

Polynomial diffeomorphisms of \mathbf{C}^2

III. Ergodicity, exponents and entropy of the equilibrium measure

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0 Introduction

Let f be a polynomial diffeomorphism of \mathbf{C}^2 which is not conjugate to an affine map or a generalized shear. These are the polynomial diffeomorphisms of \mathbf{C}^2 with nontrivial dynamics. We write $K^\pm \subset \mathbf{C}^2$ for the set of points in \mathbf{C}^2 bounded in forward/backward time, and we let $K = K^+ \cap K^-$. The sets K^\pm and K are invariant, and K is compact. The nontriviality of the dynamics of f is reflected in the fact that the topological entropy of $f|_K$ is equal to $\log d$ for some integer $d \geq 2$ (see [FM] and [S]). We call d the dynamic degree of f (see Sect. 1).

In [BS1] we studied the stable/unstable currents μ^\pm , which are defined by the formula

$$\mu^\pm = \frac{1}{2\pi} dd^c G^\pm$$

where G^\pm is the Green function for K^\pm . These currents have support equal to $J^\pm = \partial K^\pm$ and satisfy

$$f^* \mu^\pm = d^{\pm 1} \mu^\pm.$$

The equilibrium measure μ of K can be defined as $\mu = \mu^+ \wedge \mu^-$. This measure has finite total mass and is invariant under f . In this paper we consider the dynamics of f with respect to μ .

The results that we prove in this paper for μ parallel known results on the dynamical properties of the equilibrium measure for polynomial maps of \mathbf{C} , which

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were first investigated by Brolin in [Br] (see also [Si; T; C]). Let f_0 denote a polynomial map of \mathbb{C} which is neither constant nor affine. Let K_0 denote the set of points with bounded (forward) orbits. Let μ_0 be the equilibrium measure of K_0 . We prove (Theorem 2.1) that f is mixing with respect to μ . The analogous result for μ_0 was proved in [Br].

The proof of this theorem is based on the following characterization of μ^+ , which is given in Sect. 1. If S is a current of the form described in (1.1) then there is a constant c such that

$$\lim_{n \rightarrow \infty} (d^{-n})f^{n*}S = c\mu^+ . \tag{0.1}$$

Results of this form for certain classes of currents S appear in [BS1, 2, 3] and [FS]. The result in this paper extends these previous results.

There is a useful formula for the Lyapunov exponent λ_0 of f_0 with respect to μ_0 . If d is the degree of f_0 , then λ_0 can be described in terms of the rate of escape of the critical points. We have:

$$\lambda_0 = \log d + \sum_{c: f_0'(c)=0} \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ |f_0^n(c)| . \tag{0.2}$$

(This is formula (1) in [Pr, p. 177] with u_v replaced by Brolin's formula for the Green function.) The function λ_0 and related functions have proved useful in understanding how the dynamics of f_0 vary with the polynomial.

Since μ is ergodic for f , there are two Lyapunov exponents of f with respect to μ , which give the exponential growth rate of μ almost every tangent vector under iteration. Let A denote the larger of the two exponents. We can think of A as a function on parameter space. We hope that A might prove useful in understanding the space of polynomial diffeomorphisms. We do not have a formula for A which is analogous to (0.2). In particular it is not clear what the analog of a critical point is for a polynomial diffeomorphism. On the other hand a number of properties of λ_0 as a function on parameter space follow from formula (0.2), and we have been able to prove analogs of these for A .

For instance, it follows from (0.2) that $\lambda_0 \geq \log d$. We prove that $A \geq \log d$ in Theorem 3.2 using a Jensen type inequality. This result lets us invoke a powerful result from smooth dynamical systems to derive information periodic points in Theorem 3.4.

In Sect. 4 we prove that the measure theoretic entropy of μ is equal to $\log d$ (Theorem 4.4). The analogous result for μ_0 was proved in [FLM] and [Ly]. This involves a characterization of μ which is motivated by the construction of a measure with maximal entropy. For instance, let $\Theta = \frac{1}{4\pi} dd^c \log(1 + |x|^2 + |y|^2)$ be the Kähler form associated with the Fubini-Study metric, and let $\Theta_n = f^{*n}\Theta$ be the pullback under f^n . For any positive (1,1) current S , there is a measure $\nu_n := S \llcorner \Theta_n$, which acts on a continuous function φ with compact support as $\int \varphi d\nu_n = \langle S, \varphi \Theta_n \rangle$. In Sect. 4 we show that: *If S is as in (1.1) and $\langle S, \mu^- \rangle = c$, then $(d^{-n})f_*^j \nu_n \rightarrow c\mu$ as $j \rightarrow \infty$ and $n - j \rightarrow \infty$.*

There is a well-known relation between Hausdorff dimension, Lyapunov exponents and entropy. In our case, we apply a result of Lai-Sang Young to compute the Hausdorff dimension of the equilibrium measure (Corollary 4.6).

Another consequence of (0.2) is that λ_0 is a plurisubharmonic function on the space of the parameters and is pluriharmonic in the parameter region where f is an expanding map. In §5 we establish corresponding plurisubharmonicity and pluriharmonicity results for Λ (Theorems 5.5 and 5.7).

In Sect. 6 we extend the parameter space to include certain non-invertible maps. These maps are essentially 1-dimensional and conjugate to polynomial maps of C . We show that for diffeomorphisms f_a converging to a 1-dimensional map f_0 , the measure μ_a converges to its 1-dimensional analogue, μ_0 and $\limsup_{a \rightarrow a_0} \Lambda_a = \lambda_0$, i.e. $a \mapsto \Lambda_a$ and $a \mapsto \mu_a$ are well behaved on the extended parameter space. Since the 1-dimensional maps are relatively well understood, we are able to get some information about maps f_a which are close to singular. We note that this approach has been used by Fornæss and Sibony [FS] and Hubbard and Oberste-Vorth [HO] to describe the geometry and topology of J and K for certain perturbations of 1-dimensional maps.

It also follows from (0.2) that $\lambda_0 = \log d$ for precisely those maps f_0 for which K_0 is connected. We prove a partial analog of this result in Theorem 6.7 and Corollary 6.8.

We have found the analogy between one and two dimensional complex dynamics to be useful in guiding our investigations. The notation of [HO] suggests $J = J^+ \cap J^-$ as the two-dimensional analogue of the Julia set $J_0 = \partial K_0$. The results given below suggest that $J^* = \text{support}(\mu)$ may better carry through this analogy. By [BT], J^* is the Shilov boundary of K , which parallels the 1-dimensional case, since J_0 coincides with both the topological and Shilov boundaries of K_0 . Further, by Theorem 3.4, periodic saddle points are dense in J^* . This is consistent with the result of Fatou and Julia that expanding periodic points are dense in J_0 . We know that $J^* \subset J$, but we do not have an example where equality does not hold. It is shown in [BS1] that $J^* = J$ in the hyperbolic case.

1 Characterization of the stable current

We recall some useful results from earlier papers. By [FM] a polynomial diffeomorphism is conjugate either to an affine map, to a generalized shear or to a map of the form $f = f_m \circ \dots \circ f_1$, where $f_j(x, y) = (y, p_j(y) - a_j x)$, $a_j \in C$, $a_j \neq 0$, and $p_j(y)$ is a polynomial of degree at least 2. We will consider maps of the last type. Without loss of generality we may assume that f is not simply conjugate to a map of this last form but actually equal to a map of this form. We let d_j denote the degree of p_j , and it follows that $d = d_1 \dots d_m$ is equal to the (conjugacy invariant) dynamical degree $d(f) = \lim_{n \rightarrow \infty} (\deg f^n)^{1/n}$. The functions

$$G^\pm(q) = \lim_{n \rightarrow \infty} (\deg f)^{-n} \log(1 + |f^{\pm n}(q)|)$$

give the exponential rate of escape of the point q in forward/backward time. We define the stable/unstable currents as

$$\mu^\pm := \frac{1}{2\pi} dd^c G^\pm .$$

(This agrees with the definition given in [BS3] but differs from [BS1, 2] by a factor of 2π .)

This section is devoted to proving equation (0.1) for a large class of currents. We consider currents of the form

$$S = \psi T: T \text{ is a positive current on } \Omega \subset \mathbb{C}^2, \text{ and}$$

$$\psi \text{ is a test function on } \Omega \text{ with } \text{spt } \psi \cap \text{spt } \partial T = \emptyset, \tag{1.1}$$

and we will show (Theorem 1.6) that (0.1) holds for these currents.

Let us establish some notation and terminology on currents. Recall that a (1,1) current on \mathbb{C}^2 acts as a linear functional on the (1,1) test forms. If T is a positive current, then it is representable by integration, i.e. there is a matrix of signed Borel measures $(\mu_{i,j})$ such that if $\varphi = \sum \varphi_{i,j} dz_i \wedge d\bar{z}_j$ is a test form, then $T(\varphi) = \sum \int \varphi_{i,j} d\mu_{i,j}$. Since we can multiply Borel measures by continuous functions we can make sense of $\theta \wedge T$ where θ is a form with continuous coefficients. We recall that the boundary operator is defined on (1,1) currents as $\int (\partial T) \wedge \eta = - \int T \wedge d\eta$, where d is the usual exterior differential operator on 1-forms.

Let us define the expression $\mu^- \wedge \psi T$, which generalizes the construction of $\mu = \mu^+ \wedge \mu^-$ in [BS1]. Let G_ε^- be a sequence of psh smoothings of G^- with $G_\varepsilon^- \rightarrow G^-$ uniformly as $\varepsilon \rightarrow 0$, and let $\mu_\varepsilon^- = \frac{1}{2\pi} dd^c G_\varepsilon^-$. Thus μ_ε^- is a smooth form converging to μ^- in the sense of currents as $\varepsilon \rightarrow 0$. We have

$$dd^c(G_\varepsilon^- \wedge \psi T) = dd^c G_\varepsilon^- \wedge \psi T + d^c G_\varepsilon^- \wedge d\psi \wedge T + dG_\varepsilon^- \wedge d^c \psi \wedge T + G_\varepsilon^- dd^c \psi \wedge T$$

The (1,1) parts of $d^c G_\varepsilon^- \wedge d\psi \wedge T$ and $dG_\varepsilon^- \wedge d^c \psi \wedge T$ have opposite signs, so these terms cancel. Thus we are left with

$$\mu_\varepsilon^- \wedge \psi T = dd^c(G_\varepsilon^- \wedge \psi T) - G_\varepsilon^- dd^c \psi \wedge T.$$

Taking limits gives:

$$\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon^- \wedge \psi T = \lim_{\varepsilon \rightarrow 0} dd^c(G_\varepsilon^- \wedge \psi T) - G_\varepsilon^- dd^c \psi \wedge T = dd^c(G^- \wedge \psi T) - G^- dd^c \psi \wedge T.$$

The terms in the right-hand side of this equation are well defined because T can be defined on continuous test forms and G^- is continuous. This limit is independent of the sequence G_ε^- , and we may use the right-hand side of this equation to define $\mu^- \wedge \psi T$.

If φ is a test form, then $|\varphi| = \sup_x |\varphi(x)|$, where $|\varphi(x)|$ denotes the euclidean norm of the k -form $\varphi(x)$. The mass norm of a current T is then given by

$$M[T] = \sup_{|\varphi| \leq 1} T(\varphi).$$

If X is an open subset of an analytic manifold with finite area, and if $T = [X]$ is the current of integration on X , then $M[T]$ is the usual area of X . And if ψ is a function with compact support, then

$$M[[X] \llcorner \psi] = \int_X \psi dA$$

is the integral of ψ with respect to area measure.

For a current T and a test form ψ , we let $T \llcorner \psi$ denote the current defined by $(T \llcorner \psi)(\varphi) = T(\psi \wedge \varphi)$. If ψ is continuous function we write ψT for $T \llcorner \psi$. If S is a Borel set, and T is representable by integration, we will use the notation $T|_S$ to denote $T \llcorner \chi_S$, where χ_S is the characteristic function of S .

The (1,1) form $\Theta = \frac{1}{2}dd^c \log(1 + |x|^2 + |y|^2)$ dominates a multiple of the standard Kähler form at every point of \mathbb{C}^2 . Thus for a bounded set $U \subset \mathbb{C}^2$ there is a positive constant C such that for any positive (1,1) current T

$$M[T|_U] \leq C \int \Theta \wedge T.$$

Lemma 1.1. *Let φ be a test function on \mathbb{C}^2 . Then there is a constant C such that*

$$M[(f^{n*}\psi T) \llcorner \varphi] \leq Cd^n.$$

Proof. We may assume that $\psi \geq 0$, for otherwise we may write $\psi = \psi_1 - \psi_2$ as the difference of nonnegative test functions and treat the two terms separately.

$$\begin{aligned} d^{-n}M[(f^{n*}\psi T) \llcorner \varphi] &\leq Cd^{-n} \int \varphi f^{n*}(\psi T) \wedge \Theta \\ &\leq C \|\varphi\| \int \psi T d^{-n}f_*^n \Theta. \end{aligned}$$

But by [BS1] we know that the functions

$$d^{-n}f_*^n \frac{1}{2} \log(1 + |x|^2 + |y|^2) = d^{-n} \frac{1}{2} \log(1 + |f^n|^2)$$

converge uniformly on compact subsets to G^- . Thus in the last integral above we may integrate by parts and pass to the limit to obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} C \|\varphi\| \int \psi T (d^{-n}) \frac{1}{2} dd^c \log(1 + |f^n|^2) \\ = \lim_{n \rightarrow \infty} C \|\varphi\| \int dd^c \psi \wedge T (d^{-n}) \frac{1}{2} \log(1 + |f^n|^2) \\ = C \|\varphi\| \int dd^c \psi \wedge T \wedge G^-, \end{aligned}$$

which completes the proof.

Our next result gives a relation between the mass of a current and the mass of its boundary. Let us note that there is an operator J on 1-forms such that for any function φ , $J d\varphi = d^c\varphi$. J is an \mathbb{R} -linear operator, which in coordinates is given by $J(dz_j) = id\bar{z}_j$, or $J(dx_j) = dy_j$, $J(dy_j) = -dx_j$. This defines an inner product $\langle \xi, \eta \rangle := \int \xi \wedge J\eta \wedge T$ on test 1-forms. The corresponding Schwarz inequality is $\langle \xi, \eta \rangle^2 \leq \langle \xi, \xi \rangle \langle \eta, \eta \rangle$.

Lemma 1.2. *Let ψ and T be as in (1.1), and let $\psi_1 \geq 0$ be a test function on Ω with $\psi_1 = 1$ in a neighborhood of $\text{spt } \psi$. Then for any test function χ on \mathbb{C}^2 ,*

$$M[(\partial f^{n*}(\psi T)) \llcorner \chi]^2 \leq \|d\psi\|^2 M[f^{n*}(\psi_1 T) \llcorner \chi] M[(\psi_1 T) \llcorner f^{n*}\chi].$$

Proof. By the definition of ∂ , the mass norm on the left-hand side may be evaluated as

$$\begin{aligned} M[\chi f^{n*}(\partial\psi T)] &= \sup_{|\varphi| \leq 1} \int \chi \wedge f^{n*}\partial[\psi T] \wedge \varphi \\ &= \sup_{|\varphi| \leq 1} \int \chi \varphi \wedge f^{n*}[d\psi \wedge T] \\ &= \sup_{|\varphi| \leq 1} \int (f_*^n \chi \varphi) \wedge d\psi \wedge \psi_1 T \end{aligned}$$

$$\begin{aligned} &\leq \sup_{|\varphi| \leq 1} \left(\int f_*^n \chi \varphi \wedge J(f_*^n \chi \varphi) \wedge \psi_1 T \right)^{1/2} \left(\int d\psi \wedge d^c \psi \wedge \psi_1 T \right)^{1/2} \\ &\leq \sup_{|\varphi| \leq 1} \left(\int f_*^n \chi \varphi \wedge f_*^n J(\chi \varphi) \wedge \psi_1 T \right)^{1/2} \left(\int d\psi \wedge d^c \psi \wedge \psi_1 T \right)^{1/2} \\ &\leq M[\chi^2 f^{n*}(\psi_1 T)]^{1/2} \|d\psi\| M[\psi_1 T]^{1/2} . \end{aligned}$$

The third line follows because f_* is the adjoint of f^* ; the fourth line is the Schwarz inequality for the positive current $\psi_1 T$; the fact that f is holomorphic gives us $J(f_* d\varphi) = f_*(J d\varphi)$ in the fifth line; the sixth line follows because $|J\varphi| = |\varphi| \leq 1$ and usual estimations of integrals. The Lemma follows.

Lemma 1.3. *Let ψ, T be as in (1.1), and let v denote any limit of a subsequence of $\{(d^{-n})f^{n*}(\psi T)\}$. Then v is closed, i.e. $\partial v = 0$.*

Proof. Let χ be a test function on \mathbf{C}^2 , and let ψ_1 be a test function on Ω such that $\psi_1 = 1$ on $\text{spt } \psi$. By Lemma 1.1, the sequence $(d^{-n})f^{n*}(\psi T)$ has locally bounded mass, so

$$M[f^{n*}(\psi_1 T) \llcorner \chi] \leq Cd^n .$$

On the other hand, by Lemma 1.2,

$$\begin{aligned} M[(\partial f^{n*}(\psi T)) \llcorner \chi] &\leq (Cd^n)^{1/2} M[\psi_1 T]^{1/2} \\ &\leq C'd^{n/2} \end{aligned}$$

Thus we see that the mass norm of $(d^{-n})(\partial f^{n*}(\psi T)) \llcorner \chi$ tends to zero as $n \rightarrow \infty$. Thus $\partial v = 0$.

Our next step will be to show that these currents $d^{-n}f^{n*}(\psi T)$ actually converge to a multiple of μ^+ . Let $\mathcal{S}(\psi T)$ denote the set of all currents that arise as limits of subsequences of $\{d^{-n}f^{n*}(\psi T)\}$. By Lemma 1.3, $\mathcal{S}(\psi T)$ consists of closed currents. It is evident that $(d^{-1})f^*\mathcal{S}(\psi T) = \mathcal{S}(\psi T)$. Further ψT has compact support in a large polydisk $V = \{|x|, |y| < R\}$, and by Lemma 2.2 of [BS1] we may choose R large enough that $f^{-n}V \subset V \cup \{|x| < |y|\}$. By [BS1, Lemma 2.4] the currents of $\mathcal{S}(\psi T)$ are all supported in K^+ .

We note that if $\mathcal{S}(\psi T)$ consists of a single current, then by Theorem 1.6 of [FS] it follows that this current is a multiple of μ^+ . And in the present case, our use of Lemma 1.5 was motivated by [FS].

Let us recall some results from Section 4 of [BS3]. We will let \mathcal{L}^2 denote 2-dimensional Lebesgue measure. For any $v \in \mathcal{S}(\psi T)$, we may disintegrate the measure $v \llcorner \frac{i}{2} dx \wedge d\bar{x}$ with respect to Lebesgue measure. That is, for \mathcal{L}^2 a.e.

x there is a measure v_x on \mathbf{C} with the property that $v \llcorner \frac{i}{2} dx \wedge d\bar{x} = v_x \mathcal{L}^2(x)$. Let

us define the function

$$U_v(x, y) := \int \log |y - \xi| v_x(\xi) \mathcal{L}^2(x) .$$

A priori, the function U_v is defined only almost everywhere in x , but it may be made upper semicontinuous and psh on \mathbf{C}^2 in a unique way via a smoothing argument. Further, U_v is the unique function satisfying $dd^c U_v = v$ and

$$U_v(x, y) = c \log |y| + o(1) \tag{1.2}$$

for $|x| < R$ and $|y|$ large (cf. [BS3, Lemma 4], or Lelong [L] and Skoda [Sk]). The constant c in this equation corresponds to the fact that the total mass of ν_x is $\frac{c}{2\pi}$ a.e. x .

Lemma 1.4. *The constant in (1.2) is given by $c = \int \psi T \wedge \mu^-$, and is independent of ν .*

Proof. Let $n_j \rightarrow \infty$ be a sequence such that $\lim_{j \rightarrow \infty} d^{-n_j} f^{n_j*}(\psi T) = \nu$, and let $\theta = \frac{1}{2} dd^c \log(1 + |x|^2)$. Then

$$\begin{aligned} \lim_{j \rightarrow \infty} \int d^{-n_j} f^{n_j*}(\psi T) \wedge \theta &= \int \nu \wedge \theta \\ &= \frac{c}{2\pi} \int \theta = c. \end{aligned}$$

On the other hand, as in the proof of Lemma 1.1, we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \int d^{-n_j} f^{n_j*}(\psi T) \wedge \theta &= \int d^{-n_j} \psi T \wedge f_*^{n_j} \theta \\ &= \int \psi T \wedge (d^{-n_j} \frac{1}{2} \log(1 + |f^{-n_j}|^2)) \\ &= \int \psi T \wedge \mu^-, \end{aligned}$$

and so c is given as claimed.

We may now apply the proof of Proposition 1 of [BS3] and conclude that $U_\nu(x, y) = cG^+(x, y)$ for all $(x, y) \in \mathbb{C}^2 - K^+$. By the upper semicontinuity of U_ν , it follows that $U_\nu \geq 0$ on ∂K^+ , and by the maximum principle it follows that $U_\nu \leq 0$ on K^+ . We will show that in fact $U_\nu = 0$ on K^+ . Let us suppose, to the contrary, that there is a current $\nu \in \mathcal{S}(\psi T)$ such that $U_\nu < -1$ holds on a domain ω with $\bar{\omega} \subset \text{int } K^+$. Since $\omega \subset K^+$, it follows that $f^n(\omega)$ remains in a large polydisk $\{|x|, |y| < R\}$. For $n > 0$ there exists $\nu_n \in \mathcal{S}(\psi T)$ such that $\nu = (d^{-n})_* f^{n*} \nu_n$, and so

$$(d^n) U_\nu = U_{\nu_n}(f^{-n}).$$

Thus

$$f^n(\omega) \subset \{|x|, |y| < R\} \cap \{U_\nu < -d^n\}. \tag{1.3}$$

Let us recall the following fact about potentials.

Lemma 1.5. *If $R > 0$, then there exists a constant $C_R > 0$ such that for any positive Borel measure ν_x with total mass $\frac{c}{2\pi}$, then the potential U_{ν_x} satisfies*

$$\mathcal{L}^2(\{U_{\nu_x} < -\lambda\} \cap \{|y| < R\}) \leq C_R \exp\left(-\frac{\lambda}{c}\right).$$

Proof. For $|y|$ large, $c^{-1} U_{\nu_x} = \log |y| + o(1)$. Thus the function $c^{-1} U_{\nu_x} + \lambda/c$ is no greater than the Green function of the set $\{U_{\nu_x} < -\lambda\}$. It follows that the Robin constant of this set is at least as large as λ/c , and so the capacity is no greater than $\exp(-\lambda/c)$. The fact that the area of the set is dominated by the capacity (see [Tsuji, Theorem III.10]) completes the proof.

We may apply this inequality to the functions $U_v(x, y)$ for x fixed, and integrate it with respect to $\mathcal{L}^2(x)$ and obtain

$$\text{Volume}(\{|x|, |y| < R\} \cap \{U_v < -\lambda\}) \leq \pi R^2 C_R \exp\left(\frac{-\lambda}{c}\right).$$

By (1.3), then, $\text{Volume}(f^n(\omega)) \leq \pi R^2 C_R \exp(-d^n/c)$. On the other hand, the Jacobian of a polynomial automorphism is a constant, so it follows that the volume of $f^n(\omega)$ is $|\delta|^{2n}$ times the volume of ω . This is a contradiction, so we conclude that $U_v = 0$ on $\text{int } K^+$.

Recall that a current T is representable by integration if there are Borel measures $T_{I,J}$ such that $T = \sum T_{I,J} dz^I \wedge d\bar{z}^J$. This is equivalent to the existence of a constant C_K for every compact K such that $|T(\varphi)| \leq C \sup |\varphi|$ for all test forms φ supported on K . We say that a sequence $\{T_j\}$ of such currents converges to T as currents representable by integration if $\lim_{j \rightarrow \infty} T_j(\varphi) = T(\varphi)$ for all compactly supported forms φ with continuous coefficients.

Theorem 1.6. *If ψT is as in (1.1), then*

- (i) $\lim_{n \rightarrow \infty} (d^{-n}) f^{n*}(\psi T) = c\mu^+$ for $c = \int \psi T \wedge \mu^-$.
- (ii) $\lim_{n \rightarrow \infty} (d^{-n}) f^{n*}(T \wedge d\psi) = \lim_{n \rightarrow \infty} (d^{-n}) f^{n*}(T \wedge d^c\psi) = 0$.
- (iii) $\lim_{n \rightarrow \infty} (d^{-n}) f^{n*}(T \wedge dd^c\psi) = 0$.
- (iv) *The limits in (i), (ii), and (iii) hold in the sense of currents representable by integration.*

Proof. (i) It follows from the discussion above that there is a constant c such that $U_v = cG^+$ for all $v \in \mathcal{S}(\psi T)$. However, c is determined by the total mass of any slice v_x , which is independent of v and x . Applying $\frac{1}{2\pi} dd^c$, we have only one element, $\frac{c}{2\pi} dd^c G^+$ in $\mathcal{S}(\psi T)$, and so the first limit exists and is equal to a constant multiple of μ^+ .

(ii) Let φ be a test function. Then as in the proof of Lemma 1.3, we may bound the mass norm: $M[(f^{n*}(d\psi \wedge T)) \llcorner \varphi] \leq Cd^{n/2}$. It follows that the masses of the currents in (ii) tend to zero locally on compacts as $n \rightarrow \infty$. Thus they tend to zero as currents representable by integration.

(iii) We observe that

$$\begin{aligned} M[(f^{n*}(dd^c\psi \wedge T)) \llcorner \varphi] &= \sup_{|\alpha| \leq 1} \int (f_*^\alpha \varphi) \alpha \wedge T \wedge dd^c\psi \\ &\leq \|\varphi\| M[dd^c\psi \wedge T]. \end{aligned}$$

is bounded independently of n . Thus the mass of the current in (iii) tends to zero on compact sets as $n \rightarrow \infty$.

(iv) It remains only to discuss (i). Without loss of generality, we may assume that $\psi \geq 0$. Thus the currents in (i) are positive, and it is well known that if a sequence of positive currents converges, then it converges in the sense of currents representable by integration.

Remark. We can extend the class of currents for which the Theorem holds.

• Theorem 1.6 continues to hold for finite linear combinations of currents of the form (1.1). If α is a constant (1,0)-form, and if ψ is a positive test function on Ω , then $i\psi\alpha \wedge \bar{\alpha}$ may be identified with a positive (1,1)-current of the form (1.1). Taking

linear combinations we see that (0.1) holds for all currents that are represented by smooth (1,1)-forms with compact support.

- All points in $\{|x| > R, |y| < |x|\}$, R large, tend to infinity under f^{-1} , so the part of the current S that lies in this set plays no role. Thus it suffices to assume that $\text{spt}(\psi) \cap \{|y| \leq |x|\}$ is compact rather than assume that $\text{spt}(\psi)$ itself is compact.

- If S is a current, and if there are currents S_n^\pm of the form (1.1) such that $S_n^- \leq S \leq S_n^+$ and $|\int S_n^+ \wedge \mu^- - \int S_n^- \wedge \mu^-| \rightarrow 0$ then the (0.1) holds for S with $c = \lim \int S_n^+ \wedge \mu^-$. In particular the Theorem 1.6 holds for forms of the form ψT where ψ is continuous. Or more generally, it suffices that ψ is bounded, measurable, and $T \wedge \mu^-$ a.e. point is a point of continuity of ψ .

Corollary 1.7. *Let V be an open subset of \mathbb{C}^2 , and let S satisfy (1.1). If $(S \wedge \mu^-)(\partial V) = 0$, then*

$$\lim_{n \rightarrow \infty} (d^{-n})f^{*n}(S|_V) = c\mu^+ .$$

Example. Let X be a 1-dimensional complex submanifold in \mathbb{C}^2 , and let $\mathcal{D} \subset X$ be a relatively compact open subset. Then $d^{-n}f^{*n}[\mathcal{D}]$ converges to a multiple of μ^+ if the measure $[\mathcal{D}] \wedge \mu^-$ puts no mass on $\partial\mathcal{D}$.

2 Mixing and applications

We will show that f is (strong) mixing, and we will use these results to prove some topological properties of K and J^* . A measure is (strong) mixing if for Borel sets A and B

$$\lim_{n \rightarrow \infty} \mu(A \cap f^{-n}B) = \mu(A)\mu(B) .$$

This is equivalent to the condition that

$$\lim_{n \rightarrow \infty} \int (f^{n*}\phi)\psi d\mu = \int \phi d\mu \int \psi d\mu \tag{2.1}$$

for all ϕ and ψ in $L^2(\mu)$. The following result was proved in [BS1] in the special case that f is hyperbolic. We now prove it for general f .

Theorem 2.1. *The mapping f is mixing on μ .*

Proof. It suffices to verify the condition (2.1) when ϕ and ψ are test functions because the set of test functions is dense in $L^2(\mu)$.

We note that for a test function $\psi, \mu \llcorner \psi = (\mu^+ \llcorner \psi) \wedge \mu^-$. Thus what we need to prove is that for any test function χ

$$\lim_{n \rightarrow \infty} \int \chi f_*^n(\mu \llcorner \psi) = \int \chi d\mu \int \psi d\mu .$$

However, the left-hand side of this equation is

$$\begin{aligned} \int \chi f^{n*}(\mu^+ \llcorner \psi) \wedge f^{n*}\mu^- &= \int \chi(d^{-n})f^{n*}(\mu^+ \llcorner \psi) \wedge dd^c G^- \\ &= \int dd^c \chi(d^{-n})f^{n*}(\mu^+ \llcorner \psi) \wedge G^- \\ &\quad + \int d\chi(d^{-n})f^{n*}(\mu^+ \llcorner d^c\psi) \wedge G^- \\ &\quad - \int d^c \chi(d^{-n})f^{n*}(\mu^+ \llcorner d\psi) \wedge G^- . \end{aligned}$$

The first equality follows from the functional equation $f^{n*}\mu^- = (d^{-n})dd^cG^-$, and the next one is an integration by parts. Now we pass to the limit as $n \rightarrow \infty$ and use Theorem 1.6. The second and third terms converge to zero. The first term gives the limit

$$\int dd^c\chi \wedge (c\mu^+) \wedge G^- = c \int \chi\mu^+ \wedge \mu^- = \int \psi d\mu \int \chi d\mu,$$

which completes the proof.

Recall that a mapping f is *ergodic* if whenever E is a Borel set such that $f^{-1}(E) = E$ it follows that E is either of full measure or of measure zero. Mixing implies ergodicity so we have the following corollary.

Corollary 2.2. *f is ergodic.*

The following proposition gives a connection between measure theory and topology.

Proposition 2.3. *The measure μ puts positive mass on any nonempty open and closed subset of K .*

Proof. Let K_1 be an open and closed subset of K and let K_2 be the complement of K_1 in K . Let $P(K)$ denote the function algebra obtained by taking the uniform closure of the holomorphic polynomials on K . Since K is holomorphically convex, it follows from the Oka-Weil Theorem that $P(K)$ coincides with the uniform closure of the algebra of holomorphic functions in a neighborhood of K . Thus $P(K) = P(K_1) \oplus P(K_2)$ and so it follows that $\partial_s K = \partial_s K_1 \cup \partial_s K_2$. In particular, K_1 intersects $\partial_s K$. On the other hand, it was shown in [BT] that the support of μ is the Shilov boundary of K , so we have $\mu(K_1) > 0$.

Corollary 2.4. *If K is totally disconnected then $J^* = J (= K)$.*

If x is a point in a topological space X , we define the *component* of x to be the intersection of all sets U which are open and closed sets and contain x . Every component is closed. We say a component is *isolated* if it is also open.

Theorem 2.5. *Either K is connected or K has uncountably many components, none of which is isolated.*

Proof. If all components of K have measure zero then K must have uncountably many components. If some component were isolated it would be an open and closed set with zero measure but this is prohibited by Proposition 2.3. Assume now that some component C has positive measure. Since C has positive measure, $f^n(C)$ must meet C for some positive n . Components are either disjoint or equal so we have $f^n(C) = C$. Since $\bigcup f^n(C)$ is invariant and of positive measure the ergodicity of f with respect to μ implies that its complement must have measure zero. We will show that the complement is empty.

Assume that x is a point in the complement of $\bigcup f^n(C)$. There is some open and closed set U which contains x but not $\bigcup f^n(C)$. The set U is an open and closed subset of a set of zero measure. The existence of such a U contradicts Proposition 2.3.

The fact that f is mixing with respect to μ implies that C must have period 1. Thus $C = \bigcup f^n(C) = K$, and K is connected.

Remark. The same argument proves that either J^* is connected or has uncountably many components none of which is isolated. It is known (cf. [BT]) that μ puts no mass on a pluripolar set, and thus J^* is a perfect set.

It follows from the Theorem 2.5 that K is either perfect or consists of single point. In fact K must be perfect, since K is polynomially convex and regular, and thus no isolated part of K can be polar.

3 Lyapunov exponents

Let $p \in \mathbb{C}^2$ and let $v \in T_p$. The characteristic exponents, which determine the exponential growth rate of the vector v , are given by

$$\lambda(v, p) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(v, p)|$$

when this limit exists. The theory of Oseledets describes the behavior of this exponent for μ a.e. point. The fact that μ is ergodic makes the description of the theory simpler than it would be in the general case. Either there is a single exponent λ so that $\lambda(v, p) = \lambda$ for μ almost every point p and for every nonzero $v \in T_p$ or there are two exponents $\lambda_1 > \lambda_2$ and a measurable splitting of the tangent bundle of \mathbb{C}^2 of the form $T_p = E_p^1 \oplus E_p^2$ at μ almost every point p so that for $v \in E^i$ we have $\lambda(v, p) = \lambda_i$. In the single exponent case it is convenient to define $\lambda_1 = \lambda_2 = \lambda$.

In this section we will estimate λ_1 . It is easier to work with the following integral than to work with λ_1 directly:

$$A = \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|Df^n(x)\| \mu_K(x) ,$$

where $\|Df^n(x)\|$ denotes the operator norm of the linear transformation $Df^n(x)$. It is easily seen that $A = \lambda_1$. Since f has constant determinant δ we have the relation $\lambda_1 + \lambda_2 = \log |\delta|$, so λ_2 is determined by λ_1 .

Here we will show that in fact $A \geq \log d$ for all choices of the parameter. We will make use of the following identity which is related to Jensen's formula.

Lemma 3.1. *Let $K \subset \mathbb{C}$ be a compact subset, and let*

$$G_K(y) = \log |y| + \rho_K + O(|y|^{-1})$$

be the Green function of K , and let $\mu_K = \frac{1}{2\pi} dd^c G_K$ be the equilibrium measure of K . If $p(y) = y^N + \dots$ is a monic polynomial of degree N , then

$$\int_K \log |p| \mu_K + N\rho_K = \sum_{\{c: p(c)=0\}} G_K(c) .$$

Proof. We will apply Green's formula several times. First, we note that we may assume that ∂K is smooth, so that the effect of integrating the equilibrium measure $dd^c G_K$ over K is the same as integrating the 1-form $d^c G_K$ over ∂K . Thus we have

$$\begin{aligned} \int_K \log |p| \mu_K &= \frac{1}{2\pi} \int_K (\log |p| - N G_K) dd^c G_K \\ &= \frac{1}{2\pi} \int_{-\partial(\mathbb{C}-K)+\{\infty\}} (\log |p| - N G_K) d^c G_K + \frac{1}{2\pi} \int_{-\{\infty\}} (\log |p| - N G_K) d^c G_K \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{2\pi} \int_{\mathbb{C}^-_K} d(\log|p| - NG_K) \wedge d^c G_K - \frac{1}{2\pi} \int_{-\{\infty\}} (N\rho_K + O(|y|^{-1})) d^c G_K \\ &= -\frac{1}{2\pi} \int_{\mathbb{C}^-_K} dG_K \wedge d^c(\log|p| - NG_K) - N\rho_K \end{aligned}$$

Since the function $\log|p| - NG_K$ is smooth at ∞ , we may integrate by parts to obtain

$$\begin{aligned} \int_K \log|p| \mu_K &= \frac{1}{2\pi} \int_K (\log|p| - NG_K) dd^c G_K \\ &= \frac{1}{2\pi} \int_{\mathbb{C}^-_K} G_K dd^c \log|p| - N\rho_K \\ &= \sum_{\{c: \rho(c)=0\}} G_K(c) - N\rho_K . \end{aligned}$$

Remark. We conclude from this Lemma that if $\rho_K = 0$, and if $p(y)$ is a monic polynomial, then

$$\int \log|p| \mu_K \geq 0 .$$

Theorem 3.2. $\lambda \geq \log d$.

Proof. Let $v_p = \partial_y(p)$ be the vertical vector at the point p . We have that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|Df^n(v_p)\| = \lambda(v_p) \leq \lambda_1 .$$

This gives

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|Df^n(v_p)\| d\mu(p) \leq \lambda_1 .$$

Let $X = \{x = 0\}$ denote the y -axis. Let us write $K_0^+ = X \cap K^+$. Thus we also have $K_0^+ = \{y: f^n(0, y) \in K^+\}$.

By [BS1] we know that $(d^{-n})[f^n X] \wedge \mu^+$ converges to μ_K as $n \rightarrow \infty$. Thus we see that

$$\int \frac{1}{k} \log \|Df^k(v)\| \mu_K = \lim_{n \rightarrow \infty} \frac{1}{k} \int (d^{-n}) \log \|Df^k(v)\| [f^n X] \wedge \mu^+ .$$

To estimate this integral from below, we note that if $f = (f_{(1)}, f_{(2)})$, then $\|Df^k(v)\| \geq |\partial_y f_{(2)}^k|$. Further, $\partial_y f_{(2)}^k = d^k y^{dk-1} + \dots$, where the dots represent terms of lower degree.

We note that the current $[f^k X] \wedge \mu^+$ is the same as the current μ^+ restricted to the submanifold $[f^k X]$. Since $f^k X$ is in fact tangent at infinity to the y -axis, we see that $G^+|_{f^k X}$ is the Green function of K_0^+ inside the variety $f^k X$. Thus $[f^k X] \wedge \mu^+$ is the same as the equilibrium measure $\mu_{K_0^+}$ of K_0^+ in X pushed forward by f^k to $f^k X$. We use this now to evaluate our integral:

$$\begin{aligned} \int \log \|Df^k(v_p)\| \mu_K &\geq \int \log |\partial_y f_{(2)}^k| \mu_K \\ &= \lim_{n \rightarrow \infty} \int \log |\partial_y f_{(2)}^k| (d^{-n})[f^n X] \wedge \mu^+ \end{aligned}$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \int \log |\partial_y f_{(2)}^k| f_*^n [X] \wedge f_*^n \mu^+ \\
 &= \lim_{n \rightarrow \infty} \int f^{n*} (\log |\partial_y f_{(2)}^k|) [X] \wedge \mu^+ \\
 &= \lim_{n \rightarrow \infty} \int \log |(\partial_y f_{(2)}^k) \circ f^n| \mu_{K_0^+} .
 \end{aligned}$$

We observe further that

$$(\partial_y f_{(2)}^k) \circ f^n = d^k y^{(dk-1)d^n} + \dots$$

so that

$$\log |(\partial_y f_{(2)}^k) \circ f^n| = k \log d + \log |p(y)| ,$$

where $p(y)$ denotes a monic polynomial. Thus we have

$$\begin{aligned}
 \int \frac{1}{k} \log \|Df^k(v_p)\| d\mu_K &\geq \frac{1}{k} \int \log |(\partial_y f_{(2)}^k) \circ f^n| \mu_{K_0^+} \\
 &\geq \log d + \frac{1}{k} \int \log |p| \mu_{K_0^+} \\
 &\geq \log d
 \end{aligned}$$

by the Lemma above. The Theorem follows upon letting $k \rightarrow \infty$.

Corollary 3.3. $\lambda_1 > 0 > \lambda_2$.

Proof. We have $\lambda_1 \geq \log d$ and $\lambda_2 = \log |\delta| - \lambda_1 \leq -\log d$.

We say an ergodic measure μ is *hyperbolic* if no Lyapunov exponent is equal to zero. Hyperbolic measures have some of the properties of hyperbolic sets.

Theorem 3.4. *Periodic saddle orbits are dense in J^* .*

Proof. Katok proves [K, Theorem 4.2] that for a non-atomic ergodic hyperbolic measure μ the closure of the set of periodic saddle points contains the support of μ .

Corollary 3.5. *If K is totally disconnected then periodic points are dense in K .*

Proof. This is Corollary 2.4 combined with Theorem 3.4.

4 The entropy of μ

In this section we will show (Theorem 4.4) that the measure theoretic entropy of μ is $h_\mu(f) = \log d$ and derive some consequences about the Hausdorff dimension of the measure μ .

We begin by defining topological and measure theoretic entropy and discussing the variational principle which relates the two quantities. We will discuss Misiurewicz’s proof of the variational principle and extract from this proof an idea which we will use in the proof of Theorem 4.4. We then introduce measures σ_n and μ_n which will be used in the proof. We prove two Lemmas which describe convergence properties of these measures, and then we give the proof of the Theorem.

First we define the topological entropy of a continuous map f of a compact set X . For each n set

$$d_n(x, y) = \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)) .$$

A set $E \subset X$ is (n, ϵ) -separated if the distance between distinct points in E is at least ϵ in the d_n metric. Let $s_n(\epsilon)$ denote the cardinality of the largest (n, ϵ) -separated set. Define $s(\epsilon)$ by the following formula:

$$s(\epsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(\epsilon)$$

The topological entropy is given as follows:

$$h_{\text{top}}(f) = \lim_{\epsilon \rightarrow 0} s(\epsilon)$$

Recall from [FM] and [S] that the topological entropy of $f|_K$ is equal to $\log d$.

We now define measure theoretic entropy. Let m be an invariant probability measure on X . Let $\mathcal{A} = \{A_1 \dots A_k\}$ be a partition of X , and let $\bigvee_{i=0}^{n-1} f^{-i}\mathcal{A}$ be the partition generated by $\mathcal{A}, f^{-1}\mathcal{A}, \dots, f^{-n+1}\mathcal{A}$. If we set

$$H(\mathcal{A}) = - \sum_{i=1}^k m(A_i) \log m(A_i) ,$$

$$h(\mathcal{A}, f) = \lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{A} \right) ,$$

then the measure theoretic entropy is given by $h_m(f) = \sup_{\mathcal{A}} h(\mathcal{A}, f)$.

The variational principle states that

$$h_{\text{top}}(f) = \sup_m h_m(f) ,$$

where the supremum is taken over all invariant probability measures. Theorem 4.4 thus states that μ is a measure of maximal entropy.

Our proof of Theorem 4.4 will use some ideas from Misiurewicz’s proof of the variational principle (see [W, p. 189]). We begin by reviewing his proof and extracting the results we will use. The proof of the equality $h_{\text{top}}(f) = \sup_m h_m(f)$ follows from the proof of two inequalities. We will consider the proof that $h_{\text{top}} \leq \sup_m h_m$.

For each n , choose an (n, ϵ) -separated set E_n of cardinality $s_n(\epsilon)$. Let

$$\sigma_n = \frac{1}{s_n(\epsilon)} \sum_{x \in E_n} \delta_x .$$

For any partition \mathcal{A} into sets of diameter less than ϵ we have

$$H_{\sigma_n} \left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{A} \right) = \log s_n(\epsilon) .$$

This equality follows from the fact that each set in $\bigvee_{i=0}^{n-1} f^{-i}\mathcal{A}$ contains at most one element of E_n .

Now we write

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_*^i(\sigma_n)$$

and let n_i be a sequence of natural numbers so that $\lim_{j \rightarrow \infty} \frac{1}{n_j} \log s_{n_j}(\epsilon) = s(\epsilon)$ and μ_{n_j} converges to μ^* . The definition of μ_n insures that any such μ^* is an invariant measure. Misiurewicz proves $h_{\mu^*} \geq s(\epsilon)$.

In the course of the proof Misiurewicz proves that for any sequence of probability measures σ_n which satisfy

$$H_{\sigma_n} \left(\bigvee_{i=0}^{n-1} f^{-i} \mathcal{A} \right) \geq \log c_n$$

the limit measure μ^* satisfies

$$h_{\mu^*}(f) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log c_n. \tag{4.1}$$

It is this fact that we wish to use in the proof of Theorem 4.4.

For our measure μ we start with a current S as in (1.1). Let L denote a \mathbb{C}^2 , psh function on \mathbb{C}^2 such that

$$L(x, y) = \log |y| + O(1)$$

where the $O(1)$ is uniformly bounded for $|x| \leq O(|y|^{1/d_1})$ and $y \rightarrow \infty$. Thus

$\Theta := \frac{1}{2\pi} dd^c L$ is a positive (1,1) form on \mathbb{C}^2 . For each n we set $\Theta_n = f^{*n} \Theta$, and we define a measure

$$\nu_n = S \llcorner \Theta_n,$$

where the integral of a compactly supported, continuous function φ is given by

$$\int \varphi d\nu_n = \langle S, \varphi \Theta_n \rangle.$$

Lemma 4.1. *Let $\{j_n\}$ be a sequence such that $1 < j_n < n$, and $j_n \rightarrow \infty$ and $n - j_n \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\lim_{n \rightarrow \infty} d^{-n} f_*^{j_n} \nu_n = c \mu,$$

where

$$c = \int S \wedge \mu^-. \tag{4.2}$$

Proof. We compute

$$\begin{aligned} (d^{-n}) f_*^{j_n} \nu_n &= (d^{-n}) (f_*^{j_n} S \llcorner f_*^{j_n} \Theta_n) \\ &= (d^{-j_n}) f_*^{j_n} S \llcorner (d^{j_n - n}) f_*^{n - j_n} \Theta \\ &= (d^{-j_n}) f_*^{j_n} S \llcorner (d^{j_n - n}) \frac{1}{2\pi} dd^c (L \circ f^{n - j_n}). \end{aligned}$$

Now by §3 of [BS1] the functions $G_{n-j_n}^+ := (d^{j_n - n}) L \circ f^{n - j_n}$ converge uniformly to G^+ on compact subsets of \mathbb{C}^2 as $n - j_n \rightarrow \infty$. By Theorem 1.6 $(d^{-j_n}) f_*^{j_n} S$ converges weakly in the sense of currents to $c \mu^-$ as $j_n \rightarrow \infty$. Without loss of generality we may assume that S is positive, so the product $\frac{1}{2\pi} G_{n-j_n}^+ (d^{-j_n} f_*^{j_n} S)$ converges weakly in the sense of currents to $\frac{c}{2\pi} G^+ \mu^-$. Applying dd^c , we obtain the sum of four terms

$$\begin{aligned} &\frac{1}{2\pi} dd^c G_{n-j_n}^+ (d^{-j_n} f_*^{j_n} S) + \frac{1}{2\pi} dG_{n-j_n}^+ (d^{-j_n}) f_*^{j_n} (d^c \psi \wedge T) - \frac{1}{2\pi} d^c G_{n-j_n}^+ (d^{-j_n}) f_*^{j_n} (d\psi \wedge T) \\ &+ \frac{1}{2\pi} G_{n-j_n}^+ (d^{-j_n}) f_*^{j_n} (dd^c \psi \wedge T). \end{aligned}$$

The last three currents tend to 0 as n and $j_n \rightarrow \infty$. To see this, we act upon them by a test function and integrate by parts to remove the d or d^c from G_{n-j}^+ . The result then follows by (iv) of Theorem 1.6. Finally, the first term is equal to $d^{-n} f_*^j \nu_n$, which converges to $c\mu^+ \wedge \mu^- = c\mu$. This completes the proof.

Now we set

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \frac{d^{-n}}{c} f_*^j (S \llcorner \Theta_n)$$

with the constant c in (4.2) being assumed to be nonzero.

Lemma 4.2. $\lim_{n \rightarrow \infty} \mu_n = \mu$.

Proof. Choose a sequence $k_n \rightarrow \infty$ such that $k_n/n \rightarrow 0$. Then

$$\mu_n = \frac{1}{n} \left(\sum_{j=0}^{k_n} + \sum_{j=k_n+1}^{n-k_n} + \sum_{j=n-k_n+1}^{n-1} \right) f_*^j S \llcorner \Theta_n .$$

The total mass of the first and the third sums on a fixed compact set vanishes as $n \rightarrow \infty$. We recall from the proof of Lemma 4.1 that $f_*^j S \llcorner \Theta_n = \frac{1}{2\pi} dd^c(G_{n-j}^+ \wedge \mu_j^-)$.

Thus the middle summation is given by dd^c of

$$\left(\frac{1}{n} \sum_{j=k_n+1}^{n-k_n} G_{n-j}^+ \mu_j^- \right) = \left(G^+ \frac{1}{n} \sum_{j=k_n+1}^{n-k_n} \mu_j^- \right) + \frac{1}{n} \sum_{j=k_n+1}^{n-k_n} (G_{n-j}^+ - G^+) \mu_j^- .$$

Since on a compact subset of \mathbf{C}^2 the functions $(G_{n-j}^+ - G^+)$ converge uniformly to zero, and μ_j^- has uniformly bounded mass, the last term on the right converges to zero as $n - k_n + 1 \leq n - j \rightarrow \infty$. The Lemma then follows since $\frac{1}{n} \sum_{j=k_n+1}^{n-k_n} \mu_j^- \rightarrow \mu^-$.

Now we construct the sequence of measures that will be used in the proof of Theorem 4.4. Let us use the notation $V^- = \{(x, y): |y| > R \text{ and } |y| > |x|\}$, $V^+ = \{(x, y): |x| > R \text{ and } |y| < |x|\}$, and $V = \{(x, y): |x| \leq R \text{ and } |y| \leq R\}$. For R large, we have the filtration properties

$$f(V^-) \subset V^-, \quad f(V^- \cup V) \subset V^- \cup V \tag{4.3}$$

$$f^{-1}(V^+) \subset V^+, \quad f^{-1}(V^+ \cup V) \subset V^+ \cup V \tag{4.4}$$

(cf. [BS1, Section 2]).

Let $\mathcal{D} = \{x = 0, |y| < R\}$ denote the disk of radius R in the y -axis, and let $i: \mathcal{D} \rightarrow V$ denote the inclusion map. Let L be a smooth, subharmonic function of $|y|$ such that $L(y) = \log |y|$ for $|y| > R$, and set $\Theta = \frac{1}{2\pi} dd^c L$. If we let $\pi_y(x, y) = y$ be the projection to the y -axis, then $\pi_y \circ f^n \circ i: \mathcal{D} \rightarrow \mathbf{C}$, and we may define the measure

$$\alpha_n = (\pi_y \circ f^n \circ i)_* \Theta|_{\mathcal{D}} = [\mathcal{D}] \llcorner \Theta_n ,$$

where we first pull the form Θ back to \mathbf{C}^2 and then restrict to \mathcal{D} . By the choice of R , $\pi_y f^n i \mathcal{D}$ covers \mathcal{D} with multiplicity d^n , so $\int_{\mathcal{D}} \alpha_n = d^n \int_{\mathcal{D}} \alpha_1$. Also by the choice of R , we have $\int \mu^- \wedge [\mathcal{D}] = 1$, and thus the measure $\sigma_n = d^{-n} \alpha_n$ is a probability measure. If we define

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j \sigma_n ,$$

then

Corollary 4.3. $\lim_{n \rightarrow \infty} \mu_n = \mu$.

The following result was proved for f a hyperbolic diffeomorphism in [BS1]. We now prove it for general f .

Theorem 4.4. $h_\mu(f) = \log d$.

Proof. By the variational principle we have $h_\mu(f) \leq \log d$. Thus it suffices to prove $h_\mu(f) \geq \log d$. Let σ_n and μ_n be defined as above, and let \mathcal{A} be a partition of V so that the μ measure of the boundary of each element of \mathcal{A} is zero and each element of \mathcal{A} has diameter less than ε .

We will use a result of Yomdin [Yd] to estimate $H_{\sigma_n}(\bigvee_{j=0}^{n-1} T^{-j}\mathcal{A})$. Yomdin's result implies that the rate of area growth of pieces of the disk which remain in ε balls goes to zero as ε goes to zero.

Any element of $\bigvee_{j=0}^{n-1} T^{-j}\mathcal{A}$ is contained in an ε -ball B in the d_n -metric. The measure σ_n is given by

$$\sigma_n(B) = d^{-n} \int_{B \cap \mathcal{L}} f^{*n}\Theta = d^{-n} \int_{f^n(B \cap \mathcal{L})} \Theta .$$

It is evident that Θ is bounded above on C^2 , and thus the right hand integral is dominated by $C \text{Area}(f^n(B \cap \mathcal{D}))$. We let $v^0(f, t, n, \varepsilon)$ denote the supremum of the area of $f^n(B \cap \mathcal{D})$ over all ε -balls B . Thus the σ_n measure is bounded above by $Cd^{-n}v^0(f, t, n, \varepsilon)$. This gives

$$H_{\sigma_n} \left(\bigvee_{j=0}^{n-1} T^{-j}\mathcal{A} \right) \geq -\log C + n \log d - \log v^0(f, t, n, \varepsilon) .$$

By Corollary 4.3 $\mu_n \rightarrow \mu$, so the Misiurewicz result (4.1) gives:

$$h_\mu \geq \log d - \limsup_{n \rightarrow \infty} v^0(f, t, n, \varepsilon) .$$

Now let $v^0(f, \varepsilon) = \limsup_{n \rightarrow \infty} v^0(f, t, n, \varepsilon)$. Yomdin [Yd, Theorem 1.8] shows that $v^0(f, \varepsilon)$ goes to zero as $\varepsilon \rightarrow 0$ so that $h_\mu \geq \log d$. This completes the proof.

In [S] it was shown that the topological entropy of f restricted to K is $\log d$. Using Theorem 4.4 we can sharpen that result.

Corollary 4.5. *The topological entropy of f restricted to J or J^* is $\log d$.*

Proof. $\log d = h_{\text{top}}(f|_K) \geq h_{\text{top}}(f|_J) \geq h_{\text{top}}(f|_{J^*}) \geq h_\mu(f) = \log d$.

The Hausdorff dimension of a measure m , written $\text{HD}(m)$, is defined to be the infimum of the Hausdorff dimensions of sets of full m measure.

Corollary 4.6. *If $|\delta| \leq 1$, the Hausdorff dimension of the measure μ is given by:*

$$\text{HD}(\mu) = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) \log d = \left(\frac{1}{\Lambda} + \frac{1}{\Lambda - \log|\delta|} \right) \log d . \tag{4.5}$$

Proof. This follows from [Yg, Corollary 4.1].

Remark. The Hausdorff dimension of the set J^* is at least as large as (4.5).

Corollary 4.7. $\text{HD}(\mu) \leq 2$; in fact $\text{HD}(\mu) < 2$ unless $|\delta| = 1$ and $\Lambda = \log d$.

Proof. Without loss of generality, we may assume that $|\delta| \leq 1$. Thus $\lambda_1 = A \geq \log d$ and $-\lambda_2 = A + \log(1/|\delta|) \geq \log d$. Thus the only way that $\text{HD}(\mu) = 2$ can hold in (4.5) is if $A = \log d$ and $\log(1/|\delta|) = 0$.

5 Dependence of A on parameters

Here we let A denote an open subset of \mathbf{C}^j , and we let $f = f_a$ depend holomorphically on $a \in A$. In this case we will write $A = A(a)$. We note that we have also used a to denote one specific parameter, the complex Jacobian determinant of f , but there should be no confusion when we use a to denote a general parameter.

Let us write

$$A_n = \frac{1}{n} \int \log \|Df^n(x)\| \mu(x) .$$

We recall from the chain rule that

$$Df^n(x) = \prod_{i=1}^n Df(x_{i-1})$$

where $x_i = f^i(x)$. It follows from the chain rule and the submultiplicativity of the operator norm $\| \cdot \|$ that

$$\|Df^{m+n}(x)\| \leq \|Df^m(x)\| \|Df^n(x_{m-1})\| .$$

Taking logarithms and using the invariance of the measure μ , we have

$$(m + n)A_{m+n} \leq mA_m + nA_n . \tag{5.1}$$

Lemma 5.1. *The limit defining A exists, and $a \mapsto A_a$ is upper semicontinuous (usc).*

Proof. It is well known (see [W, p. 87]) that the condition (5.1) implies that $\lim_{n \rightarrow \infty} A_n$ exists. It is also easily seen that $k \mapsto A_{2^k}$ is monotone decreasing. Now for fixed n , the map

$$a \mapsto \frac{1}{n} \int \|Df_a^n\| \mu_a$$

is continuous. Taking the limit as $n \rightarrow \infty$ through the values $n = 2^k$, we see that A_a is a decreasing limit of continuous functions and is thus usc.

Now we define the direct image of a function under a proper holomorphic mapping. We let X and Y be manifolds, and we let $g: X \rightarrow Y$ denote a smooth, proper mapping. If ϕ is a function on X , we define

$$g_* \phi(\zeta) = \sum_{x \in g^{-1}(\zeta)} \phi(x) .$$

If ϕ is continuous or has compact support then the same holds for $g_* \phi$. Since g is proper, $g(X) = Y$, and there is a positive integer p such that a generic point $y \in Y$ has p preimages. If we let $U \subset X$ denote the points where g is a local diffeomorphism, then there are disjoint open subsets $U_j \subset U$, $1 \leq j \leq p$, with the properties that $U = \bigcup U_j$ and $Y - g(U_j)$ have measure zero, and $g: U_j \rightarrow g(U_j)$ is a diffeomorphism. Thus for any volume form θ on Y , we may apply the change of variables formula to each U_j to obtain

$$p \int \phi g^* \theta = \int g_* \phi \theta .$$

Lemma 5.2. *Let $g_a(x)$ and $h_a(x)$ depend holomorphically on $a \in \mathbb{C}^j$ and $x \in \mathbb{C}^n$, and for fixed a let g_a map \mathbb{C}^n properly to itself. If $\psi(x)$ is continuous and psh on \mathbb{C}^n , and if*

$$I(a) = \int \psi(h_a(x))(dd^c \log^+ |g_a(x)|)^n,$$

then $I(a)$ is continuous and psh on \mathbb{C}^j .

Proof. Let us consider a sequence of psh smoothings \log_ε of \log . Then the currents $\bigwedge^n dd^c \log_\varepsilon^+ |g_a(x)|$ which we write as $(dd^c \log_\varepsilon^+ |g_a(x)|)^n$ are pullbacks under g_a of the volume forms $(dd^c \log_\varepsilon^+ |\zeta|)^n$. It follows from the change of variables formula given above that

$$I_\varepsilon(a) = \int h_{a*}(\psi \circ h_a)(\zeta)(dd^c \log_\varepsilon^+ |\zeta|)^n.$$

It is evident that the direct image $(g_a)_*(\psi \circ h_a)$ is psh as a function of x and a jointly, and thus it is psh in a . Thus we see that $I_\varepsilon(a)$ is an average of psh functions and is thus psh itself. The result follows upon letting ε tend to zero.

We will need to use the following Lemmas 5.3 and 5.4, which are easy consequences of §3 of [BS1].

Lemma 5.3. *The sequence $G_n := (d^{-n})\log^+ |f^n - f^{-n}|$ converges uniformly on compact subsets of \mathbb{C}^2 to the function $G = \max(G^+, G^-)$.*

Lemma 5.4. *The mapping $g_{a,n} := f^n - f^{-n}$ is a proper, holomorphic mapping of \mathbb{C}^2 to itself.*

Theorem 5.5. *$a \mapsto A(a)$ is psh.*

Proof. Let us consider

$$A_{m,n}(a) := \frac{1}{m} \int \log \|Df_a^m\| (dd^c G_n)^2.$$

We let ψ be defined on $\mathbb{C}^4 = \mathbb{C}^{2 \times 2}$ by $\psi(\zeta) = \log \|\zeta\|$, where we identify ζ with a 2 by 2 matrix, and $\|\zeta\|$ denotes the matrix (operator) norm. We let $h_a(x) = Df_a^m(x)$ be the jacobian matrix of f_a^m . Since $g_{a,n}$ is a proper mapping and G_n has the correct form, we see from Lemma 5.2 that $A_{m,n}(a)$ is psh in a for all m and n .

If we take the limit as $n \rightarrow \infty$, we have by Lemma 5.4 that $(dd^c G_n)^2$ converges to μ_a , and so $A_{m,n}(a)$ converges to the function $A_m(a)$ defined above. Now if $a \mapsto A_m(a)$ is psh in a then if we take the limit $m \rightarrow \infty$ the functions $A_m(a)$ decrease to $A(a)$, and so $a \mapsto A(a)$ is psh, which completes the proof of the Theorem.

Corollary 5.6. *If ψ is psh on \mathbb{C}^2 , then $a \mapsto \int \psi \mu_a$ is psh.*

Proof. Apply Lemma 5.2 with $h_a(x) = x$ and $g_{a,n} = f^n - f^{-n}$. Then take the limit as $n \rightarrow \infty$.

We can derive a stronger conclusion if a is a value of the parameter for which f_a is hyperbolic. Recall that $X \subset \mathbb{C}^2$ is a hyperbolic set for the map f if there is a continuous splitting of the tangent bundle over X into subspaces E^s and E^u and constants c and $0 < \lambda < 1$ such that

$$\|Df^n|_{E^s}\| < c\lambda^n, \quad \|Df^{-n}|_{E^u}\| < c\lambda^n \quad n > 0.$$

In the terminology of [BS1] f is said to be *hyperbolic* if J is a hyperbolic set for f . In [BS3] it is shown that this assumption is equivalent to the assumption that the chain recurrent set $R(f)$ is a hyperbolic set for f . Let $\mathcal{H} \subset A$ be the set of parameters for which f_a is hyperbolic. In [BS1] it is shown that \mathcal{H} is open.

Theorem 5.7. $A(a)$ is pluriharmonic for $a \in \mathcal{H}$.

Proof. For a hyperbolic mapping we can choose an adapted metric on the tangent bundle over J so that

$$\|Df|_{E^s}\| < \lambda', \quad \|Df^{-1}|_{E^u}\| < \lambda'$$

for some $\lambda' < 1$. We may also assume that E^s and E^u are perpendicular subspaces. With these hypotheses we have $\|Df(p)\| = \|Df|_{E_p^u}\|$.

For $a \in \mathcal{H}$ and a positive integer N , we set

$$S(N) = \{(a, p) \in \mathcal{H} \times \mathbb{C}^2: f_a^N(p) = p\},$$

and we let $S(N, a_0) = S(N) \cap \{a = a_0\}$. It is shown in [BS1] that, for $a \in \mathcal{H}$, μ_a is the limit of the average of the point masses over the periodic saddle points. Thus if we set

$$\mu_{a_0}^N := \#S(N, a_0)^{-1} \sum_{p \in S(N, a_0)} \delta_p,$$

we have

$$\mu_{a_0} = \lim_{N \rightarrow \infty} \mu_{a_0}^N.$$

For $p \in S(N, a_0)$, we have

$$\log \|Df_a^k(p)\| = \sum_{i=1}^k \log |Df_a|_{E_a^u(f^{i-1}(p))}|.$$

The $p_i(a)$ depend algebraically on a . Away from those values where periodic points coalesce the $p_i(a)$ depend holomorphically on a . To any periodic point p of period N we can assign a multiplicity which is just the multiplicity of p as a root of the fixed-point equation $f^n(x) - x = 0$. The sum of the multiplicities of points in $S(N, a)$ is constant. When distinct points coalesce they yield a point of multiplicity greater than one. For $a \in \mathcal{H}$ the $p_i(a)$ are in saddle orbits thus one eigenvalue of $Df(p)$ is greater than one in absolute value and one eigenvalue is smaller. This shows that $Df^n(p) - I$ is nonsingular. $Df^n(p) - I$ is the differential of the function $f^n(x) - x$. Since it is nonsingular p is a regular point for the function $f^n(x) - x$. This implies that $p_i(a)$ has multiplicity one. We conclude that for $a \in \mathcal{H}$ the $p_i(a)$ depend holomorphically on a .

Since for $p \in S(N, a)$, the expanding subspaces are determined by the condition $Df_p^N(E_p^u) = E_p^u$, we see that $E_{p(a)}^u$ varies holomorphically in a . By the formula above, then, $a \mapsto \log \|Df_a^k(p(a))\|$ is pluriharmonic on A . Thus we conclude that

$$\frac{1}{k} \int \log \|D^k f_a\| \mu_{N,a} = k^{-1} \#S(N, a)^{-1} \int \sum_{i=1}^k \sum_{p \in S(N, a)} \log |Df_a|_{E_a^u(f^{i-1}(p(a)))}|$$

is pluriharmonic. The Theorem then follows by letting $k \rightarrow \infty$ and then letting $n \rightarrow \infty$.

6 Degeneration to 1-dimensional mappings

Fix integers d_1, \dots, d_m where $d_i \geq 2$. We consider the family of mappings $f = f_m \circ \dots \circ f_1$ such that $f_a(x, y) = (y, p_j(y) - a_j x)$ where p_j is a monic polynomial of degree d_j . When we wish to stress the dependence on parameters we write

$f_a = f_{m,a} \circ \dots \circ f_{1,a}$ where $a = (a_1, \dots, a_m, p_1, \dots, p_m)$. We will consider a to be a point in \mathbf{C}^J . As $\delta(a) = a_1 \dots a_m \rightarrow 0$, f_a approaches a mapping of rank 1, which is essentially 1-dimensional. In this section we show that μ^\pm and μ converge to the corresponding one dimensional objects. We also show that the Lyapunov exponent is well behaved as a function on extended parameter space.

If $\delta(a) \neq 0$, f_a is invertible, and the inverse is given by $f^{-1} = f_1^{-1} \circ \dots \circ f_m^{-1}$, where

$$f_j^{-1}(x, y) = (a_j^{-1} p_j(x) - a_j^{-1} y, x). \tag{6.1}$$

By repeating the proofs of Lemmas 2.1 and 2.2 of [BS1], we see that for any compact subset $A_0 \subset \mathbf{C}^J$ there exists R such that (4.3) and (4.4) hold for $f = f_a$ such that $a \in A_0$ and $\delta(a) \neq 0$. We may also repeat the proof of Proposition 3.4 of [BS1] to see that $(a, x, y) \mapsto G_a^+(x, y)$ is continuous for $a \in \mathbf{C}^J$ and $(x, y) \in \mathbf{C}^2$.

We will be interested in the behavior of f_a as $a \rightarrow a^0 = (a_1^0, \dots, a_m^0, \dots)$, where $a_j^0 = 0$ for some j with $1 \leq j \leq m$. We note that the mapping $a \mapsto f_a$ is injective on the set where $\delta(a) \neq 0$ by [FM] but it is not injective in general. Now f_a is conjugate to the mapping $f_j \circ \dots \circ f_1 \circ f_m^0 \circ \dots \circ f_{j+1}$ via $\varphi = f_1 \circ \dots \circ f_{j-1}$. Thus, without loss of generality, we may assume that $a_m^0 = 0$. We define

$$\Gamma = \{(\zeta, p_m(\zeta)): \zeta \in \mathbf{C}\} = \{y = p_m(x)\}.$$

Thus $f_{a^0}(\mathbf{C}^2) = \Gamma$, and in fact $f_{a^0}|_\Gamma$ is conjugate to the polynomial mapping $q: \mathbf{C} \rightarrow \mathbf{C}$, where $q(\zeta)$ is defined by

$$f_{a^0}(\zeta, p_m(\zeta)) = (q(\zeta), p_m(\zeta)).$$

We let J_q denote the Julia set of q , and we let $J_{a^0} = \{(\zeta, p_m(\zeta)): \zeta \in J_q\}$ denote its graph in Γ . If $G_{J_q}(\zeta)$ is the Green function of J_q in \mathbf{C} , and if \tilde{G}_{J_q} is the lift to Γ , then

$$G_{a^0}^+ = \tilde{G}_{J_q} \circ f_{a^0}. \tag{6.2}$$

Lemma 6.1. *Let $A_0 \subset \mathbf{C}^J$ be compact. Then for a compact subset $S \subset V - \Gamma$, there exists $\delta > 0$ such that $K_a^- \cap S = \emptyset$ for $a \in A_0$ such that $|a_1| < \delta$.*

Proof. Let $\varepsilon := \min_S |p_m(x) - y| > 0$ and set $\delta = R/\varepsilon$. Then by (6.1), $f_1^{-1}(x, y) \in V^+$. Applying equations (4.3) and (4.4) to f_j^{-1} for $2 \leq j \leq m$, we have $f^{-1}(x, y) \in V^+$. By [BS1, Lemma 2.4] $f^{-1}(x, y) \notin K^-$, so $(x, y) \notin K^-$.

From (6.1) it is easily seen that

$$f^{-1}(x, y) = (\tilde{\alpha}x^d + O(x^{d-1}), O(x^{d/d_m})), \tag{6.3}$$

where

$$\tilde{\alpha} = a_1^{-1} \prod_{j=2}^m a_j^{-d_1 \dots d_{j-1}}.$$

We would like to use the results of [BS1, Section 2] directly on f^{-1} . The only difference is that the polynomials in f_j^{-1} are not monic, so there is an extra constant to keep track of. Applying Corollary 2.6 of [BS1] in this case, we have: *For any $\delta > 0$ there exists R sufficiently large such that*

$$|(1 - \delta)\tilde{\alpha}||x|^d < |\pi_x f^{-1}(x, y)| < (1 + \delta)\tilde{\alpha}||x|^d$$

holds for all $(x, y) \in V^+$. Iterating this inequality we get

$$|(1 - \delta)\tilde{\alpha}|^{d^{n-1} + \dots + d^2 + d + 1} |x|^{d^n} < |\pi_x f^{-n}(x, y)| < (1 + \delta)\tilde{\alpha}|^{d^{n-1} + \dots + d^2 + d + 1} |x|^{d^n}$$

Taking logarithms and dividing by d^n gives:

$$\begin{aligned} \left(\frac{1}{d} + \dots + \frac{1}{d^n}\right) \log |(1 - \delta)\tilde{\alpha}| + \log |x| &< \frac{1}{d^n} \log |\pi_x f^{-n}(x, y)| \\ &< \left(\frac{1}{d} + \dots + \frac{1}{d^n}\right) \log |(1 + \delta)\tilde{\alpha}| + \log |x| \end{aligned}$$

Thus

$$G^-(x, y) = \log |x| + \frac{1}{d-1} \log |\tilde{\alpha}| + o(1) \tag{6.4}$$

where the $o(1)$ tends to 0 uniformly on V^+ as $|x| \rightarrow \infty$.

Lemma 6.2. *Taking limits through values $a \in C^J$ such that $\delta(a) \neq 0$, we have*

$$\lim_{a \rightarrow a^0} \left(G_a^-(x, y) - \frac{1}{d-1} \log |\tilde{\alpha}| \right) = \frac{1}{d_m} \log |p_m(x) - y|,$$

where the convergence is uniform on compact subsets of $C^2 - \Gamma$, and

$$\lim_{a \rightarrow a^0} \mu_a^- = [\Gamma]$$

in the sense of currents on C^2 .

Proof. By (6.1) we see that for x_0 fixed

$$f^{-1}(x_0, y) = ((-1)^{d_1 \dots d_{m-1}} \tilde{\alpha} y^{d/d_m} + O(y^{d/d_{m-1}}), O(y^{d/(d_1 d_m)})),$$

and the O terms are uniform for $|x_0| \leq R$. Thus by (6.4) we have

$$\begin{aligned} G_a^-(x_0, y) - \frac{1}{d-1} \log |\tilde{\alpha}| &= \frac{1}{d} G_a^-(f^{-1}(x_0, y)) - \frac{1}{d-1} \log |\tilde{\alpha}| \\ &= \frac{1}{d} (\log |\tilde{\alpha} y^{d/d_m}| + \frac{1}{d-1} \log |\tilde{\alpha}|) - \frac{1}{d-1} \log |\tilde{\alpha}| + o(1) \\ &= \frac{1}{d} \left(\frac{d}{d_m} \log |y| + \frac{d}{d-1} \log |\tilde{\alpha}| \right) - \frac{1}{d-1} \log |\tilde{\alpha}| + o(1) \\ &= \frac{1}{d_m} \log |y| + o(1). \end{aligned}$$

It follows that $G_a^-(x_0, y)$ is d_m^{-1} times the Green function of $K_a^- \cap \{x = x_0\}$. By Lemma 6.2 $K_a^- \cap \{x = x_0\} \subset \{|y - p_m(x_0)| < r(x_0, a)\}$, and $r(x_0, a) \rightarrow 0$ as $a \rightarrow a^0$. Thus $G_a^-(x_0, y)$ converges uniformly on compact subsets of $C^2 - \Gamma$ to

$$\frac{1}{d_m} \log |p_m(x) - y| \text{ as } a \rightarrow a^0.$$

In fact, as $a \rightarrow a^0$, $G_a^-(x_0, y)$ is a family of (normalized) potentials of measures whose supports decrease to the point $(x_0, p_m(x_0))$. Thus G_a^- converges to

$\frac{1}{d_m} \log |y - p_m(x)|$ locally in L^1 . The second statement then follows by applying dd^c

to $\frac{1}{2\pi} G_a^-$, taking the limit in the sense of currents, and using the Poincaré-Lelong formula $\frac{1}{2\pi} dd^c \log |p_m(x) - y| = [\Gamma]$.

Now let ν_q denote the equilibrium measure of J_q , and let μ_{a^0} be the lift of ν_q to Γ . We let Λ_q denote the Lyapunov exponent of q with respect to ν_q . Since the Lyapunov exponent is a conjugacy invariant, this is the same as the Lyapunov exponent of $f_{a^0}|_\Gamma$ with respect to μ_{a^0} .

Proposition 6.3. $\lim_{a \rightarrow a^0} \mu_a = \mu_{a^0}$, and $\limsup_{a \rightarrow a^0} \Lambda(a) = \Lambda_q$.

Proof. By (6.2) we have $\mu_a^\pm = f_{a^0}^* \mu_{a^0}$, where we identify the measure μ_{a^0} with a (1,1)-current supported on Γ . Since G_a^\pm converges uniformly on compacts to $G_{a^0}^\pm$, we may apply [BS1, Lemma 5.8] and Lemma 6.3 to obtain

$$\begin{aligned} \lim_{a \rightarrow a^0} \mu_a &= \lim_{a \rightarrow a^0} \mu_a^+ \wedge \mu_a^- \\ &= f_{a^0}^* \mu_{a^0} \wedge [\Gamma] = \mu_{a^0}. \end{aligned}$$

Now we observe that

$$\begin{aligned} \Lambda_q &= \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|Dq^n\| dv_q \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|Df_{a^0}^n\| d\mu_{a^0}. \end{aligned}$$

Thus we may apply Lemma 5.1 to conclude that $\limsup_{a \rightarrow a^0} \Lambda(a) = \Lambda_q$.

Corollary 6.4. *If we set $\Lambda(a^0) = \Lambda_q$, then $a \mapsto \Lambda(a)$ is psh on $\mathbb{C}^m \times A$.*

Now let us consider the case where the map q is hyperbolic, i.e. q is uniformly expanding on J_q . It follows that a sufficiently small perturbation is hyperbolic, i.e. J_a is a hyperbolic set for f_a . This was shown by Hubbard [H] and Fornæss and Sibony [FS] in the case where $m = 1$, i.e. $f = f_1$. We include here a proof in our somewhat more general context for the sake of the completeness of our exposition.

Proposition 6.5. *If f_{a^0} is hyperbolic, then there exists $\delta > 0$ such that f_a is hyperbolic as a 2-dimensional mapping if $|a - a^0| < \delta$.*

Proof. A convenient way to prove hyperbolicity is to show the existence of invariant cone fields \mathcal{C}^s and \mathcal{C}^u such that f is uniformly expanding on \mathcal{C}^u and uniformly contracting on \mathcal{C}^s (cf. [N, Theorem 2.2]). That is, for a point $p \in J_a$ there are proper, open cones \mathcal{C}_p^s and \mathcal{C}_p^u in the tangent space $T_p \mathbb{C}^2$, varying continuously with p , such that

$$Df(\mathcal{C}_p^u) \subset \mathcal{C}_{f(p)}^u, \quad \text{and} \quad Df^{-1}(\mathcal{C}_p^s) \subset \mathcal{C}_{f^{-1}(p)}^s,$$

and such that there is a constant $\lambda < 1$ such that

$$|Df^{-n}(v)| \leq \lambda |v| \text{ for } v \in Df^n \mathcal{C}_{f^{-n}(p)}^u, \text{ and} \tag{6.5}$$

$$|Df(v)| \leq \lambda |v| \text{ for } v \in Df^{-1} \mathcal{C}_{f(p)}^s, \tag{6.6}$$

where the length is taken with respect to some Riemannian metric in a neighborhood of J .

Let us show first that for any neighborhood U of \tilde{J}_q we will have $J_a \subset U$ if $|a - a^0|$ is small enough. For $\delta > 0$ we may assume by Lemma 6.2 that $K_a^- \cap \{|x| < R\}$ is contained in $\{|p_m(x) - y| < \delta\}$. Since q is hyperbolic, there are finitely many periodic sinks $\{s_1, \dots, s_J\}$, and $\text{int} K_q$ is the basin of attraction of $\{s_1, \dots, s_J\}$. In fact, for any $\varepsilon > 0$ and any open set $\omega \supset J_q$ there is a number n such that if $k \geq n$, then $\text{dist}(f^k(\zeta), \{s_1, \dots, s_J\}) < \varepsilon$ for $\zeta \in K_q - \omega$, and $|f^k(\zeta)| \geq \frac{1}{\varepsilon}$ for $\zeta \in \mathbb{C}^2 - (K_q \cup \omega)$. Thus f_{a^0} has attracting periodic orbits $\{\tilde{s}_1, \dots, \tilde{s}_J\} \subset \Gamma$; and for $|a - a^0|$ small, f_a has periodic sink orbits $\{\tilde{s}_1(a), \dots, \tilde{s}_J(a)\}$, whose basins contain ε -balls about $\tilde{s}_j(a)$. Furthermore, for $k \geq n$, we have $\text{dist}(f_a^k(x, y), \{\tilde{s}_1(a), \dots, \tilde{s}_J(a)\}) < \varepsilon$ for $\{(x, y): x \in K_q - \omega, |p_m(x) - y| < \delta\}$, and $|f_a^k(x, y)| > \frac{1}{\varepsilon}$ for $\{(x, y): x \in \mathbb{C} - (K_q \cup \omega), |p_m(x) - y| < \delta\}$. The points satisfying the first inequality are clearly in the interior of K^+ , and the ones satisfying the second inequality are in the complement of K^+ , so $J_a \subset \{(x, y): x \in \omega, |p_m(x) - y| < \delta\}$. Clearly we may take ω and δ small enough that this open set lies inside U .

To construct the cone fields, we note that mapping f_{a^0} is degenerate, and so for $p \in \Gamma$, $Df_{a^0}(T_p \mathbb{C}^2) = T_{f_{a^0}(p)} \Gamma$. For $p \in \Gamma$, we may let \mathcal{C}_p^u be a small neighborhood of $T_p \Gamma$, and we extend \mathcal{C}_p^u continuously to a neighborhood U of \tilde{J}_p . Thus we will have $Df_a \mathcal{C}_p^u \subset \mathcal{C}_{f_a(p)}^u$ for $p, f(p) \in U$ if $|a - a^0|$ is small. If we let \mathcal{C}^s be any conical neighborhood of the vector $D(f_m^{-1} \circ \dots \circ f_2^{-1}) \left(\frac{\partial}{\partial x} \right)$, then $Df_a^{-1} \mathcal{C}_p^s \subset \mathcal{C}_{f(p)}^s$ holds whenever $p, f(p) \in U$ if $|a - a^0|$, and thus $|a_1|$, is small.

Since q is hyperbolic, we may choose an adapted metric on Γ such that $|Dq(v)| > \lambda^{-1}|v|$ for all $p \in \Gamma$ and tangent vectors $v \in T_p \Gamma$. We may extend this metric to the tangent space $TC^2|_{\tilde{J}_q}$ by defining it in an arbitrary way on the normal bundle to Γ , and then we may extend this Riemannian metric continuously to a neighborhood U of \tilde{J}_q in \mathbb{C}^2 . Since we extended an adapted metric, (6.5) will hold on $U \cap J_a$ if $|a - a^0|$ is small. Property (6.6) holds without hyperbolicity. From the existence of the cone fields on a neighborhood of J_a we conclude that f_a is hyperbolic.

We let \mathcal{H}_1 denote the set of parameters $a^0 \in \mathbb{C}^J$ for which f_{a^0} is singular and corresponds to 1-dimensional hyperbolic mapping, and we will let $\mathcal{H}^* = \mathcal{H} \cup \mathcal{H}_1$ denote the parameter values $a \in \mathbb{C}^J$ such that f_a is a hyperbolic diffeomorphism or a singular hyperbolic mapping. By Proposition 6.5, \mathcal{H}^* is an open subset of the parameter space \mathbb{C}^J , and as in Corollary 6.4, we have

Corollary 6.6. $a \mapsto \Lambda(a)$ is pluriharmonic on \mathcal{H}^* .

Proposition 6.7. Let \mathcal{H}' be a connected component of extended hyperbolic parameter space \mathcal{H}^* such that there is a point $a^0 \in \mathcal{H}' \cap \mathcal{H}_1$. Then the Julia set of f_{a^0} is connected if and only if $\Lambda(a) = \log d$ for all $a \in \mathcal{H}'$. Otherwise, $\Lambda(a) > \log d$ for all $a \in \mathcal{H}'$.

Proof. We have seen that $a \mapsto \Lambda(a)$ is pluriharmonic on \mathcal{H}' . By Theorem 3.2 $\Lambda(a) \geq \log d$ so by the minimum principle for harmonic functions, either Λ is identically equal to $\log d$ on \mathcal{H}' , or it is everywhere greater than $\log d$ on \mathcal{H}' . Manning [M] showed that J_q is connected if and only if $\Lambda_q = \log d = \Lambda(a^0)$, which completes the proof.

We may view Proposition 6.7 as a characterization of when certain components \mathcal{H}' of hyperbolic parameter space have the property that $\Lambda = \log d$ on \mathcal{H}' . In the case of $m = 1$, [H] and [FS] showed that if $a^0 \in \mathcal{H}' \cap \mathcal{H}_1$, then the topology of J^+ is related to the topology of J_q for q corresponding to f_{a^0} . In fact, the intersection of J^+ with a transversal is locally connected if and only if J_q is connected. Since f_a is hyperbolic, it follows from the local product structure that J^+ is itself locally connected in this case. Combined with Proposition 6.7, this gives the result:

Corollary 6.8. *If $m = 1$, and if $\mathcal{H}' \cap \mathcal{H}_1 \neq \emptyset$, then Λ is equal to $\log d$ at some point in \mathcal{H}' if and only if J^+ is locally connected.*

References

- [BS1] Bedford, E., Smillie, J.: Polynomial diffeomorphisms of \mathbb{C}^2 : Currents, equilibrium measure and hyperbolicity. *Invent. Math.* **87** (1990)
- [BS2] Bedford, E., Smillie, J.: Fatou-Bieberbach domains arising from polynomial automorphisms. *Indiana Univ. Math. J.* **40**, 789–792 (1991)
- [BS3] Bedford, E., Smillie, J.: Polynomial diffeomorphisms of \mathbb{C}^2 . II. Stable manifolds and recurrence. *J. Am. Math. Soc.* **4**, 657–679 (1991)
- [BT] Bedford, E., Taylor, B.A.: Fine topology, Silov boundary and $(dd^c)^n$. *J. Funct. Anal.* **72**, 225–251 (1987)
- [Br] Brolin, H.: Invariant sets under iteration of rational functions. *Ark. Mat.* **6**, 103–144 (1965)
- [C] Carleson, L.: Complex dynamics. U.C.L.A. course notes 1990
- [FHY] Fathi, A., Herman, M., Yoccoz, J.-C.: A proof of Pesin's stable manifold theorem. In: *Geometric dynamics*. J. Palis (ed.). (Lect. Notes Math. vol. 1007, pp. 177–215) Berlin Heidelberg New York: Springer 1983
- [FS] Fornæss, J.-E., Sibony, N.: Complex Hénon mappings in \mathbb{C}^2 and Fatou Bieberbach domains. *Duke Math. J.* **65**, 345–380 (1992)
- [FLM] Freire, A., Lopes, A., Mañé, R.: An invariant measure for rational maps. *Bol. Soc. Bras. Mat.* **6**, 45–62 (1983)
- [FM] Friedland, S., Milnor, J.: Dynamical properties of plane polynomial automorphisms. *Ergodic Theory Dyn. Syst.* **9**, 67–99 (1989)
- [H] Hubbard, J.: personal communication
- [HO] Hubbard, J.H., Oberste-Vorth, R.: Hénon mappings in the complex domain. preprint
- [K] Katok, A.: Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publ. Math. Inst. Hautes Etud. Sci.* **51**, 137–174 (1980)
- [LS] Ledrappier, F., Strelcyn, J.-M.: A proof of the estimation from below in Pesin's entropy formula. *Ergodic Theory Dyn. Syst.* **2**, 203–219 (1982)
- [L] Lelong, P.: Fonctions plurisousharmoniques d'ordre fini dans \mathbb{C}^n . *J. Anal. Math.* **12**, 365–407 (1964)
- [Lj] Ljubich, M.Ju.: Entropy properties of rational endomorphisms of the Riemann sphere. *Ergodic Theory Dyn. Syst.* **3**, 351–385 (1983)
- [M] Manning, A.: The dimension of the maximal measure for a polynomial map. *Ann. Math.* **119**, 425–430 (1984)
- [N] Newhouse, S.: Lectures on dynamical systems, in *Dynamical Systems*, C.I.M.E. Lectures Bressanone, Italy, 1978. *Progress in Mathematics 8*. Basel Boston Stuttgart: Birkhäuser 1980
- [Pr] Przytycki, F.: Riemann map and holomorphic dynamics. *Invent. Math.* **85**, 439–455 (1986)
- [R] Ruelle, D.: Ergodic theory of differentiable dynamical systems. *Publ. Math. Inst. Hautes Etud. Sci.* **50**, 275–306 (1979)
- [Sk] Skoda, H.: Sous ensembles analytiques d'ordre fini ou infini dans \mathbb{C}^n . *Bull. Soc. Math. Fr.* **100**, 353–408 (1972)

- [Si] Sibony, N.: Iteration of polynomials U.C.L.A. lecture notes
- [S] Smillie, J.: The entropy of polynomial diffeomorphisms of \mathbf{C}^2 , Ergodic Theory Dyn. Syst. **10**, 823–827 (1990)
- [T] Torrat, P.: Aspects potentialistes de l'itération des polynômes. Séminaire de Théorie du Potential, Paris, No. 8 (Lect. Notes Math., vol. 1235) Berlin Heidelberg New York: Springer 1987
- [Ts] Tsuji, M.: Potential theory, New York: Chelsea 1975.
- [W] Walters, P.: An introduction to ergodic theory. Berlin Heidelberg New York: Springer 1982
- [Yd] Yomdin, Y.: Volume growth and entropy. Isr. J. Math. **57**, 285–300 (1987)
- [Yg] Young, L.-S.: Dimension, entropy and Lyapunov exponents, Ergodic Theory Dyn. Syst. **2**, 109–124 (1982)