum_{a \in f^m h \cap f^{-n} v} \delta_a,$$

*where $\delta_a$ denotes the point measure supported at $a$.*

The set $f^m h \cap f^{-n} v$ contains an $(\epsilon, n+m)$-separated set with approximately $\rho^{n+m}$ points, and so it follows that the entropy of $\mu$ is at least as large as $\log \rho$. Since $\log \rho$ was an upper bound, it must be equal to the entropy of $\mu$.

**The family** $\{g_{\alpha,\beta}\}$. We will require that $\beta \neq 0$; the case $\beta = 0$ has lower entropy. Let us start with the complexification $X = \mathbf{P}^2$, and let $G_{\alpha,\beta} : \mathbf{P}^2 \to \mathbf{P}^2$ denote the extension of $g_{\alpha,\beta}$ to a birational map of complex projective space $\mathbf{P}^2$. The indeterminacy set is $\mathcal{I}(G) = \{[0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0]\}$. These points are given in homogeneous coordinates on $\mathbf{P}^2$, so $[x : y : 1]$ denotes the point $(x, y) \in \mathbf{C}^2 \subset \mathbf{P}^2$; and $[1 : 0 : 0]$ and $[0 : 1 : 0]$ are the points at infinity at the ends of the $x$- and $y$-axes, respectively. The exceptional set is $\mathcal{E}(G) = X \cup Y \cup \mathcal{C}$, where $X = \{y = 0\}$ is the $x$-axis, $Y = \{x = 0\}$ is the $y$-axis, and $\mathcal{C} = \{(x-1)(y-1) = 1\}$. The condition (3) fails for $G$ for three reasons:

$$G(\mathcal{C}) = (0, \alpha + 1) \in Y, \text{ and } G^2(\mathcal{C}) = [0:1:0] \in \mathcal{I}(G),$$
$$G(Y) = [0:1:0] \in \mathcal{I}(G),$$
$$G(X) = [1:\beta:0] \text{ and } G^2(X) = [1:0:0] \in \mathcal{I}(G).$$

By Theorem 2.3, there is a complex surface $\hat{X}$ such that $\hat{G} : \hat{X} \to \hat{X}$ satisfies (3). In [6] we construct such a space $\hat{X}$ with a projection $\hat{\pi} : \hat{X} \to X$, by performing two blow-ups over $[0:1:0]$ and one blow-up over each of the points $[1:\beta:0]$ and $[1:0:0]$. Write $D_0 = \hat{\pi}^{-1}([0:1:0])$, $E_0 = \hat{\pi}^{-1}([1:\beta:0])$, and $F_0 = \hat{\pi}^{-1}([1:0:0])$. The projection

$$\hat{\pi} : \hat{X} - (D_0 \cup E_0 \cup F_0) \to \mathbf{P}^2 - \{[0:1:0], [1:\beta:0], [1:0:0]\}$$

is biholomorphic, and thus nothing has been changed over any point of $\mathbf{C}^2$.

The indeterminacy set for the new map is $\mathcal{I}(\hat{G}) = \{[0:0:1], \hat{p}\}$, where $\hat{p} \in F_0$, and the exceptional set is now $\mathcal{E}(\hat{G}) = X \cup \mathcal{C}$. We have $\hat{G}(\mathcal{C}) = (0, \alpha + 1)$ as before, but now $\hat{G}(X) = \hat{q} \in E_0$ is a point in the fiber over $[1:\beta:0]$. One effect of the modification is that $Y$ is no longer exceptional, and the indeterminacy at $[0:1:0]$ has disappeared — there is no point of indeterminacy in $D_0$. Condition (3) is now equivalent to the condition that

$$\hat{G}^n(0, \alpha + 1) \neq (0, 0) \text{ and } \hat{G}^n(\hat{q}) \neq \hat{p} \text{ for all } n \geq 0.$$

This is equivalent to the condition that $\beta \neq 0$ and $\alpha \neq \frac{n-1}{n+1}$ and $\alpha \neq \frac{1-n}{2n}$ for $n \geq 0$.

We compute that the spectral radius $\rho$ of $\hat{G}$ is as in Theorem 1.3. $\hat{G}$ thus has an invariant measure $\mu$ according to Theorem 1.1. In order to prove Theorem 1.3, we find lines $h$ and $v$ for which $\#(G^n h \cap G^{-m} v) \sim \rho^{n+m}$. It follows as in Theorem 3.1 that $\mu$ is given by the distribution of point masses on $G^n h \cap G^{-m} v$. Thus $\mu$ is supported on $\mathbf{R}^2$ and has entropy $\log \rho$.

## 5. Examples

Perhaps it is useful to illustrate the preceding discussion with a number of examples.

Example 1. (Complex Hénon map) We start with the mapping

$$h : \mathbf{C}^2 \to \mathbf{C}^2, \; h(x_1, x_2) = (x_2, x_2^2 + x_1), \; h^{-1}(x_1, x_2) = (-x_1^2 + x_2, x_1).$$

Let $\mathbf{P}^2 = \{[x_0 : x_1 : x_2] : (x_0, x_1, x_2) \neq (0,0,0)\}$ denote complex projective space, where we use the notaton $[x_0 : x_1 : x_2] = [\lambda x_0 : \lambda x_1 : \lambda x_2]$ (when $\lambda \neq 0$) for homogeneous coordinates. We choose the imbedding $\mathbf{C}^2 \subset \mathbf{P}^2$ given by the mapping $(x_1, x_2) \mapsto [1 : x_1 : x_2]$. Thus $\mathbf{P}^2$ is a compactification of $\mathbf{C}^2$, and we have $\mathbf{P}^2 = \mathbf{C}^2 \cup \{x_0 = 0\}$, where $\{x_0 = 0\} = \{[0 : x_1 : x_2], (x_1, x_2) \neq (0,0)\}$ is naturally equivalent to $\mathbf{P}^1$ and appears as the complex line at infinity of $\mathbf{C}^2$.

We define the extension $\hat{h} : \mathbf{P}^2 \to \mathbf{P}^2$ by asserting that its graph must be the (topological) closure $\hat{\Gamma}_h$ of the graph $\Gamma_h$ of $h$. The graph of $h$ is given by

$$\Gamma_h = \{(x, y) = (x_1, x_2, y_1, y_2) \in \mathbf{C}^2 \times \mathbf{C}^2 : y_1 = x_2, y_2 = x_2^2 + x_1\}.$$

To write $\hat{\Gamma}_h$ as a variety, we add the (redundant) equation $y_1^2 + x_1 = y_2$ to the definition of $\Gamma_h$ and then convert the defining equations to homogeneous coordinates. This is done by replacing the (inhomogeneous) coordinates by the quotient of homogeneous coordinates: $x_j$ is replaced by $X_j/X_0$ and $y_j$ by $Y_j/Y_0$, and then we clear denominators. We obtain

$$\hat{\Gamma}_h = \{(X, Y) = ([X_0 : X_1 : X_2], [Y_0 : Y_1 : Y_2]) \in \mathbf{P}^2 \times \mathbf{P}^2 :$$
$$X_2 Y_0 = X_0 Y_1, \quad Y_2 X_0^2 = X_2^2 Y_0 + X_1 X_0 Y_0, \quad Y_1^2 X_0 + X_1 Y_0^2 = Y_2 X_0 Y_0\}.$$

The third equation, which was redundant for $\Gamma_h$, is necessary for $\hat{\Gamma}_h$. It gives us $X_0 = X_2 = 0 \Rightarrow Y_0 = 0$. Without it, we would have $\{[0 : 1 : 0]\} \times \mathbf{P}^2 \subset \hat{\Gamma}_h$, but $\{[0 : 1 : 0]\} \times \mathbf{P}^2$ is clearly not in the closure of $\Gamma_h$. We note that $\hat{\Gamma}_h$ is singular at points lying over $[X_0 : X_1 : X_2] = [0 : 1 : 0]$.

The induced mapping on $\mathbf{P}^2$ is written in homogeneous coordinates as

$$\hat{h}([X_0 : X_1 : X_2]) = [X_0^2 : X_0 X_2 : X_2^2 + X_1 X_0].$$

It will now be convenient to write our variables in lower case letters. We note that this formula gives a well-defined element $\hat{h}(x) \in \mathbf{P}^2$ unless all three coordinates vanish, which happens if $x_0$ and $x_2$ both vanish. Such a point $x$ is in the indeterminacy locus: $I(\hat{h}) = \{[0 : 1 : 0]\}$. The critical locus is $\mathcal{C}(\hat{h}) = \{x_0 = 0\}$, which is the hyperplane at infinity. We see that $\hat{h}(\mathcal{C}) = [0 : 0 : 1]$, which is a fixed point of $\hat{h}$. Thus the critical variety is blown down to a fixed point, which is not indeterminate, and so (3) holds. It follows that $\mathbf{P}^2$ is a natural compactification for $h$ in the sense that $(\hat{h}^*)^n = (\hat{h}^n)^*$.

Note that in homogeneous coordinates $\hat{h}^{-1}[x_0 : x_1 : x_2] = [x_0^2 : -x_1^2 + x_0 x_2 : x_0 x_1]$. Thus $\hat{h} \circ \hat{h}^{-1} = [x_0^4 : x_0^3 x_1 : x_0^3 x_2]$, so $\hat{h} \circ \hat{h}^{-1} x = x$ if $x_0 \neq 0$, but $\hat{h} \circ \hat{h}^{-1}$ fails to be defined on $\{x_0 = 0\}$. This is related to the fact that we had to add an extra equation in the definition of $\hat{\Gamma}_h$. Note, too, that $\deg(\hat{h}) = \deg(\hat{h}^{-1}) = 2 > \deg(\hat{h} \circ \hat{h}^{-1}) = 1$.

The cohomology of complex projective space is given by $H^{1,1}(\mathbf{P}^2; \mathbf{Z}) = \mathbf{Z}$. The class of any complex line serves as a generator. It is convenient to use the line (= hyperplane) at infinity $\{x_0 = 0\}$. The pullback under $\hat{h}$ is $\hat{h}^{-1}(\{x_0 = 0\}) = \{x_0^2 = 0\}$, which is the same hyperplane but with multiplicity 2. Thus $\hat{h}^*$ multiplies the cohomology class of the hyperplane at infinity by 2, so the spectral radius $\rho(\hat{h}^*) = 2$.

On the other hand, we might have taken $X = \mathbf{P}^1 \times \mathbf{P}^1$ as our compactification of $\mathbf{C}^2$. Let $\tilde{h} : X \to X$ denote the induced birational map. In this case, we have $I(\tilde{h}) = \{(\infty, \infty)\}$. Further, $\tilde{h}(I(\tilde{h})) = \{\infty\} \times \mathbf{P}^1$, and $\tilde{h}(\{\infty\} \times \mathbf{P}^1)$. The critical locus is $\mathcal{C}(\tilde{h}) = \mathbf{P}^1 \times \{\infty\}$. We have

$$\tilde{h}(\mathcal{C}(\tilde{h})) = (\infty, \infty) \in I(\tilde{h}),$$

so $X$ is not a natural compactification for $h$.

Example 2. (Fatou map) Let us consider

$$F : \mathbf{C}^2 \to \mathbf{C}^2, \quad F(x_1, x_2) = (x_2, x_2^2 + x_1 x_2 - x_1),$$
$$F^{-1}(x_1, x_2) = \left(\frac{x_2 - x_1^2}{x_1 - 1}, x_1\right).$$

We extend to $\mathbf{P}^2$ by writing $(x_1, x_2) \leftrightarrow [1 : x_1 : x_2] = [1 : X_1/X_0 : X_2/X_0]$ in the formula for $F$:

$$[1 : \frac{X_1}{X_0} : \frac{X_2}{X_0}] \mapsto [1 : \frac{X_1}{X_0} : \frac{X_2^2}{X_0^2} + \frac{X_1 X_2}{X_0^2} - \frac{X_1}{X_0}]$$
$$= [X_0^2 : X_2 X_0 : X_2^2 + X_1 X_2 - X_1 X_0].$$

It follows, then, that $\hat{F}$ defines a point of $\mathbf{P}^2$ unless $X \in I(\hat{F}) = \{X_0 = 0\} \cap \{X_2(X_2 + X_1) = 0\}$. Thus

$$I(\hat{F}) = \{[0 : 0 : 1], [0 : 1, -1]\}.$$

For the critical locus, we first note that the line at infinity is critical: $\hat{F}(\{X_0 = 0\}) = [0 : 1 : 0]$. For the critical locus in $\mathbf{C}^2$, we take the derivative in (usual) affine coordinates:

$$F' = \begin{pmatrix} 0 & 1 \\ x_2 - 1 & 2x_2 + x_1 \end{pmatrix}.$$

The determinant of $F'$ vanishes at $x_2 - 1 = 0$, and we have $F(\{x_2 = 1\}) = (1, 1)$. Thus

$$\mathcal{C}(F) = \{X_0 = 0\} \cup \{X_2 = X_0\}.$$

To see that (3) holds, observe that $[0:1:0]$ is a fixed point, and that $(1,1) \in \mathbf{C}$, so that $F^n(1,1) \in \mathbf{C}^2$ for all $n \geq 0$. Thus $F^n(\mathcal{C}) \cap I(F) = \emptyset$ for all $n \geq 0$.

Now let us write $F^{-1}$ in homogeneous coordinates:

$$[X_0 : X_1 : X_2] \mapsto [1 : \frac{\frac{X_2}{X_0} - \frac{X_1^2}{X_0^2}}{\frac{X_1}{X_0} - 1} : \frac{X_1}{X_0}]$$
$$= [X_0(X_1 - X_0) : X_0X_2 - X_1^2 : X_1(X_1 - X_0)].$$

Thus $I(F^{-1}) = \{[0:0:1], [1:1:1]\}$, and $\mathcal{C}(F^{-1}) = \{X_1 - X_0 = 0\} \cup \{X_0 = 0\}$. We have $F^{-1}(\{X_1 - X_0 = 0\}) = [0:1:0]$ and $F^{-1}(\{X_0 = 0\}) = [0:1:-1]$. This information about $F^{-1}$ tells us what $F$ does to its points of indeterminacy: $F[0:1:0] = \{x_1 = 1\}$, and $F[0:1:0] = \{X_0 = 0\}$.

**Blowing-up.** A compactification may be thought of as a way of adding points at infinity. For the one-point compactification of $\mathbf{C}^2$, the lines

$$L_{\alpha,\beta} = \{x_2 = \alpha x_1 + \beta\} = \{[X_0 : X_1 : X_2] : X_2 = \alpha X_1 + \beta X_0\}$$

all intersect at the point at infinity. For the compactification $\mathbf{P}^2$, $L_{\alpha,\beta}$ intersects the line at infinity in the point $[0:\alpha:1]$. Thus we may consider $\alpha \in \mathbf{C}$ as a parameter on the portion of the line at infinity given by $\{[0:\alpha:1] : \alpha \in \mathbf{C}\} = \{x_0 = 0\} - [0:1:0]$.

Let us describe how to modify a compactification by blowing up. Recall that the blow-up of $\mathbf{C}^2$ at the point $(0,0)$ is defined as

$$\hat{\mathbf{C}}^2 = \{((x_1, x_2), [\xi_1 : \xi_2]) \in \mathbf{C}^2 \times \mathbf{P}^1 : x_1\xi_2 = x_2\xi_1\},$$

with the projection $\hat{\pi} : \hat{\mathbf{C}}^2 \to \mathbf{C}^2$ given by $\hat{\pi}(x, \xi) = x$. It follows that $\hat{\mathbf{C}}^2$ is a complex manifold, that the exceptional fiber over $(0,0)$ is $\hat{\pi}^{-1}(0,0) = \mathbf{P}^1$, and that

$$\hat{\pi} : \hat{\mathbf{C}}^2 - \hat{\pi}^{-1}(0,0) \to \mathbf{C}^2 - (0,0)$$

is a biholomorphism.

The blowing up construction involves only a neighborhood of $(0,0)$ in $\mathbf{C}^2$, so it may be performed at any point of a complex manifold. For instance, let us denote by $\hat{\pi}_\alpha : \hat{X} \to \mathbf{P}^2$ the blow up of $\mathbf{P}^2$ at the point $[0:\alpha:1]$ (in the line at infinity). It follows that the lines $L_{\alpha,\beta}$ all land at different points of the fiber of $\hat{\pi}^{-1}([0:\alpha:1])$. Thus we may use $\mathbf{C} \ni \beta \mapsto L_{\alpha,\beta} \cap \hat{\pi}^{-1}([0:\alpha:1])$ as a coordinate on (most of) the blow-up fiber $\hat{\pi}^{-1}([0:\alpha:1])$.

EXAMPLE 3. (Shears) Let us next consider

$$s : \mathbf{C}^2 \to \mathbf{C}^2, \quad s(x,y) = (x, y + x^2), \quad s^{-1}(x,y) = (x, y - x^2).$$

These mappings are dynamically trivial because $s^n(x,y) = (x, y + nx^2)$. Let us see how they fit into the framework we have described above. Here we use coordinates $(x,y)$ in $\mathbf{C}^2$ and $[x : y : t]$ on $\mathbf{P}^2$, and we imbed $\mathbf{C}^2$ into $\mathbf{P}^2$ by the map $(x,y) \mapsto [x : y : 1]$. Thus the line at infinity is now $\{[x : y : 0]\} = \{t = 0\}$. By $s$ we denote the extension $s : \mathbf{P}^2 \to \mathbf{P}^2$ of $s$ to $X = \mathbf{P}^2$. As above, we see that $s^{\pm 1}$ may be written in homogeneous coordinates as $[x : y : t] \mapsto [xt : yt \pm x^2 : t^2]$. The sets of indeterminacy are $I(s) = \{[0 : 1 : 0]\} = I(s^{-1})$. The line at infinity is critical since $s(t = 0) = [0 : 1 : 0]$ is a point; and since $s$ is a diffeomorphism of $\mathbf{C}^2$, line at infinity is all of $\mathcal{C}$. Observing that $s(\mathcal{C}) = [0 : 1 : 0]$ belongs to $I(s)$, we see that (3) fails.

To find a space on which (3) holds, we start by blowing up $s(\mathcal{C})$. Let $\eta : X_1 \to \mathbf{P}^2$ denote the blow-up of $\mathbf{P}^2$ at the point $[x : y : t] = [0 : 1 : 0]$, the point where the $y$-axis intersects the line at infinity. Let $D_1 = \eta^{-1}[0 : 1 : 0]$ denote the exceptional fiber. Thus $D_1$ is equivalent to $\mathbf{P}^1$, and $\eta : X_1 - D_1 \to \mathbf{P}^2 - [0 : 1 : 0]$ is biholomorphic. Let $s_1 : X_1 \to X_1$ denote the bimeromorphic extension of $s$ to $X_1$. We will find the set of indeterminacy and critical locus of $s_1$.

For $\beta \in \mathbf{C}$, let $V_\beta$ denote the closure of the vertical line $\{(x,y) \in \mathbf{C}^2 : x = \beta\}$ inside $X_1$. Thus $V_\beta$ is a complex curve which intersects $D_1$ in a point, which we denote by $\hat{\beta}$. A neighborhood in $\mathbf{C}^2$ of a point $\hat{\beta}_0 \in D_1$ is given by $\{(x,y) : |x - \beta_0| < \epsilon, |y| > \epsilon^{-1}\}$, where $\epsilon > 0$ is taken small. It follows that $s(x,y) = (x_1, y_1)$ satisfies $|x_1 - \beta_0| < \epsilon$ and $|y_1| > \epsilon^{-1} - o(\epsilon^{-1})$. We conclude that $s(x,y)$ converges to $\hat{\beta}_0$ as $(x,y) \to \beta_0$. Thus $s_1(\hat{\beta}) = \hat{\beta}$ for all $\beta \in \mathbf{C}$.

Let $T$ denote the closure of $\{[x : y : 0] : x \neq 0\} \subset X_1 - D_1$ in $X_1$. Thus $T$ coincides with the line at infinity in $\mathbf{P}^2$ over all points different from $[0 : 1 : 0]$, and $D_1 \cap T = \{\hat{\infty}\}$. We have seen that $s_1 : D_1 - T \to D_1 - T$ is the identity map. It follows that $I(s_1) = \{\hat{\infty}\} = T \cap D_1$.

Since $s$ is a diffeomorphism, the critical locus of $s_1$ does not intersect $\mathbf{C}^2$. We have seen that $D_1 - \hat{\infty}$ is not critical. It remains to check $T - \{\hat{\infty}\}$. For $\alpha \in \mathbf{C}$, let $L_\alpha$ denote the closure of the line $\{y = \alpha x\} \subset \mathbf{C}^2$ in $X_1$. Thus $L_\alpha \cap T \in X_1$ is a point which we denote by $[1 : \alpha : 0]$. If we parametrize $L_\alpha$ by $t \mapsto (t, \alpha t)$, then $s_1(L_\alpha)$ is parametrized by $t \mapsto (t, \alpha t + t^2)$. If we set $\tau = t^2 + \alpha t$, then we may parametrize this curve by $\tau \mapsto (o(\tau), \tau)$, which intersects $D_1$ in the point $\hat{0}$. Thus for all $\hat{\alpha} \in T - D_1$, we have $s_1([1 : \alpha : 0]) = \hat{0}$. In other words, $s_1(\mathcal{C}) =$

$s_1(T - D_1) = \hat{0} \in D_1$. Thus we have $\mathcal{C}(s_1) = T - \hat{\infty}$. Since $\hat{0}$ is fixed under $s_1$, we have

$$s_1^n(T - D_1) = \hat{0} \notin I(s_1) \text{ for all } n \geq 0,$$

and so (3) holds.

Next let us compute the action of $s_1^*$ on $H^{1,1}(X_1)$. Perhaps it is useful first to make a couple of comments on cohomology. And for this we discuss the general problem of transfering a curve from $\mathbf{P}^2$ to a curve of $X_1$. Given a curve $\Gamma$ in $\mathbf{P}^2$, there are two curves in $X_1$ that correspond to $\Gamma$. The first (which is minimal) is called the proper transform of $\Gamma$ and is simply the closure of $\eta^{-1}(\Gamma - [0:1:0])$ in $X_1$. The second one (which is maximal) is called the total transform and is $\eta^{-1}(\Gamma)$. An advantage of the total transform is that it is continuous with respect to the topology of currents. If $L$ is a line in $\mathbf{P}^2$ that does not meet $[0:1:0]$, then the proper and total transforms of $L$ coincide, and without ambiguity we can say that $L$ is a line in $X_1$. On the other hand, the proper transform of the line $\{t = 0\} \subset \mathbf{P}^2$ is $T$, but the total transform is $T \cup D_1$.

We will find it useful to view the cohomology of $X_1$ from the point of view of the pullback map $\eta^* : H^{1,1}(\mathbf{P}^2) \to H^{1,1}(X_1)$, which corresponds to the total transform. Recall that $H^{1,1}(\mathbf{P}^2)$ is generated by the cohomology class determined by a general line $L$ in $\mathbf{P}^2$, and the pullback $\eta^*(L)$ determines a cohomology class in $X_1$. Taking the line $L = \{t = 0\}$, we have $\eta^*(L) = T + D_1$. This is not equal to $T$; $H^{1,1}(X_1)$ is two-dimensional, and we may take $T$ and $D_1$ as a basis.

We compute that $s^{-1}(T) = T$, and $s^{-1}D_1 = s^{-1}(\hat{0}) \cup s^{-1}(D_1 - \hat{0}) = T \cup D_1$. Thus, with respect to this basis, we have

$$s^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

We see that the spectral radius is $\rho(s^*) = 1$.

The reader may have observed already that if we use the compactification $Y_1 := \mathbf{P}^1 \times \mathbf{P}^1$, then (3) holds. The reason we avoided the "obvious" compactification and proceeded via $\mathbf{P}^2$ and $X_1$ is because it illustrates a procedure that works in greater generality.

Let us conclude this example with the observation that $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$ are equivalent via a birational map. To see this, let $\eta_2 : X_2 \to X_1$ denote the blowing up of $X_1$ at the point $[1:0:0]$, and let $\pi_2 : Y_2 \to Y_1$ denote the blowing up of $Y_1$ at the point $(\infty, \infty)$. The spaces $X_2$ and $Y_2$ are biholomorphically equivalent: the map $\iota : \mathbf{C}^2 \to \mathbf{C}^2$, $\iota(x, y) = (x, y)$ extends to a biholomorphic map $\iota : X_2 \to Y_2$. Thus $\eta \circ \eta_2 \circ \iota^{-1} \circ \pi_2^{-1} : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ is birational.

Example 4. (Recurrence Equations) We consider mappings of the form

$$p(x,y) = (y, ay + \frac{b}{y^m} + \sum_{j=0}^{m-1} \frac{b_j}{y^j} - cx)$$

with $b \neq 0$ and $c \neq 0$. Note that conjugation by the map $(x,y) \mapsto (\delta x, \delta y)$ leaves $a$ and $c$ unchanged but changes $b$ in the formula to $b\delta^{-m+1}$. Thus if $\delta$ is any $(m-1)$-st root of $b$, then we have conjugated $p$ to a mapping in which $b = 1$.

Before discussing these examples, let us remark that this and Example 1 are special cases of the family of birational mappings which arise from two-term recursion formulas. Specifically, for a map of the form $p(x,y) = (y, f(y) - cx)$, then we may consider $p^n(x,y) = (x_{n-1}, x_n)$ as arising from the recurrence $x_{n+1} + cx_{n-1} = f(x_n)$, $n \in \mathbf{Z}$, with $x_{-1} = x$, $x_0 = y$. Thus $p^{-1}(x,y) = (c^{-1}(f(x) - y), x)$. If $c = 1$, then $p$ is reversible, which is to say that there is an involution $\tau$ which conjugates $f$ to $f^{-1}$, i.e. $f^{-1} = \tau \circ f \circ \tau$; in the case here we may take $\tau(x,y) = (y,x)$.

In homogeneous coordinates, $p$ takes the form

$$[x : y : t] \mapsto [y^{m+1} : ay^{m+1} + bt^{m+1} + \sum_{j=0}^{m-1} t^{j+1} y^{m-j} - cxy^m : ty^m].$$

It is evident, then, that $I(p) = \{[1:0:0]\}$. Checking the line at infinity, we find that $p([x:y:0]) = [y^{m+1} : ay^{m+1} - cxy^m : 0] = [y : ay - cx : 0]$ for $y \neq 0$. Thus $\{t = 0\}$ is not critical. The $x$-axis $\{y = 0\}$ is critical, however, since $p(\{y=0\}) = \{[0:1:0]\}$. Thus $\mathcal{C} = \{y = 0\}$.

Arguing similarly with $p^{-1}$, we find that $I(p^{-1}) = \{[0:1:0]\}$ and $\mathcal{C}(p^{-1}) = \{x = 0\}$. It follows that $p([1:0:0]) = \mathcal{C}(p^{-1}) = \{x = 0\}$.

Let us determine $p^* : H^{1,1}(\mathbf{P}^2) \to H^{1,1}(\mathbf{P}^2)$. We recall that $H^{1,1}(\mathbf{P}^2)$ is generated by a generic line $L = \{Ax + By + Ct = 0\}$. The preimage $p^*L = p^{-1}L = \{Ay^{m+1} + B(ay^{m+1} + bt^{m+1} + \dots) + Cty^m = 0\}$ is a curve of degree $m+1$, which is equal to $(m+1)L$ in $H^{1,1}(\mathbf{P}^2)$. Thus $p^*$ acts as multiplication by $m+1$.

The condition (3) is equivalent to the condition that $p^{n+1}(\mathcal{C}) = p^n[0 : 1 : 0] \neq [1:0:0]$ for all $n \geq 0$. Let us set $M = \begin{pmatrix} 0 & 1 \\ -c & a \end{pmatrix}$. Then $p^n[x:y:0] = [x_n : y_n : 0]$, where $[x_n : y_n] = M^n[x:y]$ corresponds to matrix multiplication. The lower left hand entry of $M^n$ is a polynomial $q_n(a,c)$ of degree $n+1$. The condition (3) is equivalent to the condition that $q_n(a,c) \neq 0$ for all $n \geq 0$, which is a condition on $a$ and $c$ alone and requires that $(a,c)$ avoid a countable family of varietes.

EXAMPLE 5. (Failure of Condition (7)) Let us consider a mapping $p$ as in Example 4, with $c = 1$. It follows that $p|_{\{t=0\}}$ is given by the matrix $M = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}$. Thus $M$ acts as a linear (fractional) transformation on $\{t = 0\}$. If $-2 < a < 2$, then $M$ is conjugate to a rotation of the Riemann sphere. This conjugacy takes the real line to a circle, which is rotated through an angle of $2\pi\chi$. The points $[0 : 1 : 0]$ and $[1 : 0 : 0]$, respectively, are taken to points with angles $2\pi\theta_0$ and $2\pi\theta_1$ on this circle. Thus $M$ acts on the circle by translation: $\theta \mapsto \theta + \chi \bmod 1$. It is evident that the distance from $M^n[0 : 1 : 0]$ to $[1 : 0 : 0]$ in $\mathbf{P}^2$ is comparable to the distance from $\theta_0 + n\chi$ (mod 1) to $\theta_1$ in the circle. As was shown in [12], we may choose $\chi$ so that (7) diverges. Observe that this depends only on $a$, and the values of $a$ for which (7) fails are dense in the interval $[-2, 2]$. On the other hand, observe that if $c \neq 1$, or if $a \notin [-2, 2]$, then (7) holds.

EXAMPLE 6. We consider a special case of Example 4:

$$p(x, y) = (y, y + \frac{1}{y} - x).$$

(This is closely related to the so-called discrete Painlevé II map.) For this map we have $p|_{\{t=0\}} = [y : y - x : 0]$, so $(p|_{\{t=0\}})^3 = id$. Condition (3) fails because $p([0 : 1 : 0]) = [1 : 1 : 0]$, and $p([1 : 1 : 0]) = [1 : 0 : 0] \in I(p)$ i.e. $p^3(\mathcal{C}) \in I(p)$.

We may obtain a compactification for which (3) holds by performing a series of blowups. We start with the modification $\eta_0 : X_0 \to \mathbf{P}^2$ which is obtained by blowing up $\mathbf{P}^2$ at the orbit of the critical locus, i.e. the points $[1 : 0 : 0]$, $[1 : 1 : 0]$, and $[0 : 1 : 0]$. We denote the fibers by: $D_0 = \eta_1^{-1}([0 : 1 : 0])$, $E_0 = \eta_1^{-1}([1 : 1 : 0])$, and $F_0 = \eta_1^{-1}([1 : 0 : 0])$. Let us introduce a complex parameter $\beta$ on the each fiber: $\hat{\beta} \in D_0$ corresponds to the point where the closure of the line $x = \beta$ intersects $D_0$; $\hat{\beta} \in E_0$ corresponds to the point where the closure of the line $y = x + \beta$ intersects $E_0$; and $\hat{\beta} \in F_0$ corresponds to the point where the closure of the line $y = \beta$ intersects $F_0$.

By $p_0 : X_0 \to X_0$ we denote the mapping induced by $p$. Where is the indeterminacy locus of $p_0$? The set $\{|x| > \epsilon^{-1}, |y| < \epsilon\}$ is a basic neighborhood of $\hat{0} \in D_1$ inside $\mathbf{C}^2$. With such neighborhoods, it follows that $p_0$ cannot take a limit as $(x, y) \to \hat{0} \in D_0$ inside $X_0$. Thus $\hat{0} \in I(p_0)$. It is easy to see that $p_0$ is well behaved away from this point, so $I(p_0) = \{\hat{0} \in D_0\}$.

Now we describe the behavior of $p_0$ on $D_0 \cup E_0 \cup F_0$. Curves of the form $\{y=\beta\}$ are taken to $\{x=\beta\}$. Thus

$$F_0 \ni \hat{\beta} \mapsto p_0(\hat{\beta}) = \hat{\beta} \in D_0.$$

Similarly, the path $s \mapsto (\beta, s)$, which approaches $\hat{\beta} \in D_0$ as $s \to \infty$, is taken under $p_0$ to the path of $s \mapsto (s, s-\beta+o(1))$. Thus we have $D_0 \ni \hat{\beta} \mapsto p_0(\hat{\beta}) = \widehat{-\beta} \in E_0$, and a similar consideration shows that $E_0 \ni \hat{\beta} \mapsto p_0(\hat{\beta}) = \widehat{\beta} \in F_0$.

It follows that none of the exceptional varieties $D_0$, $E_0$ or $F_0$ is critical. Now we check where $p_0$ sends $\{y=0\}$. A point $(x,0)$ is the limit of a path $\epsilon \mapsto (x,\epsilon)$ as $\epsilon \to 0$. Applying $p$ to this path, we find that it is mapped to $\epsilon \mapsto (\epsilon, \epsilon + a\epsilon^{-1} - x)$. Writing $s = \epsilon + a\epsilon^{-1} - x$, we have $s \to \infty$ as $\epsilon \to 0$. This path may be written as $s \mapsto (s^{-1}+o(s^{-1}), s)$, which tends to $\hat{0} \in D_0$ as $s \to \infty$. Thus $\{y=0\}$ is critical for $p_0$. We may summarize:

$$\mathcal{C}(p_0) = \{y=0\}, \quad p_0(\{y=0\}) = \hat{0} \in D_0,$$
$$p_0(\hat{0} \in D_0) = \ \hat{0} \in E_0, \text{ and } \ p_0^3(\{y=0\}) \in I(p_0).$$

In particular, $p_0$ does not satifsy (3).

It is a useful exercise at this stage to determine the action of $p_0^*$ on $H^{1,1}(X_0)$. Let $L' \in H^{1,1}(\mathbf{P}^2)$ denote the class of a complex line. Let $L_0 = \eta_0^*(L')$ be the class in $H^{1,1}(X_0)$ defined by the pullback map $\eta_0^* : H^{1,1}(\mathbf{P}^2) \to H^{1,1}(X_0)$, i.e. the total transform. Note that if $L'$ is represented by a line in $\mathbf{P}^2$ which is disjoint from the blown up points $[0:1:0]$, $[1:1:0]$, and $[1:0:0]$, then the total and proper transforms of $L'$ coincide and define the same element of $H^{1,1}(X_0)$. On the other hand, the total transform of the critical locus $\{y=0\} \subset \mathbf{P}^2$ is $\eta_0^*(\{y=0\}) = \mathcal{C} + F_0$. Since all lines in $\mathbf{P}^2$ are equal in $H^{1,1}(\mathbf{P}^2)$, we have $L_0 = \mathcal{C} + F_0$.

Now let us determine the class $p_0^* L_0$. The pullbacks commute: $p_0^*\eta_0^* = \eta_0^* p_0^*$. Thus we have $p_0^* L_0 = p_0^*\eta_0^* L' = \eta_0^* p^* L'$. We may represent $L'$ by a generic line in $\mathbf{P}^2$. Thus $L'$ intersects the $y$-axis $\{x=0\}$ with multiplicity 1. Since the $y$-axis is the image of the point of indeterminacy $[1:0:0] \in I(p) \subset \mathbf{P}^2$, it follows that $[1:0:0] \in p^{-1}L'$ is a point with multiplicity 1. Now by Example 4, $p^{-1}L' = 2L' \in H^{1,1}(\mathbf{P}^2)$, so $p_0^*(L_0) = \eta_0^*(p^{-1}L') = 2L_0 + F_0$, the coefficient of $F_0$ being the multiplicity of $[0:1:0]$.

We have seen that $p_0(\mathcal{C}) = \hat{0} \in D_0$. Thus $p_0^*(D_0) = F_0 + \mathcal{C} = L_0$. Since there is no critical locus outside of $\mathcal{C}$, the other exceptional curves are easier: $p_0^*(E_0) = D_0$, and $p_0^*(F_0) = E_0$. We take $D_0, E_0, F_0, L_0$ as a basis of $H^{1,1}(X_0)$, and with respect to this basis we have

$$p_0^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

In order to have a space for which (3) holds, we need to continue this process, performing blow-ups at two more levels. We define the space $\eta_1 : X_1 \to X_0$ to be the space $X_0$ blown up along the critical orbit $\hat{1} \in E_0$, $\hat{1} \in F_0$, $\hat{0} \in D_0$. Condition (3) does not hold for the induced map $p_1 : X_1 \to X_1$. Now we let $\eta_2 : X_2 \to X_1$ be the space $X_1$ blown up along the critical orbit of $p_1$. It now turns out that the induced map $p_2 : X_2 \to X_2$ is biholomorphic. The space $H^{1,1}(X_2)$ has dimension 10, and the spectral radius of $p_2^*$ is 1. This construction is described in [18].

Example 7. (Hietarinta-Viallet map [14]) This is like Example 6 but with the $y^{-1}$ replaced by $y^{-2}$:

$$(x, y) \mapsto (y, y + \frac{1}{y^2} - x).$$

This may be analyzed along the lines of Example 6, except that two more levels of blowing up are involved. This is carried out in [17]-[19].

## References


[1] N. Abarenkova, J.-Ch. Anglès d'Auriac, S. Boukraa, S. Hassani and J.-M. Maillard, *Real Arnold complexity versus real topological entropy for birational transformations*, J. Phys. A: Math. Gen. **33** (2000), 1465–1501.

[2] N. Abarenkova, J.-Ch. Anglès d'Auriac, S. Boukraa and J.-M. Maillard, *Growth complexity spectrum of some discrete dynamical systems*, Physica D **130** (1999), 27–42.

[3] ———, *Real topological entropy versus metric entropy for birational measure-preserving transformations*, Physica D **144** (2000), 387–433.

[4] E. Bedford and J. Diller, *Energy and invariant measure for birational surface maps*, preprint.

[5] ———, *Real and complex dynamics of a family of birational maps of the plane: The golden mean subshift*, preprint.

[6] ———, in preparation.

[7] E. Bedford and J. Smillie, *Polynomial diffeomorphisms of* $\mathbf{C}^2$. *VIII: Quasi-expansion*, Amer. J. Math. **124** (2002), 221–271.

[8] ———, *Real polynomial diffeomorphisms with maximal entropy: Tangencies*, Math. DS/0103038.

[9] R. Devaney and Z. Nitecki, *Shift automorphisms in the Hénon family*, Comm. Math. Phys. **67** (1979), 48–137.

[10] J. Diller, *Dynamics of birational maps of* $\mathbf{P}^2$, Indiana U. Math. J. **45** (1996), 721–772.

[11] J. Diller and C. Favre, *Dynamics of bimeromorphic maps of surfaces*, Amer. J. of Math. **123** (2001), 1135–1169.

[12] C. Favre, *Points périodiques d'applications birationnelles de* $\mathbf{P}^2$, Ann. Inst. Fourier (Grenoble) **48** (1998), 999–1023.

[13] S. Friedland, *Entropy of rational self-maps of projective varieties*, in Dynamical Systems and Related Topics (Nagoya, 1990), pages 128–140. World Sci. Publishing, River Edge, NJ, 1991.

[14] J. Hietarinta and C. M. Viallet, *Singularity confinement and chaos in discrete systems*, Phys. Rev. Lett. **81** (1997), 325–328.

[15] S. Morosawa, Y. Nishimura, M. Taniguchi and T. Ueda, *Holomorphic Dynamics*, Cambridge Univ. Press, 2000.

[16] R. Oberste-Vorth, *Complex horseshoes and the dynamics of mappings of two complex variables*, Ph.D dissertation, Cornell Univ., Ithaca, NY, 1987.

[17] T. Takenawa, *A geometric approach to singularity confinement and algebraic entropy*, J. Phys. A: Math. Gen. **34** (2001), L95–L102.

[18] ———, *Algebraic entropy and the space of initial values for discrete dynamical systems*, J. Phys. A: Math. Gen. **34** (2001), 10533–10545.

[19] ———, *Discrete dynamical systems associated with root systems of indefinite type*, Comm. Math. Phys. **224** (2001), 657–681.

Department of Mathematics
Indiana University
Bloomington, IN 47405
*E-mail*: bedford@indiana.edu